The toric code

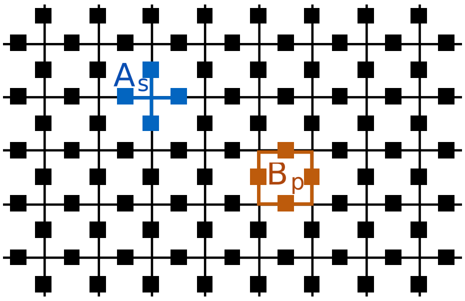

Figure 1: The toric code is a model defined on the square lattice with one spin 1/2 on every bond (black square). The A_s operators (star operators) are defined on each site of the lattice and act on the four surrounding spins with the operator σ^z. The B_p operators (plaquette operators) are defined on each plaquette and act on the corresponding four spins with the operator σ^x.

We consider the toric code defined on a square lattice with $L_x \times L_y$ sites, with a spin 1/2 on each bond, and periodic boundary conditions along both x and y directions (see Figure 1). The Hamiltonian is written in terms of star and plaquette operators A_s and B_p, which are both products of four individual spin operators:

$$H = -\sum_s A_s - \sum_p B_p \tag{1}$$

where

$$A_s = \prod_{i\in s} \sigma_i^z \tag{2}$$

$$B_p = \prod_{i\in p} \sigma_i^x \tag{3}$$

σ_i^x, σ_i^z are Pauli matrices acting on the spin i.

1) Show that any star operator A_s commutes with any plaquette operator B_p: $\forall (s,p) \in [1, L_x L_y]^2,\ [A_s, B_p] = 0$.
   Is this property still true if some of the stars have a number of branches different from 4?

2) What is the energy E_0 of the ground state of this model? What are the constraints on A_s and B_p such that a quantum state Ψ verifies $H\Psi = E_0\Psi$?

3) The system defined in Figure 1 with periodic boundary conditions is a torus. Let us compute the degeneracy of the ground state

- What is the total number of spin configurations in this system (i.e. the dimension of the Hilbert space)?
- Show that $\prod_s A_s = I$ and $\prod_p B_p = I$, where I is the identity operator.
- How many linearly independent equations can be written from the constraints of question 2)?
- What is the degeneracy of the ground state of the toric code on the torus?

4) Follow the same reasoning as question 3) to predict the degeneracy of the ground state of the toric code on the sphere geometry, and on a higher genus surface (for example, a surface with two "holes").

5) Statistics of the excitations of the toric code
We define the paths l^e and l^m which connect spins on bonds respectively parallel (l^e) and perpendicular (l^m) to them. e (respectively m) excitations can be created at the ends of a l^e (resp. l^m) path by acting on the ground state with a product of σ^x (resp. σ^z) operators along an open path l^e (l^m). e and m have trivial self-statistics, and non-trivial (-1) mutual statistics, as summarized in Figure 2.

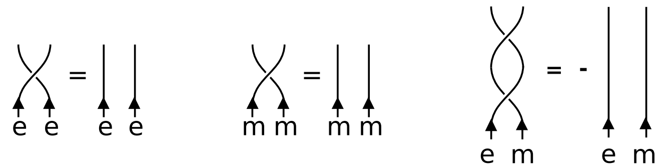

Figure 2: Exchange statistics of the excitations e and m of the toric code

-Using a drawing similar to Figure 2, can you guess what is the self-statistics of the combined particle $f = e \times m$?
-Prove the above result by considering l^e paths of σ^x operators and l^m paths of σ^z operators on the toric code of Figure 1. This reasoning is the same as the one used in the lecture to show the self-statistics of the e and m particles, which used l^e and l^m loops separately.