JUMPS, CUSPS AND FRACTALS, IN TIME-EVOLUTION PDES

LYONELL BOULTON (HERIOT-WATT UNIVERSITY)
CURSILLO UNIVERSIDAD DE LOS ANDES[1]
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1. Preliminaries

The phenomenon of *revivals* in linear dispersive equations was discovered first experimentally in optics, in around 1834, then rediscovered several times by theoretical and experimental investigations. While the concept has been used systematically and consistently by many authors, there is no consensus on a rigorous definition. Several have described it by stating that a given periodic time-dependent boundary-value problem exhibits *a Talbot effect or dispersive quantisation*, if the solution evaluated at a certain dense subset of times, is given by finite superposition of translated copies of the initial condition. When this initial condition has jump discontinuities at time zero, these propagate and remain present in the solution at the times which are a rational multiple of the period.

The complementary phenomenon to revivals is *fractality*. One instance of this is that, for initial conditions with jump discontinuities, the solution is continuous in space for almost every time, but its graph is a curve of high fractal dimension.

In these cursillo notes, I fist present the classical framework, in the context of harmonic analysis, number theory and fractal geometry. Then, I report on the presence of revivals and fractality for three types of perturbations of the original equation, in the context of spectral theory. Concretely, I change the boundary conditions, add a potential term and make the equation non-local. The presentation is self-contained and I tried to calibrate the material to the level of a graduate student in Mathematical Sciences.

Section 2 summarises features of the theory developed by a number of authors over the last 25 years. Sections 3-5 report on research that I conducted in the last 5 years. The motivation to consider the different models was the PhD project of my former student George Farmakis of the MIGSAA Centre for Doctoral Training, who I co-supervised with my friend Beatrice Pelloni. It follows an earlier collaboration with my colleagues Dave Smith and Peter Olver. The part corresponding to the Benjamin-Ono equation was developed last Summer as an internship project for the 4th year undergraduate student of Heriot-Watt University, Breagh MacPherson. The final Appendix A contains exercises with full solutions. Some of these were developed for a graduate lecture course that I delivered at the Maxwell Institute for the Mathematical Sciences in Winter 2024.

I am very grateful to Monika Winklmeier, who organised such a stimulating event at the Universidad de Los Andes. I also wish to thank the participants of the cursillo, who contributed with so many interesting ideas to simplify many of my original proofs.

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1.1. **Conventions and notation.** Let $e_n(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{inx}$. Here and everywhere below, we write the Fourier coefficients of a periodic distribution F with one of the usual scalings on $\mathbb{T} = (-\pi, \pi]$, as

$$\widehat{F}(n) = \frac{1}{\sqrt{2\pi}}\langle F, e_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-iny} F(y)\,\mathrm{d}y.$$

See [19, Chapter 11] or [10, Section 9.3]. This choice makes either series in the expression

$$F(x) \sim \sum_{n\in\mathbb{Z}} \langle F, e_n\rangle e_n(x) = \sum_{n\in\mathbb{Z}} \widehat{F}(n)\mathrm{e}^{inx},$$

convenient for long calculations. Recall that $\{e_n\}_{n\in\mathbb{Z}} \subset \mathrm{L}^2(\mathbb{T})$ is an orthonormal basis of eigenfunctions for the Laplacian,

$$-\partial_x^2 : \mathrm{H}^2(\mathbb{T}) \longrightarrow \mathrm{L}^2(\mathbb{T}).$$

Indeed,

$$-e_n'' = n^2 e_n,$$

for all $n \in \mathbb{Z}$.

The following function spaces will be considered throughout. The definitions and relevant properties, are given in the next subsection and in the text.

- $\mathrm{BV}(\mathbb{T})$ functions of *bounded variation*,
- $\mathrm{AC}(\mathbb{T})$ *absolutely continuous* functions,
- $\mathrm{C}^\alpha(\mathbb{T})$ *Hölder continuous* functions of regularity $\alpha \in (0,1)$,
- $\mathrm{H}^\alpha(\mathbb{T})$ functions in the L^2 *Sobolev space* with regularity $\alpha \geq 0$,
- $\mathrm{B}_p^\alpha(\mathbb{T})$ distributions in the ℓ^∞-L^p *Besov space* with regularity $\alpha \in \mathbb{R}$ and $1 \leq p \leq \infty$.

On two occasions, instead of $\mathbb{T}$, we will use $(0, \pi]$ not identifying 0 with π. As the mapping of the definitions to this subsegment and the properties that we will require are standard, we omit any further details on this.

1.2. **Connections and properties of the classical function spaces.** Recall the definitions of $\mathrm{BV}(\mathbb{T})$ and $\mathrm{AC}(\mathbb{T})$, given in any classical analysis monograph such as [16, p.9 and p.47]. We know that

$$f \in \mathrm{AC}(\mathbb{T}) \qquad \Longleftrightarrow \qquad f' \in \mathrm{L}^1(\mathbb{T}).$$

We also know that $f \in \mathrm{BV}(\mathbb{T})$, if and only if f' is a complex-valued Radon measure on $\mathbb{T}$. The latter means that its real and imaginary parts, are finite signed measures on the Borel σ-algebra of $\mathbb{T}$, compatible with the topology. Moreover, if $f \in \mathrm{BV}(\mathbb{T})$, then

$$f = f_{\mathrm{ac}} + f_{\mathrm{s}},$$

for $f_{\mathrm{ac}} \in \mathrm{AC}(\mathbb{T})$ with $f'_{\mathrm{ac}} \in \mathrm{L}^\infty(\mathbb{T})$, and f'_{s} a singular measure with support of Lebesgue measure 0. For the proofs of these statements, see the two theorems on [16, p.53].

Problem 1. *Let $f \in \mathrm{BV}(\mathbb{T})$. Show that there exists a constant $C > 0$, such that*

$$|\widehat{f}(n)| \leq \frac{C}{|n|}$$

for all $n \neq 0$.

Let $f : \mathbb{T} \longrightarrow \mathbb{C}$ and $\alpha \geq 0$. We will write $f \in \mathrm{H}^\alpha(\mathbb{T})$, whenever

$$\sum_{n\in\mathbb{Z}} (1+n^2)^\alpha \left|\widehat{f}(n)\right|^2 < \infty.$$

It is easy to see that

$$f' \in \mathrm{H}^\alpha(\mathbb{T}) \iff f \in \mathrm{H}^{\alpha+1}(\mathbb{T}).$$

Let $f : \mathbb{T} \longrightarrow \mathbb{C}$ and $\alpha \in (0,1)$. We will write $f \in \mathrm{C}^\alpha(\mathbb{T})$, whenever

$$\sup_{\substack{x\in\mathbb{T}\\ h\neq 0}} \frac{|f(x+h)-f(x)|}{|h|^\alpha} < \infty.$$

Adding $\|f\|_\infty$ to this expression, defines a norm in $\mathrm{C}^\alpha(\mathbb{T})$ and makes it a Banach space, but we will neither use nor prove this fact. See [14, §11.3].

Problem 2. *Let $f : \mathbb{T} \longrightarrow \mathbb{R}$ be given by*

$$f(x) = |x| \log \frac{1}{|x|}.$$

Show that $f \in \mathrm{C}^\alpha(\mathbb{T})$ for all $0 < \alpha < 1$. Show that f is not a Lipschitz function. Is $f \in \mathrm{AC}(\mathbb{T})$?

Problem 3. *Show that $\mathrm{H}^1(\mathbb{T}) \subseteq \mathrm{C}^{\frac{1}{2}}(\mathbb{T})$. Hint: use the Cauchy-Schwarz inequality. Is $\mathrm{H}^1(\mathbb{T}) = \mathrm{C}^{\frac{1}{2}}(\mathbb{T})$?*

2. The Schrödinger equation

The study of different modifications of the following simple dispersive equation is the main goal of this cursillo,

$$\begin{aligned} &i\partial_t u(x,t) = -\partial_x^2 u(x,t) && x\in\mathbb{T} \quad t \in \mathbb{R} \\ &u(x,0) = f(x) && x \in \mathbb{T}. \end{aligned} \tag{A}$$

The regularity of the solutions will play a central role in our analysis. It is routine to seen that

$$u(x,t) = \sum_{j\in\mathbb{Z}} \mathrm{e}^{-ij^2 t} \widehat{f}(j) e_j(x) = \sum_{j=-\infty}^{\infty} \widehat{f}(j) \mathrm{e}^{-ij^2 t + ijx},$$

for all $t \in \mathbb{R}$. Therefore, the solution does not change its Sobolev norm as t evolves.

Problem 4. *Let $\alpha \geq 0$. Show that*

$$\|u(\cdot,t)\|_{\mathrm{H}^\alpha(\mathbb{T})} = \|f\|_{\mathrm{H}^\alpha(\mathbb{T})}$$

for all $t \in \mathbb{R}$.

Quite remarkably, in this quarter of a Century, it has been discovered that the regularity properties of u, beyond the Sobolev scale, are intimately connected with the best approximation of t in continued fractions. Our first theorem illustrates this in a concrete manner. It assembles results first formulated in [12] and [17]. Some of the original proofs were simplified in [18], [15] and [5].

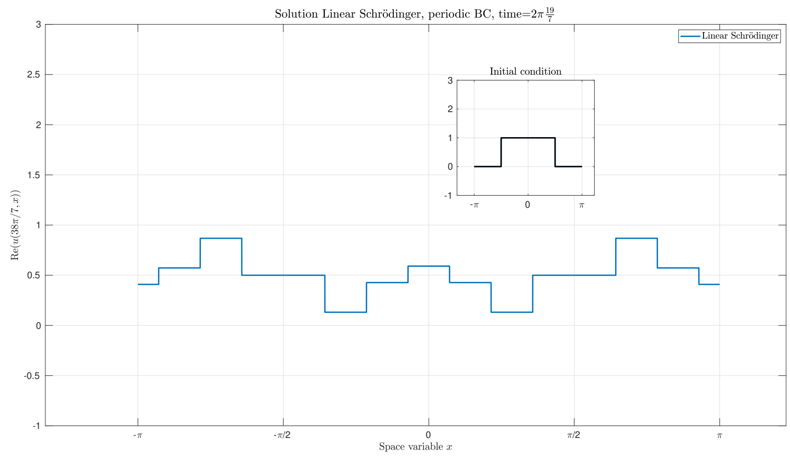


FIGURE 1. Revivals: solution of (A) for $f(x) = \mathbb{1}_{[-\frac{\pi}{2},\frac{\pi}{2}]}(x)$ at time $t = 2\pi\frac{19}{7}$.

Theorem A. *Let* $f \in \mathrm{L}^2(\mathbb{T})$*. Let* u *be the solution to* (A).

a) *If* $p, q \in \mathbb{N}$ *are co-prime, then*

$$u\Big(x, 2\pi\frac{p}{q}\Big) = \frac{1}{q}\sum_{m=0}^{q-1}\left[\sum_{k=0}^{q-1} \mathrm{e}^{2\pi i\frac{km-pm^2}{q}} f\Big(x - 2\pi\frac{k}{q}\Big)\right].$$

b) *There exists a subset* $\mathcal{K} \subset \mathbb{R}$ *with* $|\mathcal{K}^c| = 0$*, such that the following is valid. For all* $t \in \mathcal{K}$*,*

$$f \in \mathrm{BV}(\mathbb{T}) \qquad \Rightarrow \qquad u(\cdot,t) \in \mathrm{C}^\alpha(\mathbb{T}) \quad \forall \alpha < \frac{1}{2}. \tag{I}$$

Additionally, for almost all $t \in \mathcal{K}$*,*

$$\max\big\{s > 0 \,:\, f \in \mathrm{H}^s(\mathbb{T})\big\} = \frac{1}{2} \qquad \Rightarrow \qquad \dim\begin{bmatrix}\text{Graph of} \\ \operatorname{Re} u(\cdot,t)\end{bmatrix} = \frac{3}{2}. \tag{II}$$

Here $|\mathcal{S}|$ for $\mathcal{S} \subset \mathbb{R}$ indicates the Lebesgue measure and "dim" denotes the fractal dimension. For the latter see the Subsection 2.4.

This theorem prescribes that the regularity of the solution in the space variable, changes significantly with time, when seen from a perspective different than that of the Sobolev norm. For example, if f is a step function, the solution is a finite linear combination of step functions whenever $\frac{t}{2\pi} \in \mathbb{Q}$, while it is continuous but fractal for almost every $\frac{t}{2\pi} \notin \mathbb{Q}$. We can call this a *revivals/fractality dichotomy*, and illustrate it in figures 1 and 2. Note that Figure 2 suggests that in the fractal regime, the solution is nowhere differentiable. To the best of my knowledge the latter has not been proved analytically.

2.1. **Proof of Theorem A-a).** The proof of the first statement in Theorem A is as follows.

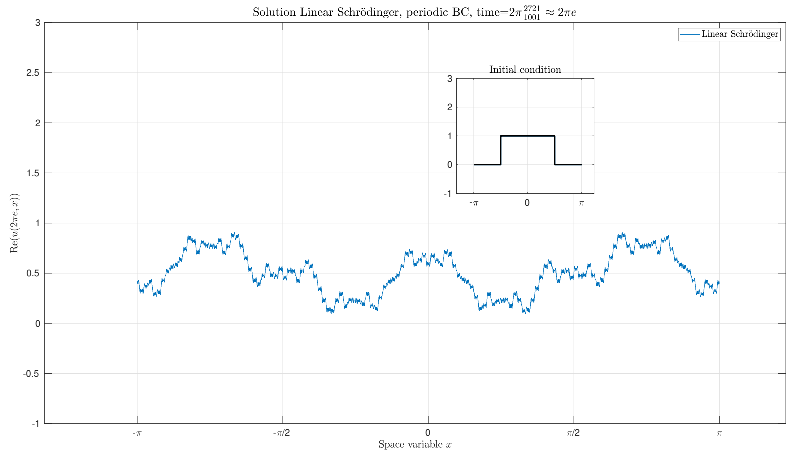


FIGURE 2. Fractality: solution of (A) for $f(x) = \mathbb{1}_{[-\frac{\pi}{2},\frac{\pi}{2}]}(x)$ at time $t \approx 2\pi\mathrm{e}$.

Let $t = 2\pi\frac{p}{q}$. Take $j \underset{q}{\equiv} m$ so that $\mathrm{e}^{ij^2t} = \mathrm{e}^{im^2t}$. Then,

$$u(x,t) = \frac{1}{2\pi}\sum_{m=0}^{q-1}\mathrm{e}^{-im^2t}\sum_{\substack{j\in\mathbb{Z}\\ j\underset{q}{\equiv}m}}\langle f,\mathrm{e}^{ij(\cdot)}\rangle\mathrm{e}^{ijx}.$$

Now,

$$\sum_{k=0}^{q-1}\mathrm{e}^{2\pi i(m-j)\frac{k}{q}} = \begin{cases} q & j\underset{q}{\equiv}m \\ 0 & j\underset{q}{\not\equiv}m. \end{cases} \tag{1}$$

Thus,

$$\begin{aligned}\sum_{\substack{j\in\mathbb{Z}\\ j\underset{q}{\equiv}m}}\langle f,\mathrm{e}^{ij(\cdot)}\rangle\mathrm{e}^{ijx} &= \sum_{j\in\mathbb{Z}}\left(\frac{1}{q}\sum_{k=0}^{q-1}\mathrm{e}^{2\pi im\frac{k}{q}}\right)\mathrm{e}^{-2\pi i\frac{k}{q}j}\langle f,\mathrm{e}^{ij(\cdot)}\rangle\mathrm{e}^{ijx}\\ &= \frac{1}{q}\sum_{k=0}^{q-1}\mathrm{e}^{2\pi im\frac{k}{q}}\sum_{j\in\mathbb{Z}}\left\langle f\Big(\cdot-\frac{2\pi k}{q}\Big),\mathrm{e}^{ij(\cdot)}\right\rangle\mathrm{e}^{ijx}.\end{aligned}$$

From this, the statement of Theorem A-a) follows.

Problem 5. *Give the proof of the identity* (1).

Problem 6. *Let* $f\in\mathrm{L}^2(0,\pi)$. *Find the solution to*

$$\begin{aligned} &\partial_t u(x,t) = i\partial_x^2 u(x,t) && x\in(0,\pi) \quad t\in\mathbb{R}\\ &\partial_x u(0,t) = \partial_x u(\pi,t) = 0 && t\in\mathbb{R}\\ &u(x,0) = f(x) && x\in(0,\pi). \end{aligned} \tag{2}$$

Give your solution in terms of a Fourier series of f. Now, set

$$f(x) = \begin{cases} 1 & x \in [0, \frac{\pi}{2}] \\ 0 & x \in (\frac{\pi}{2}, \pi]. \end{cases}$$

Find $\tilde{t} > 0$ such that $u(\cdot, \tilde{t})$ has a discontinuity at $x = \frac{\pi}{8}$. Hint: the second part is tougher than you might think.

2.2. **Besov spaces.** The Fourier coefficients of Hölder continuous functions have a specific behaviour, which can be seen through the scale of Besov spaces. The latter give a more refined criterion for the regularity of a function than the scale of Sobolev spaces. To simplify our notation, we will write $\mathrm{B}_p^\alpha(\mathbb{T})$, for $\alpha \in \mathbb{R}$ and $p \geq 1$, to denote what is normally written as $\mathrm{B}_{p,\infty}^\alpha(\mathbb{T})$, defined as follows.

Let $\chi : \mathbb{R} \longrightarrow [0, 1]$ be a C^∞ function, such that

$$\operatorname{supp} \chi = [2^{-1}, 2] \qquad \text{and} \qquad \sum_{j=0}^{\infty} \chi(2^{-j}\xi) = 1,$$

for all $\xi \geq 1$. Let $\chi_j(\xi) = \chi(2^{-j}\xi)$ for $j \in \mathbb{N}$ and $\chi_0(\xi) = 1 - \sum_{j=1}^\infty \chi_j(\xi)$. Let the *Littlewood-Paley projections* of a periodic distribution F on $\mathbb{T}$, be given by

$$(K_j F)(x) = \sum_{n \in \mathbb{Z}} \chi_j(|n|) \widehat{F}(n) \mathrm{e}^{inx}.$$

We write $F \in \mathrm{B}_p^\alpha(\mathbb{T})$, if and only if

$$\sup_{j=0,1,\ldots} 2^{\alpha j} \|K_j F\|_{\mathrm{L}^p(\mathbb{T})} < \infty.$$

We will be mostly concerned with the case $p = \infty$.

The following two properties will be crucial in our arguments below,

$$F' \in \mathrm{B}_\infty^\alpha(\mathbb{T}) \Longleftrightarrow F \in \mathrm{B}_\infty^{\alpha+1}(\mathbb{T}) \tag{3}$$

for all $\alpha \in \mathbb{R}$ and

$$\mathrm{B}_\infty^\alpha(\mathbb{T}) = \mathrm{C}^\alpha(\mathbb{T}), \tag{4}$$

for all $\alpha \in (0, 1)$. We now give the proofs of these statements.

Let $g \in \mathcal{S}(\mathbb{R})$ be such that $\mathcal{F}g(\xi) = \chi(\xi)$, where

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} f(x) \mathrm{e}^{-i\xi x} \,\mathrm{d}x$$

is the *Fourier transform*. Then, $(\mathcal{F}g_j)(\xi) = \chi_j(\xi)$ for $g_j(x) = 2^j g(2^j x)$. If $f \in \mathcal{S}(\mathbb{R})$, *Poisson's Summation Formula* prescribes that,

$$\sum_{n \in \mathbb{Z}} f(x + n) = \sum_{n \in \mathbb{Z}} (\mathcal{F}f)(2\pi n) \mathrm{e}^{2\pi i n x}$$

for all $x \in \mathbb{R}$. Letting $\tilde{f}(x) = f(2\pi x)$, gives

$$(\mathcal{F}\tilde{f})(\xi) = \frac{1}{2\pi} (\mathcal{F}f) \left(\frac{\xi}{2\pi} \right).$$

Then,

$$\sum_{k \in \mathbb{Z}} f(z + 2\pi k) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} (\mathcal{F}f)(k) \mathrm{e}^{ikz}.$$

Hence, we can represent the projections K_j of any periodic distribution F, as

$$\begin{aligned}(K_jF)(x) &= \sum_{k\in\mathbb{Z}} \chi_j(|k|)\left(\frac{1}{2\pi}\int_{\mathbb{T}} F(y)\mathrm{e}^{-iky}\,\mathrm{d}y\right)\mathrm{e}^{ikx}\\ &= \int_{\mathbb{T}}\left(\frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\chi_j(|k|)\mathrm{e}^{ik(x-y)}\right)F(y)\,\mathrm{d}y\\ &= \int_{\mathbb{T}}\left(\sum_{k\in\mathbb{Z}} g_j(x-y+2k\pi)\right)F(y)\,\mathrm{d}y\\ &= \sum_{k\in\mathbb{Z}}\int_{\mathbb{T}} g_j(x-y+2k\pi)F(y-2k\pi)\,\mathrm{d}y\\ &= \int_{\mathbb{R}} g_j(x-y)F(y)\,\mathrm{d}y = (g_j\star F)(x).\end{aligned}$$

for all $x\in\mathbb{R}$. Here the symbol "$\star$" denotes the *convolution on* $\mathbb{R}$.

Now, according to [1, Lemma 2.1, p.52] in the case $p=\infty$, there exists a constant $C>0$ which only depends on r_1, r_2 and λ, ensuring the following estimates. For any function $u\in\mathrm{L}^\infty(\mathbb{R})$, such that

$$\operatorname{supp}(\mathcal{F}u)\subset\{\lambda\xi\in\mathbb{R}\,:\,0<r_1\le|\xi|\le r_2\},$$

we have

$$\frac{\lambda}{C}\|u\|_{\mathrm{L}^\infty(\mathbb{R})}\le\|u'\|_{\mathrm{L}^\infty(\mathbb{R})}\le C\lambda\|u\|_{\mathrm{L}^\infty(\mathbb{R})}. \tag{5}$$

This is some times called *Bernstein's Inequality.*

Problem 7. *Give the proof of* (3)*. Hint: use* (5)*.*

Proof of (4). From the definitions $f\in\mathrm{C}^\alpha(\mathbb{T})$, if and only if $S<\infty$ for

$$S=\sup_{\substack{x\in\mathbb{T}\\ h\neq 0}}\frac{|f(x+h)-f(x)|}{|h|^\alpha};$$

and $f\in\mathrm{B}^\alpha_\infty(\mathbb{T})$, if and only if $R<\infty$ for

$$R=\sup_{j=0,1,\ldots}\sup_{x\in\mathbb{T}}2^{\alpha j}|K_jf(x)|.$$

Let $f\in\mathrm{B}^\alpha_\infty(\mathbb{T})$. We show that $S<\infty$. Firstly note that, since $\alpha>0$,

$$f(x)=\sum_{j=0}^\infty K_jf(x),$$

where the convergence of the series is pointwise for all $x\in\mathbb{T}$. Indeed,

$$\sum_{j=0}^\infty\|K_jf\|_{\mathrm{L}^\infty(\mathbb{T})}\le\sum_{j=0}^\infty\frac{R}{2^{\alpha j}}<\infty.$$

Now, if

$$S_1=\limsup_{h\to 0}\left(\sup_{x\in\mathbb{T}}\frac{|f(x+h)-f(x)|}{|h|^\alpha}\right)<\infty,$$

then $S<\infty$. So we focus on S_1.

For $j=0,1,\ldots$, let

$$S_2(j)=\limsup_{h\to 0}\left(\sup_{x\in\mathbb{T}}\frac{|K_j(f(x+h)-f(x))|}{|h|^\alpha}\right).$$

Then, on the one hand,

$$S_1\leq\sum_{j=0}^\infty S_2(j).$$

On the other hand, by the Mean Value Theorem, for suitable $|h_j|<2^{-2j}$,

$$\begin{aligned}
S_2(j)&\leq \sup_{\substack{x\in\mathbb{T}\\0<|h|\leq 2^{-2j}}}\frac{|K_jf(x+h)-K_jf(x)|}{|h|^\alpha}\\
&\leq \sup_{\substack{x\in\mathbb{T}\\0<|h|\leq 2^{-2j}}}\frac{|(K_jf)'(x+h_j)||h|}{|h|^\alpha}\\
&= \sup_{0<|h|\leq 2^{-2j}}|h|^{1-\alpha}\sup_{x\in\mathbb{T}}|(g_j\star f)'(x+h_j)|\\
&\leq 2^{-2j(1-\alpha)}\|(g_j\star f)'\|_{\mathrm{L}^\infty(\mathbb{R})}\\
&\leq C2^j2^{-2j(1-\alpha)}\|g_j\star f\|_{\mathrm{L}^\infty(\mathbb{R})}\\
&= C2^{-j(1-\alpha)}2^{\alpha j}\|K_jf\|_{\mathrm{L}^\infty(\mathbb{T})}\\
&\leq CR2^{-j(1-\alpha)}.
\end{aligned}$$

Thus, indeed, $S_1<\infty$. Here we have used that $1-\alpha>0$. This confirms that $\mathrm{B}^\alpha_\infty(\mathbb{T})\subseteq\mathrm{C}^\alpha(\mathbb{T})$.

Let us now show that $\mathrm{C}^\alpha(\mathbb{T})\subseteq\mathrm{B}^\alpha_\infty(\mathbb{T})$. Assume that $f\in\mathrm{C}^\alpha(\mathbb{T})$. That is $S<\infty$. Our goal is to show that $R<\infty$.

Since $g\in\mathcal{S}(\mathbb{R})$, then there exists a constant $c_1>0$, such that

$$|g_j(x)|\leq c_1\frac{2^j}{(1+2^j|x|)^2},$$

for all $x\in\mathbb{R}$ and $j\in\mathbb{N}$. Also, for any $\varphi\in\mathbb{R}$ thought of as a constant periodic function, we have that $(g_j\star\varphi)(x)=\varphi\chi_j(0)=0$. Then,

$$(g_j\star f)(x)=(g_j\star(f+\varphi))(x).$$

Thus,

$$\begin{aligned}
|(g_j\star f)(x)|&\leq\int_{\mathbb{R}}|g_j(y)||f(x-y)+\varphi|\,\mathrm{d}y\\
&\leq c_12^j\int_{\mathbb{R}}\frac{|f(x-y)+\varphi|}{(1+2^j|y|)^2}\,\mathrm{d}y\\
&=c_1\int_{\mathbb{R}}\frac{\left|f\left(x-\frac{z}{2^j}\right)+\varphi\right|}{(1+|z|)^2}\,\mathrm{d}z,
\end{aligned}$$

for all $x\in\mathbb{R}$, $\varphi\in\mathbb{R}$ and $j\in\mathbb{N}$.

Taking $\varphi=-f(x)$ above, yields

$$2^{\alpha j}|(g_j\star f)(x)|\leq c_12^{\alpha j}\int_{\mathbb{R}}\frac{\left|f\left(x-\frac{z}{2^j}\right)-f(x)\right|}{(1+|z|)^2}\,\mathrm{d}z=A_j(x)+B_j(x),$$

where we split the integral as follows. The first term is,

$$\begin{aligned}A_j(x) &= c_1 2^{\alpha j} \int_{-2^j}^{2^j} \frac{\left|f\left(x-\frac{z}{2^j}\right)-f(x)\right|}{(1+|z|)^2}\,\mathrm{d}z \\ &= c_1 \int_{-2^j}^{2^j} \frac{|z|^\alpha \left|f\left(x-\frac{z}{2^j}\right)-f(x)\right|}{\left(\frac{|z|}{2^j}\right)^\alpha (1+|z|)^2}\,\mathrm{d}z \\ &\le c_1 S \int_{-\infty}^{\infty} \frac{|z|^\alpha}{(1+|z|)^2}\,\mathrm{d}z \le c_2 < \infty.\end{aligned}$$

Here we have used that $0<\alpha<1$. The second term is,

$$\begin{aligned}B_j(x) &= c_1 2^{\alpha j} \int_{|z|\ge 2^j} \frac{\left|f\left(x-\frac{z}{2^j}\right)-f(x)\right|}{(1+|z|)^2}\,\mathrm{d}z \\ &\le c_1 2^{\alpha j} 2 \sup_{x\in\mathbb{T}} |f(x)| \int_{|z|\ge 2^j} \frac{\mathrm{d}z}{(1+|z|)^2} \\ &\le c_3 2^{(\alpha-1)j} \le c_4 < \infty.\end{aligned}$$

Here we have used that $\alpha<1$. Then $R \le c_2+c_4<\infty$. This completes the proof of (4). ☐

Problem 8. *Show that*

$$\mathrm{B}_1^{\alpha_1}(\mathbb{T}) \cap \mathrm{B}_\infty^{\alpha_2}(\mathbb{T}) \subset \mathrm{H}^\alpha(\mathbb{T})$$

for all $\alpha < (\alpha_1+\alpha_2)/2$.

2.3. **Proof of Theorem A-b) statement** (I)**.** We will use the next lemma, which was established in [12, Corollaries 2.2 and 2.4]. The formulation with only half of the Fourier coefficients that we give here will be more convenient for later purposes.

Lemma 1. *There exists a subset* $\mathcal{K}\subset\mathbb{R}$ *with* $|\mathcal{K}^c|<\infty$*, such that the following holds true for all* $t\in\mathcal{K}$*. Given* $\delta>0$*, there exists a constant* $C>0$ *such that*

$$\sup_{x\in\mathbb{T}} \left| \sum_{n=0}^{\infty} \chi_j(n) \mathrm{e}^{in^2t+inx} \right| \le C 2^{\frac{j}{2}(1+\delta)}, \tag{6}$$

for all $j=0,1,\ldots$.

Proof. According to Dirichlet's Theorem, for every irrational number $a>0$ there are infinitely many positive co-prime integers p, $q\in\mathbb{N}$, such that

$$\left| a-\frac{p}{q} \right| \le \frac{1}{q^2}. \tag{7}$$

By virtue of [9, Lemma 4], there exists a constant $c_1>0$ such that, if the irreducible fraction $\frac{p}{q}$ satisfies (7), then

$$\left| \sum_{n=M}^{N} \mathrm{e}^{2\pi i(an^2+bn)} \right| = \left| \sum_{k=1}^{N-M} \mathrm{e}^{2\pi i(ak^2+bk)} \right| \le c_1 \left(\frac{N-M}{\sqrt{q}} + \sqrt{q} \right)$$

for all $N\in\mathbb{N}$, $0<M<N$ and $b\in\mathbb{R}$. Here c_1 is independent of a and b. Take any sequence $\{\omega_n\}$, such that $\omega_n=0$ for $n<M$ or $n>N$, and

$$\sum_{n=M}^{N} |\omega_{n+1}-\omega_n| \le d.$$

Since,

$$\begin{aligned}\left|\sum_{n=M}^{N}\omega_n \mathrm{e}^{2\pi i(an^2+bn)}\right| &= \left|\sum_{n=M}^{N}(\omega_{n+1}-\omega_n)\sum_{k=M}^{n}\mathrm{e}^{2\pi i(ak^2+bk)}\right| \\ &\leq \sum_{n=M}^{N}|\omega_{n+1}-\omega_n|\left|\sum_{k=M}^{n}\mathrm{e}^{2\pi i(ak^2+bk)}\right| \\ &\leq d \sup_{n=M,\dots,N}\left|\sum_{k=M}^{n}\mathrm{e}^{2\pi i(ak^2+bk)}\right|,\end{aligned}$$

then

$$\left|\sum_{n=M}^{N}\omega_n \mathrm{e}^{2\pi i(an^2+bn)}\right| \leq dc_1\left(\frac{N-M}{\sqrt{q}}+\sqrt{q}\right). \tag{8}$$

Let $[a_0, a_1, \dots]$ be the continued fraction expansion of the irrational number a,

$$a = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2+\frac{1}{\cdots}}}.$$

Then, the irreducible fractions,

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2+\cdots\frac{1}{a_{n-1}+\frac{1}{a_n}}}},$$

are such that (7) holds true with $\{p_n\}$ and $\{q_n\}$ increasing sequences. According to the Khinchin-Lévy Theorem, for almost every $a > 0$, the denominators q_n satisfy [13, p.66]

$$\lim_{n\to\infty}\frac{\log q_n}{n} = \rho := \frac{\pi^2}{12\log 2}.$$

If a is such that this limit exists, then for all $j \in \mathbb{N}$ sufficiently large we can find quotients $\frac{p_{n(j)}}{q_{n(j)}}$ with denominators satisfying $q_{n(j)} = 2^{j(1+r_j)}$, where $r_j \to 0$ as $j \to \infty$. Indeed, we can take $n(j)$ equal to the integer part of $j(\log 2)/\rho$. This choice yields

$$\lim_{j\to\infty}\frac{\log q_{n(j)}}{j} = \log 2.$$

Note that $\{q_{n(j)}\}$ is not a subsequence of $\{q_n\}$, since $\log 2/\rho < 1$ and therefore indices may be repeated, but this does not cause problems.

Let $\mathcal{K}$ be the set of times of the form $t = 2\pi a$, such that the quotients of $|a|$ satisfy the conditions of the previous paragraph. Let $t \in \mathcal{K}$ and fix $\delta > 0$. Let $J > 0$ be such that $|r_j| < \delta$ for all $j \geq J$. Taking $M = 2^{j-1}$, $N = 2^{j+1}$, $\omega_n = \chi_j(n)$ and

$$d = 2\sup_{\xi\in\mathbb{R}}|\chi'(\xi)|,$$

in (8), yields

$$\begin{aligned}\sup_{x\in\mathbb{T}}\left|\sum_{n=0}^{\infty}\chi_j(n)\mathrm{e}^{in^2t+inx}\right| &= \sup_{x\in\mathbb{T}}\left|\sum_{n=2^{j-1}}^{2^{j+1}}\chi_j(n)\mathrm{e}^{in^2t+inx}\right| \\ &\leq dc_1\left(\frac{2^{j+1}-2^{j-1}}{\sqrt{q_{n_j}}}+\sqrt{q_{n_j}}\right) \\ &\leq dc_1\left(\frac{2^{j-1}3}{2^{\frac{j}{2}(1-\delta)}}+2^{\frac{j}{2}(1+\delta)}\right) \\ &\leq c_2 2^{\frac{j}{2}(1+\delta)},\end{aligned}$$

for all $j\geq J$. This implies (6) for sufficiently large $C>0$. □

The proof of the statement (I) in Theorem A-b) is as follows.

Let the periodic distribution,

$$E_t(x)=\sum_{n\in\mathbb{Z}}\mathrm{e}^{inx-in^2t}.$$

Since the Fourier coefficients e^{-in^2t} of E_t have modulus 1, the series converges in the weak sense of distributions and determines E_t uniquely for all $t\in\mathbb{R}$, [19, Theorems 11.6-1 and 11.6-2]. Moreover, for all $t\in\mathbb{R}$, $E_t\in \mathrm{B}^{\beta}_{\infty}(\mathbb{T})$ for all $\beta<-1$.

Let $t\in\mathcal{K}$, with $\mathcal{K}\subset\mathbb{R}$ as in Lemma 1. Then it follows from (6) that the stronger inclusion $E_t\in\mathrm{B}^{\beta}_{\infty}(\mathbb{T})$ holds for all $\beta<-\frac{1}{2}$. Define the periodic distribution H_t by $H_t'=E_t$, namely

$$H(t)=\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{\mathrm{e}^{inx-in^2t}}{in}.$$

Then, according to (3) and (4), $H_t\in\mathrm{C}^{\beta}(\mathbb{T})$ for all $\beta<\frac{1}{2}$.

Now,

$$\widehat{f}(n)=\frac{1}{2\pi in}\int_{\mathbb{T}}\mathrm{e}^{-iny}\,\mathrm{d}f(y)=\frac{\widehat{\mu}(n)}{in},\quad n\neq 0,$$

where μ is the Lebesgue-Stieltjes measure associated to f, and $|\mu|(\mathbb{T})<\infty$. Then, the solution of (A) can be expressed in terms of H_t as follows,

$$\begin{aligned}u(x,t)&=\widehat{f}(0)+\sum_{n\in\mathbb{Z}\setminus\{0\}}\mathrm{e}^{-in^2t}\widehat{f}(n)\mathrm{e}^{inx} \\ &=\widehat{f}(0)+\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{\mathrm{e}^{-in^2t}\widehat{\mu}(n)}{in}\mathrm{e}^{inx} \\ &=\widehat{f}(0)+(H_t*\mu)(x).\end{aligned}$$

Here and elsewhere below, "$*$" denotes the *convolution on* $\mathbb{T}$,

$$(H*F)(x)=\frac{1}{2\pi}\int_{\mathbb{T}}H(x-y)F(y)\,\mathrm{d}y,$$

so that $(\widehat{H*F})(n)=\widehat{H}(n)\widehat{F}(n)$. Hence, since μ is a bounded measure, we indeed have $u(\cdot,t)\in\mathrm{C}^{\alpha}(\mathbb{T})$ for all $\alpha<\frac{1}{2}$.

This completes the proof of the first claim in Theorem A-b).

Problem 9. *Show that if* $H \in \mathrm{C}^\alpha(\mathbb{T})$ *and* F *is a bounded measure, then* $(H * F) \in \mathrm{C}^\alpha(\mathbb{T})$*.*

2.4. **Fractal dimension of a graph.** For the statement (II) in part b) of Theorem A, we first recall the notion of fractal dimension. Then, we give a formula for the dimension of the graph of a function in terms of its regularity.

Let $g : \mathbb{T} \longrightarrow \mathbb{R}$ be a continuous function. Denote the *graph of* g, by

$$\Gamma = \Big\{(x, g(x)) \in \mathbb{T} \times \mathbb{R} \,:\, x \in \mathbb{T}\Big\}.$$

The *upper Minkowski (or fractal) dimension* of Γ, is defined by the expression

$$\dim \Gamma = \limsup_{\varepsilon \to 0} \frac{\log \mathcal{N}(\varepsilon)}{\log \frac{1}{\varepsilon}},$$

where $\mathcal{N}(\varepsilon)$ is the number of squares that intersect Γ, in a grid (covering $\mathbb{T} \times \mathbb{R}$) made of squares of side ε.

First we recall the classical upper bound, formulated in [7, Corollary 11.2-(a)].

Lemma 2. *If* $g \in \mathrm{C}^\alpha(\mathbb{T})$*, then* $\dim \Gamma \leq 2 - \alpha$*.*

Proof. We simplify the notation by doing this proof in the interval $[0, 1]$ instead of $(-\pi, \pi]$. So we show that the function $h : [0, 1] \longrightarrow \mathbb{R}$, given by $h(x) = g(2\pi x - \pi)$, has a graph of fractal dimension less than or equal to $2 - \alpha$. The Hölder constant of h is α and the fractal dimension does not change with the scaling of the interval. We adapt the counting function $\mathcal{N}$, accordingly.

Let

$$R_h[a, b] = \sup_{x,y \in [a,b]} |h(x) - h(y)|.$$

Let $0 < \varepsilon < 1$ and let m be the smallest integer greater than or equal to $\frac{1}{\varepsilon}$. Then,

$$\mathcal{N}(\varepsilon) \leq 2m + \sum_{k=0}^{m-1} \frac{R_h[k\varepsilon, (k + 1)\varepsilon]}{\varepsilon}. \tag{9}$$

Indeed, split the interval $[0, 1]$ into sub-intervals $[k\varepsilon, (k + 1)\varepsilon]$ all of size ε and consider a mesh of size ε. Then, the number of squares that intersect the graph of h in each sub-interval, is at least

$$\frac{R_h[k\varepsilon, (k + 1)\varepsilon]}{\varepsilon}.$$

To find the upper bound in (9), we use the fact that h is continuous. The function may overlap another square, when entering a new sub-interval from below or leaving one from above. Hence, the maximum number of squares that intersect the graph of h, is

$$2 + \frac{R_h[k\varepsilon, (k + 1)\varepsilon]}{\varepsilon}.$$

This shows (9).

Now, from the hypothesis, it follows that for a suitable constant $c_1 > 0$,

$$R_h[k\varepsilon, (k + 1)\varepsilon] \leq c_1 \varepsilon^\alpha.$$

Then, using the upper bound in (9) and the fact that $m < 1 + \varepsilon^{-1}$, we have

$$\mathcal{N}(\varepsilon) \leq 2m + c_1 m \varepsilon^{\alpha-1} \leq (1 + \varepsilon^{-1})(2 + c_1 \varepsilon^{\alpha-1}) \leq c_2 \varepsilon^{\alpha-2}.$$

Taking logarithms, gives

$$\frac{\log \mathcal{N}(\varepsilon)}{\log \varepsilon^{-1}} \le \frac{\log c_2}{\log \varepsilon^{-1}} + 2 - \alpha.$$

Hence, taking the limsup, yields the claim of the lemma. □

Finding bounds, complementary to the one in the above lemma, turns out to be less straightforward. One of the best results currently available was obtained in the paper [6].

Lemma 3. *If* $g \notin \mathrm{B}_1^\alpha(\mathbb{T})$*, then* $\dim \Gamma \ge 2 - \alpha$.

Proof. This lemma is a direct corollary of the stronger [6, Theorem 4.2]. □

2.5. **Proof of Theorem A-b) statement** (II)**.** We first show the following lemma, which was first established in [5]. Our proof follows the rather ingenious strategy employed in that paper. The important point of this statement is the fact that the regularity of the real part of the solution (and not only of the solution itself), cannot improve beyond the regularity of f, for almost all $t \in \mathbb{R}$.

Lemma 4. *Let* $f \in \mathrm{BV}(\mathbb{T})$ *and let*

$$r_f = \sup\{s > 0 : f \in \mathrm{H}^s(\mathbb{T})\}.$$

If $r_f \in [\frac{1}{2}, 1)$*, then there exists a subset* $\mathcal{J} \subset \mathbb{R}$ *with* $|\mathcal{J}^c| = 0$*, such that*

$$\sup\left\{s > 0 \,:\, \operatorname{Re} u(\cdot, t) \in \mathrm{H}^s(\mathbb{T}\right\} \le r_f$$

for all $t \in \mathcal{J}$.

Proof. Since $\mathrm{H}^s(\mathbb{T}) \subset \mathrm{H}^r(\mathbb{T})$ for $r < s$, without loss of generality we will assume that r is such that $r_f < r < \frac{r_f+1}{2}$.

Write the real part of the solution as,

$$\begin{aligned} \operatorname{Re} u(x,t) &= \frac{1}{2}\big(u(x,t) + \overline{u(x,t)}\big) \\ &= \frac{1}{2}\sum_{j\in\mathbb{Z}} \left(\mathrm{e}^{-itj^2}\widehat{f}(j) + \mathrm{e}^{itj^2}\overline{\widehat{f}(-j)}\right)\mathrm{e}^{ijx}. \end{aligned}$$

Then, the conclusion will follow, if we find a sequence $\{J_n\}_{n=1}^\infty \subset \mathbb{N}$, such that

$$\lim_{n\to\infty} \sum_{j=1}^{J_n} j^{2r} \left|\mathrm{e}^{-itj^2}\widehat{f}(j) + \mathrm{e}^{itj^2}\overline{\widehat{f}(-j)}\right|^2 = \infty.$$

Now,

$$\left|\mathrm{e}^{-itj^2}\widehat{f}(j) + \mathrm{e}^{itj^2}\overline{\widehat{f}(-j)}\right|^2 = |\widehat{f}(j)|^2 + |\widehat{f}(-j)|^2 + 2\operatorname{Re}\left(\mathrm{e}^{-2itj^2}\widehat{f}(j)\widehat{f}(-j)\right).$$

By hypothesis, we have that

$$\sum_{j\in\mathbb{Z}} j^{2r}|\widehat{f}(j)|^2 = \infty.$$

Then,

$$\sum_{j=1}^{\infty} j^{2r}(|\widehat{f}(j)|^2 + |\widehat{f}(-j)|^2) = \infty. \tag{10}$$

Let the partial summations

$$S_L(t)=\sum_{j=1}^{L} j^{2r}\mathrm{e}^{-2itj^2}\widehat{f}(j)\widehat{f}(-j).$$

If we can find another sequence $\{L_n\}_{n=0}^{\infty}\subset\mathbb{N}$, such that $S_{L_n}(t)$ converges as $n\to\infty$, this will ensure that

$$2\,\mathrm{Re}\left[\sum_{j=1}^{L_n} j^{2r}\left(\mathrm{e}^{-2itj^2}\widehat{f}(j)\widehat{f}(-j)\right)\right]$$

converges, and we will be able to find J_n from this convergence and the divergence (10). Note that we do need to argue through the subsequence J_n, as there might be some cancellations in the intermediate terms, preventing a "uniform divergence".

So our next goal is to show that the sequence L_n exists. Since S_L is 2π-periodic, without loss of generality we can assume that $t\in\mathbb{T}$. Write $S_L(t)$ in the expanded form,

$$S_L(t)=\sum_{m=1}^{\infty}\mathrm{e}^{-itm}\left(\sum_{\substack{1\le j\le L\\ 2j^2=m}} j^{2r}\widehat{f}(j)\widehat{f}(-j)\right).$$

Since $j^2=\frac{m}{2}$ has at most two (integer) solutions, the sum inside the bracket has at most 2 terms for every positive integer m and it is equal to 0 for most of them. If

$$\sum_{j=1}^{\infty} j^{4r}\left|\widehat{f}(j)\widehat{f}(-j)\right|^2<\infty, \tag{11}$$

then $S=\lim_{L\to\infty}S_L$ is the limit of a Fourier series, convergent in the norm of $\mathrm{L}^2(\mathbb{T})$. Should this happen, by virtue of Carleson's theorem [11, Theorem 3.6.15], there would exists a subset $\mathcal{J}\subset\mathbb{R}$ whose complement is of measure 0, satisfying the following. For all $t\in\mathcal{J}$, there is a subsequence $\{L_n\}\subset\mathbb{N}$, such that the limit

$$\lim_{n\to\infty}S_{L_n}(t)=S(t)$$

converges pointwise. Therefore, according to the previous paragraph, $\mathcal{J}$ would be the needed subset in the statement of the lemma.

So we complete the proof by showing that (11) holds true. Since $f\in\mathrm{H}^{\frac{1}{2}}(\mathbb{T})$, then

$$|\widehat{f}(-j)|^2\le c_2 j^{-2}$$

for all $j\in\mathbb{N}$. Let $s<r_f$. Since $f\in\mathrm{H}^s(\mathbb{T})$, then

$$\sum_{j=1}^{\infty} j^{2s}|\widehat{f}(j)|^2<\infty.$$

Hence,

$$\begin{aligned}\sum_{j=1}^{\infty} j^{4r}\left|\widehat{f}(j)\widehat{f}(-j)\right|^2 &\le c_2\sum_{j=1}^{\infty} j^{4r-2-2s}\left(j^{2s}|\widehat{f}(j)|^2\right)\\ &\le c_2\sum_{j=1}^{\infty} j^{2s}|\widehat{f}(j)|^2<\infty,\end{aligned}$$

for any $r < \frac{s+1}{2}$. Taking $s < r_f$ close enough to r_f, we can always get r to satisfy both, this latter condition which ensures (11), and the assumption in the beginning of the proof. $\square$

To complete the proof of Theorem A-b), we proceed as follows.

Let D denote the fractal dimension of the graph of $\operatorname{Re} u(\cdot,t)$. By virtue of the statement (I) in Theorem A-b) and Lemma 2, it follows that $D \leq \frac{3}{2}$ for all $t \in \mathcal{K}$.

Now, take $r_f = \frac{1}{2}$ in the previous lemma. According to this and to the statement of Problem 8, the hypothesis of (II) imposed on f, implies that $\operatorname{Re} u(\cdot,t) \notin \mathrm{B}^r_1(\mathbb{T})$ whenever $r > \frac{1}{2}$, for all $t \in \mathcal{K} \cap \mathcal{J}$. Thus, by virtue of Lemma 3, we also have the complementary bound $D \geq \frac{3}{2}$ for all such t.

This completes the proof of Theorem A.

Problem 10. *Let u be the solution to the time-evolution equation* (2) *from Problem 6. Show that there exists $\tilde{t} > 0$ such that $u(\cdot,\tilde{t})$ is continuous. Hint: this is easier than you think.*

3. Changing the boundary conditions

Let $b \in \mathbb{R} \setminus \{1\}$. We now consider

$$\begin{aligned} &i\partial_t u(x,t) = -\partial_x^2 u(x,t) && x \in (0,\pi) \quad t \in \mathbb{R} \\ &u'(a,t) = \frac{b}{1-b} u(a,t) && a = 0,\pi \quad t \in \mathbb{R} \\ &u(x,0) = f(x) && x \in (0,\pi) \end{aligned} \tag{B}$$

which is a modification from the periodic boundary conditions of (A) to a class of Robin boundary conditions. Our goal is to determine to what extend, this modification supports any form of revival and or fractality.

If $b = 0$ we obtain the Neumann boundary conditions. So, Theorem B below generalises the solutions to the problems 6 and 10 from the previous section. For $b \in (0,1)$, the part a) of this theorem was established in [2] and the PhD dissertation [8]. The extension to the other values of b that we give here, does not present any technical improvement from that case. As far as I am aware, the part b) of the theorem is new.

Let any function $h : [0,\pi] \longrightarrow \mathbb{C}$. We denote by h_{e}, the 2π-periodic extension of the even function

$$h_{\mathrm{e}}(x) = \begin{cases} h(x) & x \in [0,\pi] \\ h(-x) & x \in (-\pi,0). \end{cases}$$

That is, the *even extension* of h to $\mathbb{R}$. Likewise, we denote by h_{o} the 2π-periodic extension of the odd function

$$h_{\mathrm{o}}(x) = \begin{cases} h(x) & x \in [0,\pi] \\ -h(-x) & x \in (-\pi,0). \end{cases}$$

That is, the *odd extension* of h to $\mathbb{R}$.

The next conventions will simplify the arguments below. Let

$$A_b = \frac{2b}{(1-b)\big(\mathrm{e}^{2\pi \frac{b}{1-b}} - 1\big)}$$

for $b \neq 0$ and $A_0 = \frac{1}{\pi}$. We denote with the unambiguous symbol ϕ, the function $\phi : \mathbb{R} \longrightarrow \mathbb{C}$ which is the 2π-periodic extension of

$$\phi(x) = \sqrt{A_b}\, \mathrm{e}^{\frac{b}{1-b}x},$$

from $x \in [0, 2\pi]$ to $\mathbb{R}$. We remark that ϕ, regarded as a function of $\mathbb{T}$, has a discontinuity at 0 but it is C^∞ at $\pm\pi$. Also, note that ϕ satisfies the boundary conditions of (B) and that $\|\phi\|_{L^2(0,\pi)} = 1$.

For any $f \in \mathrm{L}^2(0, \pi)$, we will consider an associated function $g : \mathbb{T} \longrightarrow \mathbb{C}$, given by the expression

$$g = \sqrt{\pi}\phi * (f_\mathrm{o} - f_\mathrm{e}). \tag{12}$$

Here, g is a function on the torus and the convolution is, as previously, also on $\mathbb{T}$.

Theorem B. *Let $f \in \mathrm{L}^2(0, \pi)$ and let g be as in the expression* (12)*. Let u be the solution to* (B)*.*

a) *If $p, q \in \mathbb{N}$ are co-prime, then*

$$u\Big(x, \frac{2\pi p}{q}\Big) = \mathrm{e}^{2\pi i \frac{pb^2}{q(1-b)^2}} \langle f, \phi\rangle_{\mathrm{L}^2(0,\pi)}\phi(x) + \frac{1}{q}\sum_{m=0}^{q-1}\left[\sum_{k=0}^{q-1} \mathrm{e}^{2\pi i \frac{km - pm^2}{q}} (f_\mathrm{e} + g)\Big(x - 2\pi\frac{k}{q}\Big)\right].$$

b) *There exists a subset $\mathcal{K} \subset \mathbb{R}$ with $|\mathcal{K}^c| = 0$, such that the following is valid. For all $t \in \mathcal{K}$,*

$$f \in \mathrm{BV}(\mathbb{T}) \qquad \Rightarrow \qquad u(\cdot, t) \in \mathrm{C}^\alpha(\mathbb{T}) \quad \forall \alpha < \frac{1}{2}. \tag{I}$$

Additionally, for almost all $t \in \mathcal{K}$,

$$\max\big\{s > 0 \,:\, f_\mathrm{e} \in \mathrm{H}^s(\mathbb{T})\big\} = \frac{1}{2} \qquad \Rightarrow \qquad \dim\begin{bmatrix}\text{Graph of} \\ \mathrm{Re}\, u(\cdot, t)\end{bmatrix} = \frac{3}{2}. \tag{II}$$

In the statement a) of this theorem, we highlight the contribution of f_e and of g to the revival formula, separately. The reason for this is that g is more regular than f, therefore it does not make a contribution of the fractal part of the solution in the statement b). This will be seen in the proof below.

Problem 11. *Show that the expressions*

$$\mathrm{e}^{2\pi i \frac{pb^2}{q(1-b)^2}} \langle f, \phi\rangle_{\mathrm{L}^2(0,\pi)}\phi(x)$$

and

$$\frac{1}{q}\sum_{m=0}^{q-1}\left[\sum_{k=0}^{q-1} \mathrm{e}^{2\pi i \frac{km - pm^2}{q}} g\Big(x - 2\pi\frac{k}{q}\Big)\right],$$

cancel out from the statement a) of this theorem, in the case $b = 0$. Compare with your answer to Problem 6.

3.1. **Proof of Theorem B.** Let $L : \mathrm{Dom}(L) \longrightarrow \mathrm{L}^2(0,\pi)$ be the self-adjoint operator $L = -\partial_x^2$ with domain

$$\mathrm{Dom}(L) = \left\{ g \in \mathrm{H}^2(0,\pi) \,:\, g'(0) = \frac{b}{1-b} g(0) \text{ and } g'(\pi) = \frac{b}{1-b} g(\pi) \right\}.$$

We first give the eigenfunctions and the spectrum of L.

To start with, note that

$$L\phi = -\frac{b^2}{(1-b)^2}\phi.$$

For $j \in \mathbb{N}$, let

$$\Lambda_j = \frac{b - j(1-b)i}{b + j(1-b)i}$$

and

$$\phi_j(x) = e_j(x) - \Lambda_j e_{-j}(x).$$

Then, $\|\phi_j\|_{\mathrm{L}^2(0,\pi)} = 1$. Indeed, note that $|\Lambda_j| = 1$. Moreover, $\phi_j \in \mathrm{Dom}(L)$ and $L\phi_j = j^2\phi_j$.

The family $\{\phi\} \cup \{\phi_j\}_{j=1}^{\infty}$ is an orthonormal basis of $\mathrm{L}^2(0,\pi)$. Therefore, we know that

$$\mathrm{Spec}(L) = \left\{ -\frac{b^2}{(1-b)^2} \right\} \cup \{j^2\}_{j=1}^{\infty}.$$

Problem 12. *Show that the family of eigenvectors* $\{\phi\} \cup \{\phi_j\}_{j=1}^{\infty}$ *is indeed an orthonormal basis of* $\mathrm{L}^2(0,\pi)$*. Hint: the important point here is to show that they are a complete family.*

Then, for all $f \in \mathrm{L}^2(0,\pi)$. The solution to (B) is

$$u(x,t) = \mathrm{e}^{i\frac{b^2}{(1-b)^2}t} \langle f, \phi \rangle_{\mathrm{L}^2(0,\pi)} \phi(x) + \sum_{j=1}^{\infty} \mathrm{e}^{-ij^2 t} \langle f, \phi_j \rangle_{\mathrm{L}^2(0,\pi)} \phi_j(x).$$

The crucial point in the proof of Theorem B is the following lemma, which gives a different representation of the second term on the right hand side of this expression. This was first obtained in [8].

Lemma 5. *Let* $f \in \mathrm{L}^2(0,\pi)$ *and let* g *be given by* (12)*. Let*

$$U(x,t) = \sum_{j=1}^{\infty} \mathrm{e}^{-ij^2 t} \langle f, \phi_j \rangle_{\mathrm{L}^2(0,\pi)} \phi_j(x).$$

Then $U = U_1 + U_2$*, where*

$$U_1(x,t) = \sum_{k\in\mathbb{Z}} \widehat{f}_{\mathrm{e}}(k) \mathrm{e}^{-ik^2t+ikx} \qquad \text{and} \qquad U_2(x,t) = \sum_{k\in\mathbb{Z}} \widehat{g}(k) \mathrm{e}^{-ik^2t+ikx}.$$

Proof. Begin by re-writing

$$\phi_j(x) = \sqrt{\frac{2}{\pi}} \left(\frac{1-\Lambda_j}{2} \cos(jx) + i\frac{1+\Lambda_j}{2} \sin(jx) \right).$$

Then

$$\langle f, \phi_j \rangle_{\mathrm{L}^2(0,\pi)} = \frac{1-\Lambda_j}{2} a_j + \frac{1+\overline{\Lambda_j}}{2i} b_j,$$

where a_j are the cosine Fourier coefficients of f and b_j are the sine Fourier coefficients of f, in $(0,\pi)$.

Let $\phi_{\rm r}(x)=\phi(2\pi-x)$. That is the reflection of ϕ about π. Note that,

$$\widehat{\phi}(j)=\frac{b}{2\,(b-j(1-b)i)}\qquad\text{and}\qquad\widehat{\phi_{\rm r}}(j)=\frac{b}{2\,(b+j(1-b)i)}.$$

Then, re-writing U with the above representation of the eigenfunctions and inner products, gives

$$U(x,t)=\frac{1}{2}\sum_{k=1}^{5}v_k(x,t)$$

where each of the v_k are the solution to (A), with corresponding initial conditions

$$v_1(x,0)=2f_{\rm e}(x),\qquad v_2(x,0)=-(\phi_{\rm r}+\phi)*f_{\rm e}(x),\qquad v_3(x,0)=(\phi_{\rm r}-\phi)*f_{\rm e}(x)$$
$$v_4(x,0)=(\phi-\phi_{\rm r})*f_{\rm o}(x)\quad\text{and}\quad v_5(x,0)=(\phi_{\rm r}+\phi)*f_{\rm o}(x).$$

Once this expression is obtained, we observe that $U_1(x,t)=\frac{1}{2}v_1(x,t)$ and

$$U_2(x,t)=\frac{1}{2}\sum_{k=2}^{5}v_k(x,t).$$

□

Problem 13. *Give the proof of Theorem B-a), using Lemma 5.*

We now present the proof of Theorem B-b). Observe that $\phi\in{\rm C}^\infty(0,\pi)$, so we can ignore this correction in the solution and concentrate on U.

First consider the claim (I). Let $f\in{\rm BV}(0,\pi)$. Then $f_{\rm e}$, $f_{\rm o}$ and hence $f_{\rm o}-f_{\rm e}$, are in ${\rm BV}(\mathbb{T})$. To see this, note that the even and odd extensions can, at most, introduce a jump discontinuity at $x=0$ and $x=\pm\pi$.

Invoke Lemma 5. Since U_1 is a solution to (A) with initial condition $f_{\rm e}$, according to Theorem A-b), we know that $U_1(\cdot,t)\in{\rm C}^\alpha(\mathbb{T})$ for all $\alpha<\frac{1}{2}$, provided $t\in\mathcal{K}$. Likewise, being a convolution with a regular function, $g\in{\rm BV}(\mathbb{T})$ and U_2 is the solution to (A) with this initial condition. Then, also $U_2(\cdot,t)\in{\rm C}^\alpha(\mathbb{T})$. Hence, the statement (I) in Theorem B-b) is valid.

Note that, seen as a function of $\mathbb{T}$, $\phi\in{\rm BV}(\mathbb{T})$. Indeed it has a jump discontinuity at $x=0$ for $b\neq0$, but it is ${\rm C}^\infty$ at all other points $x\in\mathbb{T}$. We are not using this fact in the previous paragraph.

Consider now the claim (II). Assume the hypothesis,

$$\max\left\{s\,:\,f_{\rm e}\in{\rm H}^s(\mathbb{T})\right\}=\frac{1}{2}.$$

By virtue of Theorem A-b), it then follows that the graph of ${\rm Re}\,U_1(\cdot,t):\mathbb{T}\longrightarrow\mathbb{C}$ has fractal dimension equal to $\frac{3}{2}$ for almost all $t\in\mathcal{K}$. But since $f_{\rm e}$ is even, also $U_1(\cdot,t)$ is even. Then, necessarily, ${\rm Re}\,U_1(\cdot,t):(0,\pi)\longrightarrow\mathbb{C}$ has also fractal dimension equal to $\frac{3}{2}$ for almost all $t\in\mathcal{K}$.

Now, $U_2(x,t)=(H_t*g)(x)$. Since $g\in{\rm C}^\alpha(\mathbb{T})$ for all $\alpha<1$, then $U_2(\cdot,t)\in{\rm C}^\alpha(\mathbb{T})$ for all $\alpha<1$ and $t>0$. Hence, ${\rm Re}\,U_2(\cdot,t)\in{\rm C}^\alpha(\mathbb{T})$ for all $\alpha<1$ and $t>0$. Therefore, by virtue of Lemma 2, the graph of ${\rm Re}\,U_2(\cdot,t)$ has fractal dimension equal to 1. Thus, the overall fractal dimensions of the graph of ${\rm Re}\,U={\rm Re}\,U_1+{\rm Re}\,U_2$ is equal to $\frac{3}{2}$. This completes the proof of Theorem B.

4. Adding a potential

Let $V:(0,\pi]\longrightarrow\mathbb{C}$ be a potential, satisfying either of the following conditions

- $V\in \mathrm{H}^2(0,\pi)$ and $\|V\|_{\mathrm{L}^\infty(0,\pi)}<\frac{3}{2}$; or
- $V:(0,\pi]\longrightarrow\mathbb{R}$, $V\in \mathrm{BV}(0,\pi)$ and $V\in \mathrm{L}^\infty(0,\pi)$.

In this short section we describe the revival property for the equation,

$$\begin{aligned} &i\partial_t u(x,t)=-\partial_x^2 u(x,t)+V(x)u(x,t) && x\in(0,\pi)\quad t\in\mathbb{R}\\ &u(0,t)=u(\pi,t)=0 && t\in\mathbb{R}\\ &u(x,0)=f(x) && x\in(0,\pi)\end{aligned}\tag{C}$$

The proof of the next theorem can be found in [3].

Theorem C. *Let $f\in \mathrm{L}^2(0,\pi)$ and let u be the solution to* (C). *Then, for all $t\in\mathbb{R}$, there exists $w(\cdot,t)\in \mathrm{C}^0(0,\pi)$ ensuring the following. If $p,q\in\mathbb{Z}$ are co-prime, then*

$$u\Big(x,2\pi\frac{p}{q}\Big)=w\Big(x,2\pi\frac{p}{q}\Big)+\frac{\mathrm{e}^{-2\pi i\langle V\rangle\frac{p}{q}}}{q}\sum_{m=0}^{q-1}\left(\sum_{k=1}^{q-1}\mathrm{e}^{2\pi i\frac{mk-m^2p}{q}}f_{\mathrm{o}}\Big(x-2\pi\frac{k}{q}\Big)\right).$$

Here $\langle V\rangle=\frac{1}{\pi}\langle V,1\rangle$ is the *mean of* V.

According to this theorem, the revivals property still holds true for the boundary-value problem (C), modulo a continuous correction.

Problem 14. *Give the proof of Theorem C for $V(x)=c$, where $c\in\mathbb{C}$. Hint: note that* (C) *has Dirichlet boundary conditions and that c does not need to satisfy the hypotheses on V stated in the bullet points.*

5. Multiplying by Hilbert's transform

In this final section we consider the striking example of the linear Benjamin-Ono equation, where a seemingly substantial change to the right hand side of the equation, still supports the revivals/fractality dichotomy that we described in Section 2.

Let the *Hilbert transform* of $g:\mathbb{T}\longrightarrow\mathbb{C}$, be given by the expression

$$\mathcal{H}g(x)=\frac{1}{2\pi}\ \mathrm{p.\,v.}\!\int_{-\pi}^{\pi}\cot\frac{x-y}{2}\,g(y)\,\mathrm{d}y,$$

assuming that the principal value integral exists. The expression of $\mathcal{H}g$ for $g\in \mathrm{L}^2(\mathbb{T})$ in terms of the Fourier coefficients, given below, implies that

$$\mathcal{H}:\mathrm{L}^2(\mathbb{T})\longrightarrow \mathrm{L}^2(\mathbb{T})$$

is a bounded operator.

Let $f:\mathbb{T}\longrightarrow\mathbb{R}$. The *linear Benjamin-Ono (BO) equation*, is the time-evolution problem

$$\begin{aligned} &\partial_t u(x,t)=\mathcal{H}\partial_x^2 u(x,t) && x\in\mathbb{T}\quad t\in\mathbb{R}\\ &u(x,0)=f(x) && x\in\mathbb{T}.\end{aligned}\tag{D}$$

As we shall see next, this equation supports a revivals/fractality dichotomy that combines properties observed in all the models discussed previously, except that it gives rise to cusps of a very specific type in the solution. This is made precise in the next theorem, formulated in [4], whose conclusion differs from that of Theorem A only in the first statement. Note that the solution of (D) are real-valued, but the statement (II) in the part b), is not an immediate consequence of any well-posedness of this equation in the Sobolev norms.

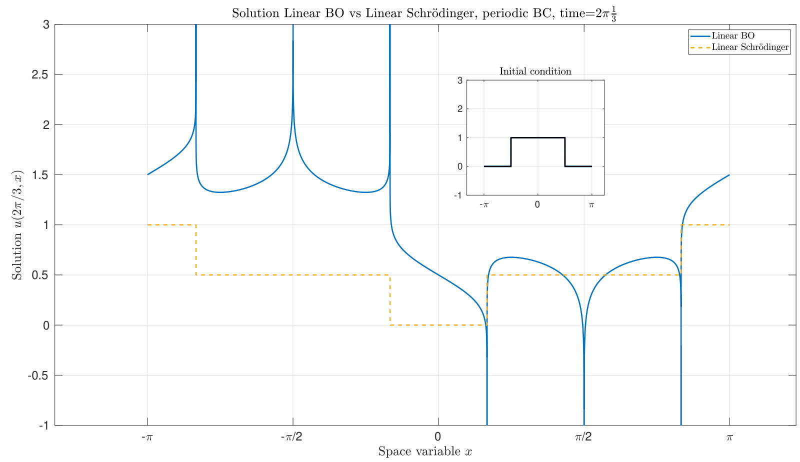


FIGURE 3. Solution of (D) for $f(x) = \mathbb{1}_{[-\frac{\pi}{2},\frac{\pi}{2}]}(x)$ at time $t = 2\pi\frac{1}{3}$ superimposed on the real and imaginary parts of the solution of (A) at $-t$. The cusp singularities in the solution of (D) correspond to jump singularities in either part of the solution of (A).

Theorem D. *Let $f \in \mathrm{L}^2(\mathbb{T})$ be real-valued. Let u be the solution to* (D).

a) *If $p, q \in \mathbb{N}$ are two co-prime integers, then*

$$u\Big(x, 2\pi\frac{p}{q}\Big) = \frac{1}{q}\sum_{m=0}^{q-1} \mathrm{Re}\left[\sum_{k=0}^{q-1} \mathrm{e}^{2\pi i\frac{km+pm^2}{q}}(I + i\mathcal{H})f\Big(x - 2\pi\frac{k}{q}\Big)\right].$$

b) *There exists a subset $\mathcal{K} \subset \mathbb{R}$ with $|\mathcal{K}^c| = 0$, such that the following is valid. For all $t \in \mathcal{K}$,*

$$f \in \mathrm{BV}(\mathbb{T}) \qquad \Rightarrow \qquad u(\cdot,t) \in \mathrm{C}^{\alpha}(\mathbb{T}) \quad \forall \alpha < \frac{1}{2}. \tag{I}$$

Additionally, for almost all $t \in \mathcal{K}$,

$$\max\left\{s > 0 \,:\, f \in \mathrm{H}^s(\mathbb{T})\right\} = \frac{1}{2} \qquad \Rightarrow \qquad \dim\begin{bmatrix}\text{Graph of} \\ u(\cdot,t)\end{bmatrix} = \frac{3}{2}. \tag{II}$$

The first statement implies that, if f has a jump discontinuity, then the solution at times $t \in 2\pi\mathrm{Q}$ will have logarithmic singularities. See the Figure 3. By contrast, quite remarkably, the solution is continuous for almost all other t. A difference between the Schrödinger and this equation, is the fact that, for all initial data of bounded variation, the solution to (A) is bounded whole the solution to (D) is not. I suspect that there is an interesting structural connection between the two, using BMO spaces, but this will be left for further investigation.

Problem 15. *Show that*

$$\mathcal{H}\mathbb{1}_{[a,b]}(x) = \frac{1}{\pi}\log\left|\frac{\sin\left(\frac{x-a}{2}\right)}{\sin\left(\frac{x-b}{2}\right)}\right|,$$

for a, $b \in \mathbb{T}$ with $-\pi \leq a < b < \pi$.

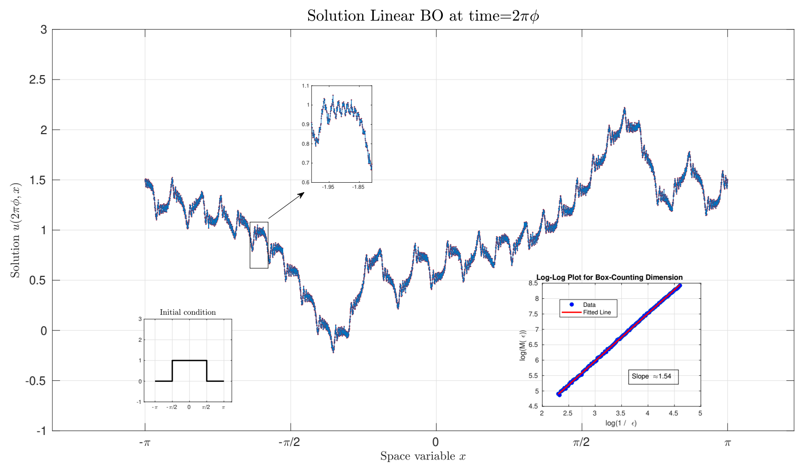


FIGURE 4. Solution for $\frac{t}{2\pi}$ a rational approximation of $\phi \sim \frac{p}{q}$ for $p = F_{16} = 2584$ and $q = F_{15} = 1597$. Note that $|\phi - \frac{p}{q}| < 1.7 \times 10^{-6}$. The estimate of the box counting dimension is $D = 1.54$.

5.1. **Proof of Theorem D.** Firstly, note that

$$\mathcal{H}e_n = \begin{cases} ie_n & \text{for } n < 0 \\ 0 & \text{for } n = 0 \\ -ie_n & \text{for } n > 0. \end{cases} \tag{13}$$

Problem 16. *Compute $\mathcal{H}e_n$ to verify the previous claim.*

Then, $\mathcal{H}$ and $-\partial_x^2$ have the same orthonormal basis of eigenfunctions. Hence,

$$\mathcal{H}g(x) = i \sum_{n=1}^{\infty} [\hat{g}(-n)\mathrm{e}^{-inx} - \hat{g}(n)\mathrm{e}^{inx}]$$

for all $g \in \mathrm{L}^2(\mathbb{T})$ and

$$\mathcal{H}\partial_x^2 g(x) = i \sum_{n=1}^{\infty} n^2 [\hat{g}(n)\mathrm{e}^{inx} - \hat{g}(-n)\mathrm{e}^{-inx}]$$

for all $g \in \mathrm{H}^2(\mathbb{T})$. In fact the latter is the domain of the integro-differential operator $\mathcal{H}\partial_x^2$. This implies that, for any $f \in \mathrm{L}^2(\mathbb{T})$, the solution to (D) is given by the expression

$$\begin{aligned} u(x,t) &= \sum_{n=-\infty}^{\infty} \mathrm{e}^{inx}\mathrm{e}^{in|n|t}\widehat{f}(n) \\ &= \widehat{f}(0) + \sum_{n=1}^{\infty} [\mathrm{e}^{inx}\mathrm{e}^{in^2 t}\widehat{f}(n) + \mathrm{e}^{-inx}\mathrm{e}^{-in^2 t}\widehat{f}(-n)]. \end{aligned}$$

Since f is real-valued, $\widehat{f}(-n) = \overline{\widehat{f}(n)}$ and so

$$\mathrm{e}^{-inx}\mathrm{e}^{-in^2 t}\widehat{f}(-n) = \overline{\mathrm{e}^{inx}\mathrm{e}^{in^2 t}\widehat{f}(n)}.$$

Hence, u is also real-valued and

$$u(x,t) = \widehat{f}(0) + 2\,\mathrm{Re}\left[\sum_{n=1}^{\infty} \mathrm{e}^{inx}\mathrm{e}^{in^2t}\widehat{f}(n)\right]. \tag{14}$$

This representation provides the link between the solutions of (D) and (A), as stated in the next lemma.

Lemma 6. *Let $f \in \mathrm{L}^2(\mathbb{T})$ be real-valued. Let $v(x,t)$ denote the solution to* (A) *with initial datum f. Then, the solutions to* (D)*, is given by the expression*

$$u(x,t) = \mathrm{Re}\left[v(x,-t) + i\mathcal{H}v(x,t)\right]. \tag{15}$$

Proof. We have that,

$$v(x,-t) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{in^2t+inx}\widehat{f}(n) = \widehat{f}(0) + \sum_{n=1}^{\infty}\left[\mathrm{e}^{inx}\mathrm{e}^{in^2t}\widehat{f}(n) + \mathrm{e}^{-inx}\mathrm{e}^{in^2t}\overline{\widehat{f}(n)}\right]$$

and

$$i\mathcal{H}v(x,-t) = \sum_{n=1}^{\infty}\left[\mathrm{e}^{inx}\mathrm{e}^{in^2t}\widehat{f}(n) - \mathrm{e}^{-inx}\mathrm{e}^{in^2t}\overline{\widehat{f}(n)}\right].$$

Then,

$$v(x,t) + i\mathcal{H}v(x,t) = \widehat{f}(0) + 2\sum_{n=1}^{\infty}\mathrm{e}^{inx}\mathrm{e}^{in^2t}\widehat{f}(n).$$

Hence, (14), yields (15). □

Note that the expression (15) can be written in operator form as

$$\mathrm{e}^{\mathcal{H}\partial_x^2 t}f = \widehat{f}(0) + 2\,\mathrm{Re}\left(\mathrm{e}^{-i\partial_x^2 t}\Pi f\right), \tag{16}$$

where $\Pi f(x) = \sum_{n=1}^{\infty}\hat{f}(n)\mathrm{e}^{inx}$ is the *modified Szegö projector*. Indeed, the latter commutes with both $\mathcal{H}$ and $-\partial_x^2$.

The combination of Lemma 6 and Theorem A, gives Theorem D as follows.

Problem 17. *Complete the proof of Theorem D-a).*

Proof of Theorem D-b) statement (I)*.* By virtue of (14), letting $\mu = f'$, we have

$$\begin{aligned} u(x,t) &= \widehat{f}(0) + 2\,\mathrm{Re}\left[\sum_{n=1}^{\infty}\frac{\mathrm{e}^{inx}\mathrm{e}^{in^2t}\widehat{\mu}(n)}{in}\right] \\ &= \widehat{f}(0) + 2\,\mathrm{Re}\left[(\widetilde{H}_t * \mu)(x)\right], \end{aligned}$$

where

$$\widetilde{H}_t = \sum_{n=1}^{\infty}\frac{\mathrm{e}^{inx}\mathrm{e}^{in^2t}}{in} = \widetilde{E}_t'(x)$$

for $\widetilde{E}_t(x) = \sum_{n=1}^{\infty}\mathrm{e}^{inx}\mathrm{e}^{in^2t}$.

By virtue of Lemma 1, $\widetilde{E}_t \in \mathrm{C}^{\alpha}(\mathbb{T})$ for all $\alpha < -\frac{1}{2}$, hence $\widetilde{H}_t \in \mathrm{C}^{\alpha}(\mathbb{T})$ for all $\alpha < \frac{1}{2}$, whenever $t \in \mathcal{K}$. As the real part of a C^{α} function is also in C^{α}, this ensures (I). □

Proof of Theorem D-b) statement (II). Recall (16). Since $f:\mathbb{T}\longrightarrow\mathbb{R}$, then $f=\Pi f+\overline{\Pi f}+\widehat{f}(0)$. The hypothesis of (II) implies that $f\in \mathrm{H}^{\frac{1}{2}}(\mathbb{T})$ but $f\notin \mathrm{H}^{\frac{1}{2}+\varepsilon}(\mathbb{T})$ for any $\varepsilon>0$. Then, Theorem A-b) gives that the fractal dimension of the graph of $\mathrm{Re}\left[\mathrm{e}^{-i\partial_x^2 t}\Pi f\right]$ is equal to $\frac{3}{2}$ for almost every $t\in\mathcal{K}$. This ensures the property (II) of Theorem D. □

Appendix A. Exercises and solutions

Problem 1. *Let* $f \in \mathrm{BV}(\mathbb{T})$. *Show that there exists a constant* $C > 0$, *such that*

$$|\widehat{f}(n)| \leq \frac{C}{|n|}$$

for all $n \neq 0$.

Solution. Use that

$$\widehat{f}(n) = \frac{\widehat{f'}(n)}{n}$$

and the representation of a bounded variation function given above. □

Problem 2. *Let* $f : \mathbb{T} \longrightarrow \mathbb{R}$ *be given by*

$$f(x) = |x| \log \frac{1}{|x|}.$$

Show that $f \in \mathrm{C}^{\alpha}(\mathbb{T})$ *for all* $0 < \alpha < 1$. *Show that* f *is not a Lipschitz function. Is* $f \in \mathrm{AC}(\mathbb{T})$*?*

Solution. For the first and second parts, use that for fixed $0 < b < 1$,

$$|y| < |y| \log \frac{1}{|y|} < |y|^b$$

in a neighbourhood of $x = 0$. For the third part, note that

$$f'(x) = \mathrm{sgn}(x)[\log \frac{1}{|x|} - 1],$$

so $f'(x) \in \mathrm{L}^1(\mathbb{T})$. □

Problem 3. *Show that* $\mathrm{H}^1(\mathbb{T}) \subseteq \mathrm{C}^{\frac{1}{2}}(\mathbb{T})$. *Hint: use the Cauchy-Schwarz inequality. Is* $\mathrm{H}^1(\mathbb{T}) = \mathrm{C}^{\frac{1}{2}}(\mathbb{T})$*?*

Solution. Let $f \in \mathrm{H}^1(\mathbb{T})$. Then $f' \in \mathrm{L}^2(\mathbb{T})$. Hence $f' \in \mathrm{L}^1(\mathbb{T})$. Then $f \in \mathrm{AC}(\mathbb{T})$. Let $g = f'$. Then,

$$\begin{aligned} |f(x) - f(y)| &= \left| \int_y^x g(z)\, \mathrm{d}z \right| \\ &\leq \left(\int_y^x \mathrm{d}z \right)^{\frac{1}{2}} \left(\int_y^x |g(z)|^2 \mathrm{d}z \right)^{\frac{1}{2}} \\ &\leq |x - y|^{\frac{1}{2}} \|g\|_{\mathrm{L}^2}. \end{aligned}$$

This ensures that $f \in \mathrm{C}^{\frac{1}{2}}(\mathbb{T})$.

The answer to the second question is "no". For example, $|\cdot|^{\frac{1}{2}} \not\in \mathrm{H}^1(\mathbb{T})$ but $|\cdot| \in \mathrm{C}^{\frac{1}{2}}(\mathbb{T})$. □

Problem 4. *Let* $\alpha \geq 0$. *Show that*

$$\|u(\cdot, t)\|_{\mathrm{H}^{\alpha}(\mathbb{T})} = \|f\|_{\mathrm{H}^{\alpha}(\mathbb{T})}$$

for all $t \in \mathbb{R}$.

Solution. Use Parseval's identity and the fact that $|\mathrm{e}^{-ij^2 t}| = 1$ for all $t \in \mathbb{R}$. □

Problem 5. *Give the proof of the identity* (1).

Solution. If $j \underset{q}{\equiv} m$, then $m-j = nq$ for some $n \in \mathbb{N}$ and hence

$$\sum_{k=0}^{q-1} \mathrm{e}^{2\pi i(m-j)\frac{k}{q}} = \sum_{k=0}^{q-1} 1 = q.$$

Otherwise, $j = nq + r$ for some $r \in \{1, \ldots, q-1\}$, so

$$\sum_{k=0}^{q-1} \mathrm{e}^{2\pi i(m-j)\frac{k}{q}} = \sum_{k=0}^{q-1} \mathrm{e}^{2\pi i r\frac{k}{q}} = \frac{1-\mathrm{e}^{2\pi i r}}{1-\mathrm{e}^{\frac{2\pi i r}{q}}} = 0.$$

Note that the denominator of the fraction is different from zero. □

Problem 6. *Let $f \in \mathrm{L}^2(0,\pi)$. Find the solution to*

$$\begin{aligned} &\partial_t u(x,t) = i\partial_x^2 u(x,t) && x \in (0,\pi) \quad t \in \mathbb{R} \\ &\partial_x u(0,t) = \partial_x u(\pi,t) = 0 && t \in \mathbb{R} \\ &u(x,0) = f(x) && x \in (0,\pi). \end{aligned} \tag{2}$$

Give your solution in terms of a Fourier series of f. Now, set

$$f(x) = \begin{cases} 1 & x \in [0, \frac{\pi}{2}] \\ 0 & x \in (\frac{\pi}{2}, \pi]. \end{cases}$$

Find $\tilde{t} > 0$ such that $u(\cdot, \tilde{t})$ has a discontinuity at $x = \frac{\pi}{8}$. Hint: the second part is tougher than you might think.

Solution. Any $f \in \mathrm{L}^2(0,\pi)$ can be expanded as

$$f(x) = \sum_{n=0}^{\infty} \widetilde{f}(n)\cos(nx) \qquad x \in (0,\pi)$$

where

$$\widetilde{f}(0) = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x \qquad \widetilde{f}(n) = \frac{2}{\pi}\int_0^{\pi} \cos(nx) f(x)\,\mathrm{d}x.$$

Then,

$$u(x,t) = \sum_{n=0}^{\infty} \mathrm{e}^{-in^2 t}\widetilde{f}(n)\cos(nx).$$

Consider now the second part of the question.

Step 1. We derive a version of Theorem A-a). From the proof and the previous part, we start with

$$u(x,t) = \sum_{n=0}^{\infty} \mathrm{e}^{-in^2 t}\widetilde{f}(n)\cos(nx)$$

and transform into exponential form. Let f_{e} denote the 2π-periodic extension of

$$f_{\mathrm{e}}(x) = \begin{cases} f(x) & x \in [0,\pi] \\ f(-x) & x \in (-\pi, 0) \end{cases}$$

By expressing the cosine in exponential form, doubling the integral of the Fourier coefficients and gathering terms, we get

$$\widetilde{f}(n) = \frac{1}{\pi}\left(\int_{-\pi}^{0} + \int_0^{\pi}\right) f_{\mathrm{e}}(x)\mathrm{e}^{-inx}\,\mathrm{d}x = \frac{1}{\pi}\langle f_{\mathrm{e}}, \mathrm{e}^{in(\cdot)}\rangle.$$

Therefore, u is a solution to (2) with datum f, if and only if the extension u_{e} is a solution to (A) with datum f_{e}. Indeed, note that a solution to (A) with even initial datum is even. Hence we have the following.

Theorem A'-a). *Let $f \in \mathrm{L}^2(0,\pi)$ and consider the solution to the equation* (2). *Then, for $\tilde{t} = \frac{2\pi p}{q}$ where p and q are co-primes,*

$$u(x,\tilde{t}) = \frac{1}{q} \sum_{k,m=0}^{q-1} \mathrm{e}^{2\pi i \frac{km - pm^2}{q}} f_{\mathrm{e}}(x - 2\pi k/q).$$

Step 2. With this result at hand, let us now find $\tilde{t}$ such that the RHS of the above revival expression has a discontinuity at $\tilde{t} = \frac{\pi}{8}$. First, note that

$$f_{\mathrm{e}}(x) = \operatorname{sgn}(\cos(x)).$$

We need an educated guess.

Taking $q = 16$ gives no discontinuity at $\tilde{x} = \frac{\pi}{8}$ despite of having the correct coefficients to play around with $k = 5$ and $k = 13$ in the above formula. See Figure 5-(a).

Taking $q = 32$ and $p = 1$, gives $\tilde{t} = \frac{\pi}{16}$. Now,

$$\frac{\pi}{8} - \frac{k\pi}{16} = \frac{\pi}{2} + 2n\pi \qquad \Longleftrightarrow \qquad k \equiv_{32} -6 \equiv_{32} 26$$

and

$$\frac{\pi}{8} - \frac{k\pi}{16} = -\frac{\pi}{2} + 2n\pi \qquad \Longleftrightarrow \qquad k \equiv_{32} 10.$$

These are the only contributing terms in the revival summation that give a jump at $\pi/8$ in the case $q = 32$. One is a jump up, the other a jump down. We need to check that these do not cancel out. Octave gives

```
octave:1> m=0:31;
octave:2> sum(exp(i*pi*(26*m-m.^2)/16))
ans =  4.4446 + 6.6518i
octave:3> sum(exp(i*pi*(10*m-m.^2)/16))
ans = -4.4446 - 6.6518i
```

Hence, at $k = 26$,

$$A = \sum_{m=0}^{31} \mathrm{e}^{-\pi \frac{26m - m^2}{16}} \approx 4.4446 + 6.6518i$$

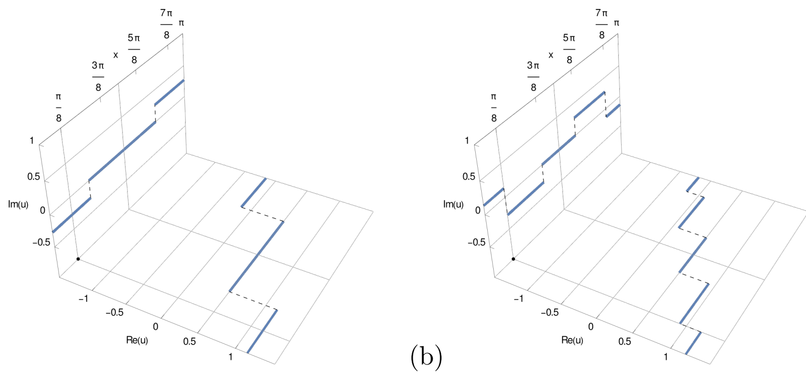
and, at $k = 10$,

$$B = \sum_{m=0}^{31} \mathrm{e}^{-\pi \frac{10m - m^2}{16}} \approx -4.4446 - 6.6518i.$$

Thus, since A is safely away from B and they are both safely away from 0, we know that there is a discontinuity at $x = \frac{\pi}{8}$. See Figure 5-(b).

□

Problem 7. *Give the proof of* (3). *Hint: use* (5).



(a) (b)

FIGURE 5. Solution to (2) for $f = \mathbb{1}_{[0,\frac{\pi}{2}]}$ at $t = \frac{2\pi p}{q}$, where: (a) $q = 16$ and $p = 1$, (b) $q = 32$ and $p = 1$.

Solution. Take $u = g_j \star F$, $\lambda = 2^j$, $r_1 = 2^{-1}$ and $r_2 = 2$ in (5). Then, the left hand side inequality yields,

$$2^{(\alpha+1)j}\|K_j F\|_{\mathrm{L}^\infty(\mathbb{T})} \leq C 2^{\alpha j}\|K_j(F')\|_{\mathrm{L}^\infty(\mathbb{T})} < \infty,$$

for $F' \in B^\alpha_\infty(\mathbb{T})$. Conversely, the right hand side inequality yields,

$$2^{\alpha j}\|K_j(F')\|_{\mathrm{L}^\infty(\mathbb{T})} \leq C 2^{(\alpha+1)j}\|K_j F\|_{\mathrm{L}^\infty(\mathbb{T})} < \infty,$$

for $F \in B^{\alpha+1}_\infty(\mathbb{T})$. □

Problem 8. *Show that*

$$\mathrm{B}^{\alpha_1}_1(\mathbb{T}) \cap \mathrm{B}^{\alpha_2}_\infty(\mathbb{T}) \subset \mathrm{H}^\alpha(\mathbb{T})$$

for all $\alpha < (\alpha_1 + \alpha_2)/2$.

Solution. Recall the definition,

$$f \in \mathrm{B}^\alpha_p(\mathbb{T}) \qquad \Longleftrightarrow \qquad 2^{\alpha j}\|K_j f\|_{\mathrm{L}^p} < C \quad \forall j \in \mathbb{N}.$$

Since,

$$\|K_j f\|^2_{\mathrm{L}^2} \leq \|K_j f\|_{\mathrm{L}^1}\|K_j f\|_{\mathrm{L}^\infty},$$

then for all $\alpha = \frac{\alpha_1+\alpha_2}{2} - \varepsilon$, we have that

$$f \in \mathrm{B}^{\alpha_1}_1(\mathbb{T}) \cap \mathrm{B}^{\alpha_2}_\infty(\mathbb{T}) \quad \Rightarrow \quad 2^{2\alpha j}\|K_j f\|^2_{\mathrm{L}^2} \leq C 2^{-\varepsilon j} \quad \forall j \in \mathbb{N}.$$

Hence $f \in \mathrm{B}^{\alpha+\varepsilon}_2(\mathbb{T})$. Thus, since by Plancherel's identity,

$$\|K_j f\|^2_{\mathrm{L}^2} = \sum_{n\in\mathbb{Z}} \chi_j(|n|)|\widehat{f}(n)|^2,$$

we have

$$\sum_{n\in\mathbb{Z}} n^{2\alpha}|\widehat{f}(n)| \leq 2^{2\alpha}\sum_{n=1}^{\infty} 2^{2\alpha j}\|K_j f\|^2_{\mathrm{L}^2} \leq C\sum_{j=1}^{\infty} 2^{-\varepsilon j} < \infty.$$

This ensures that $f \in \mathrm{H}^\alpha(\mathbb{T})$. □

Problem 9. *Show that if* $H \in \mathrm{C}^\alpha(\mathbb{T})$ *and* F *is a bounded measure, then* $(H * F) \in \mathrm{C}^\alpha(\mathbb{T})$.

Solution. Since $H \in \mathrm{C}^\alpha(\mathbb{T})$, then

$$\sup_{\substack{x\in\mathbb{T}\\ |h|>0}} \frac{|H(x+h)-H(x)|}{|h|^\alpha} < \infty.$$

Hence,

$$\sup_{\substack{x\in\mathbb{T}\\ |h|>0}} \frac{|(H*F)(x+h)-(H*F)(x)|}{|h|^\alpha} \leq \sup_{\substack{x\in\mathbb{T}\\ |h|>0}} \frac{|H(x+h)-H(x)|}{|h|^\alpha} \int_{\mathbb{T}} \mathrm{d}|F| < \infty.$$

□

Problem 10. *Let u be the solution to the time-evolution equation* (2) *from Problem 6. Show that there exists $\tilde{t} > 0$ such that $u(\cdot, \tilde{t})$ is continuous. Hint: this is easier than you think.*

Solution. From what we have discovered so far, a solution to the equation with Neumann boundary conditions is a solution to (A) with initial condition f_{e}. Since f_{e} is piecewise constant, it is of bounded variation in $\mathbb{T}$. Therefore, for almost all $\tilde{t}$ the solution is C^α for all $0 \leq \alpha < 1/2$ and hence it is continuous. □

Problem 11. *Show that the expressions*

$$\mathrm{e}^{2\pi i \frac{pb^2}{q(1-b)^2}} \langle f, \phi\rangle_{\mathrm{L}^2(0,\pi)} \phi(x)$$

and

$$\frac{1}{q}\sum_{m=0}^{q-1}\left[\sum_{k=0}^{q-1} \mathrm{e}^{2\pi i \frac{km-pm^2}{q}} g\Big(x - 2\pi\frac{k}{q}\Big)\right],$$

cancel out from the statement a) of this theorem, in the case $b = 0$. Compare with your answer to Problem 6.

Solution. Write $u(x,t) = T_1 + T_2 + T_3$, where

$$T_1 = \mathrm{e}^{2\pi i \frac{b^2}{(1-b)^2}} \langle f, \phi\rangle \phi(x), \qquad T_2 = \sum_{n\in\mathbb{Z}} \mathrm{e}^{-in^2 t} \widehat{f_{\mathrm{e}}}(n) \mathrm{e}^{inx} \quad \text{and}$$

$$T_3 = \sum_{n\in\mathbb{Z}} \mathrm{e}^{-in^2 t} \widehat{g}(n) \mathrm{e}^{inx}.$$

For $b = 0$,

$$T_1 = \frac{1}{\pi}\int_0^\pi f(x)\,\mathrm{d}x.$$

Let us compute T_3. Directly,

$$g(x) = \left[\sqrt{\pi}\frac{1}{\sqrt{\pi}} * (f_{\mathrm{e}} - f_{\mathrm{o}})\right](x) = \frac{-1}{2\pi}\int_{-\pi}^0 2f(-x)\,\mathrm{d}x = -T_1.$$

This means that T_3 is a solution to (A) with initial condition $g = -T_1$ a constant. Therefore, we should have T_3 equal to that same constant $-T_1$.

Curiously, note that this calculation gives the following identity,

$$\sum_{m=0}^{q-1}\sum_{k=0}^{q-1} \mathrm{e}^{2\pi i \frac{-m^2 p + mk}{q}} = q.$$

□

Problem 12. *Show that the family of eigenvectors* $\{\phi\} \cup \{\phi_j\}_{j=1}^{\infty}$ *is indeed an orthonormal basis of* $\mathrm{L}^2(0,\pi)$. *Hint: the important point here is to show that they are a complete family.*

Solution. It is routine that they form an orthonormal family, because L is a symmetric operator and all the eigenvalues listed above are distinct. The proof that they are complete is as follows.

Observe that Λ_j is unimodular and that we need its square root to write ϕ_j in a more symmetric manner. So write

$$\Lambda_j = \mathrm{e}^{2i\lambda_j},$$

for $\lambda_j \in (-\pi, \pi]$. Since $\Lambda_j \to -1$ as $j \to \infty$, then we can pick unambiguously $\lambda_j \to \frac{\pi}{2}$. This gives,

$$\begin{aligned}\phi_j(x) &= \frac{\mathrm{e}^{i\lambda_j}}{\sqrt{2\pi}}\big(\mathrm{e}^{i(jx+\lambda_j)} - \mathrm{e}^{-i(jx+\lambda_j)}\big)\\ &= 2i\frac{\mathrm{e}^{i\lambda_j}}{\sqrt{2\pi}}\sin(j(x+\lambda_j))\\ &= s_j\mathrm{e}^{i\lambda_j}\sqrt{\frac{2}{\pi}}\Big(\cos(jx) + t_j\sin(jx)\Big)\end{aligned}$$

for $s_j = \sin\lambda_j$ and $t_j = \cot\lambda_j$. Here $s_j \to 1$ and $t_j \to 0$ as $j \to \infty$.

Let the linear operator $R : \mathrm{L}^2(0,\pi) \longrightarrow \mathrm{L}^2(0,\pi)$, be the linear extension of the map $R : \sin(x) \mapsto \frac{1}{\sqrt{\pi}}$ and $R : \sin((k+1)x) \mapsto \cos(kx)$ for all $k \in \mathbb{N}$. Then, by Parseval's identity, it follows that R is an isometry. It is also onto, therefore it is unitary.

Let $J \in \mathbb{N}$ be such that $|t_j| \leq \frac{1}{2}$ for all $j \geq J$. Let the operator

$$F = (R+T)S : \mathrm{L}^2(0,\pi) \ominus \mathrm{Span}\{\sin(jx)\}_{j=1}^{J} \longrightarrow F(\mathrm{L}^2(0,\pi)),$$

where $T : \sin((k+1)x) \mapsto t_k\sin(kx)$ and $S : \sin((k+1)x) \mapsto s_k\sin((k+1)x)$ for all $k \geq J$. Its is immediate that $F : \sin((k+1)x) \mapsto \phi_k(x)$ for all $k \geq J$. Moreover, F is invertible with a bounded inverse. Indeed, F is invertible because R is an isometry, T is compact with $\|T\| \leq \frac{1}{2}$ and S is invertible. Furthermore, $F(\mathrm{L}^2(0,\pi))^{\perp}$ has dimension J. For this, note that the missing subspace in $F(\mathrm{L}^2(0,\pi))$ is

$$\mathrm{Span}\{\phi, \phi_1, \ldots, \phi_{J-1}\}$$

and all these functions are linearly independent, because they are eigenfunctions of L associated to distinct eigenvalues. From this, the fact that

$$\mathrm{Span}\{\phi, \phi_1, \phi_2, \ldots\} = \mathrm{L}^2(0,\pi)$$

is immediate. □

Problem 13. *Give the proof of Theorem B-a), using Lemma 5.*

Solution. Substitute $t = 2\pi\frac{p}{q}$ and use Theorem A-a) with initial data $f_{\mathrm{e}} + g$. □

Problem 14. *Give the proof of Theorem C for* $V(x) = c$, *where* $c \in \mathbb{C}$. *Hint: note that* (C) *has Dirichlet boundary conditions and that* c *does not need to satisfy the hypotheses on* V *stated in the bullet points.*

Solution. The eigenfunctions of $-\partial_x^2 + c$ are $\sin(nx)$ with corresponding eigenfunctions $c + n^2$ for all $n \in \mathbb{N}$. Hence, by following a similar argumentation as in the solution to Problem 6, we have

$$u(x,t) = \mathrm{e}^{-i2\pi c\frac{p}{q}} \sum_{m=0}^{q-1} \left(\sum_{k=1}^{q-1} \mathrm{e}^{2\pi i \frac{mk - m^2 p}{q}} f_{\mathrm{o}}\Big(x - 2\pi \frac{k}{q}\Big) \right).$$

In this case, $w(x,t) = 0$. □

Problem 15. *Show that*

$$\mathcal{H}\mathbb{1}_{[a,b]}(x) = \frac{1}{\pi} \log \left| \frac{\sin\left(\frac{x-a}{2}\right)}{\sin\left(\frac{x-b}{2}\right)} \right|,$$

for $a, b \in \mathbb{T}$ *with* $-\pi \leq a < b < \pi$.

Solution.

$$\begin{aligned}
\mathcal{H}\mathbb{1}_{[a,b]}(z) &= \frac{1}{2\pi} \text{ p.v.}\int_a^b \cot\left(\frac{z-w}{2}\right) \mathrm{d}w \\
&= \frac{1}{2\pi} \text{ p.v.}\int_a^b \frac{\cos\left(\frac{z-w}{2}\right)}{\sin\left(\frac{z-w}{2}\right)} \mathrm{d}w \\
&= -\frac{1}{\pi} \int_{\frac{z-a}{2\pi}}^{\frac{z-b}{2\pi}} \frac{(\sin(\pi x))'}{\sin(\pi x)} \mathrm{d}x.
\end{aligned}$$

□

Problem 16. *Compute* $\mathcal{H}e_n$ *to verify* (13).

Solution. First observe that

$$\cot \frac{z}{2} = 2 \sum_{n=1}^{\infty} \sin(nz), \tag{17}$$

meaning that the periodic distributions on both sides are equal. Indeed, this is true, because taking the sine Fourier coefficients of $\cot \frac{z}{2}$,

$$\widetilde{f}(n) = \frac{2}{\pi} \int_0^{\pi} \cot \frac{x}{2} \sin(nx) \,\mathrm{d}x,$$

we have

$$\pi \widetilde{f}(n) = \int_{-\pi}^{\pi} \cot \frac{x}{2} \sin(nx) \,\mathrm{d}x = A.$$

Note that there is no singularity at $x = 0$ any more because the sine vanishes there too. Writing the cotangent and the sine in exponential form, and simplifying we get,

$$\begin{aligned} A &= \frac{1}{2}\int_{-\pi}^{\pi} \frac{\mathrm{e}^{i\frac{x}{2}i} + \mathrm{e}^{-i\frac{x}{2}i}}{\mathrm{e}^{i\frac{x}{2}i} - \mathrm{e}^{-i\frac{x}{2}i}} (\mathrm{e}^{inx} - \mathrm{e}^{-inx})\,\mathrm{d}x \\ &= \int_\gamma \frac{(z^2+1)(z^{4n}-1)}{(z^2-1)z^{2n+1}}\,\mathrm{d}z \\ &= \int_\gamma (z^2+1)\sum_{k=0}^{2n-1} z^{2(k-n)-1}\,\mathrm{d}z \\ &= 2\pi, \end{aligned}$$

where γ is the semi-circle $\gamma(w) = e^{iw}$ for $w \in (-\frac{\pi}{2}, \frac{\pi}{2})$. This gives (17).

Now, let $g : \mathbb{T} \longrightarrow \mathbb{C}$ be a periodic distribution. Then,

$$\begin{aligned} \mathcal{H}g(x) &= \frac{1}{2\pi}\ \text{p.v.}\!\int_{-\pi}^{\pi} g(y)\sum_{n=1}^{\infty} 2\sin(n(x-y))\,\mathrm{d}y \\ &= \frac{-i}{2\pi}\ \text{p.v.}\!\int_{-\pi}^{\pi} g(y)\sum_{n=1}^{\infty}\left[\mathrm{e}^{in(x-y)} - \mathrm{e}^{-in(x-y)}\right]\,\mathrm{d}y \\ &= i\sum_{n=1}^{\infty}\langle g, e_{-n}\rangle e_{-n}(x) - i\sum_{n=1}^{\infty}\langle g, e_n\rangle e_n(x). \end{aligned}$$

From this, taking $g = e_k$, the statement (13) follows. Notice that if, additionally, $g \in \mathrm{L}^2(\mathbb{T})$, then $\mathcal{H}g \in \mathrm{L}^2(\mathbb{T})$, and therefore the identity is valid in the L^2 sense. Also note that contour integration without (17) does not give the answer directly, because Jordan's Lemma is not applicable for this intergation. □

Problem 17. *Complete the proof of Theorem D-a).*

Solution. Substitute the formula of the theorem, directly into the formula of the lemma. □

References

[1] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier Analysis and Nonlinear Partial Differential Equations*. Springer-Verlag, Berlin, 2011.

[2] L. Boulton, G. Farmakis, and B. Pelloni. Beyond periodic revivals for linear dispersive PDEs. *Proceedings of the Royal Society A*, 477:10241, 2021.

[3] L. Boulton, G. Farmakis, and B. Pelloni. The phenomenon of revivals on complex potential Schrödinger's equation. *Zeitschrift für Analysis und ihre Anwendungen*, 43:401–416, 2024.

[4] L. Boulton, B. MacPherson, and B. Pelloni. Jumps, cusps and fractals in the solution of the periodic linear Benjamin-Ono equation. *Proceedings of the Royal Society of Edinburgh*, to appear in 2025.

[5] V. Chousionis, M.B. Erdoğan, and N. Tzirakis. Fractal solutions of linear and nonlinear dispersive partial differential equations. *Proceedings of the London Mathematical Society*, 110(3):543–564, 2014.

[6] A. Deliu and B. Jawerth. Geometrical dimension versus smoothness. *Constructive Approximation*, 8:211–222, 1992.

[7] K. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*. John Wiley & Sons, New York, 1990.

[8] G. Farmakis. *Revivals in Time Evolution Problems*. PhD Thesis, Heriot-Watt University, Edinburgh, 2022.

[9] H. Fiedler, W. Jurkat, and O. Körner. Asymptotic expansions of finite theta series. *Acta Arithmetica*, 32(2):129–146, 1977.

[10] G. Folland. *Fourier Analysis and its Applications*. Brooks/Cole Publishing Company, California, 1992.

[11] L. Grafakos. *Classical Fourier Analysis*. Springer Verlag, New York, 2008.

[12] L. Kapitanski and I. Rodnianski. Does a quantum particle know the time? In *Emerging applications of number theory*, pages 355–371. Springer, 1999.

[13] A.YA. Khinchin. *Continued Fractions*. The University of Chicago Press, Chicago and London, 1964.

[14] G. Leoni. *A First Course in Sobolev Spaces*. The American Mathematical Society, Providence, Rhode Island, 2009.

[15] P.J. Olver. Dispersive quantization. *The American Mathematical Monthly*, 117(7):599–610, 2010.

[16] F. Riesz and B. Sz.-Nagy. *Functional Analysis*. Dover Publications, New York, 1990.

[17] I. Rodnianski. Fractal solutions of the Schrödinger equation. *Contemporary Mathematics*, 255:181–188, 2000.

[18] M. Taylor. The Schrödinger equation on spheres. *Pacific journal of mathematics*, 209(1):145–155, 2003.

[19] A.C. Zemanian. *Distribution Theory and Transform Analysis*. Dover Publications, New York, 1987.

Department of Mathematics and Maxwell Institute for the Mathematical Sciences, Heriot-Watt University, Edinburgh EH14 4AS, United Kingdom.

Email address: `l.boulton@hw.ac.uk`