Homotopy classes and braiding of anyons

Ex.1: Quantum particles in D dimensions and homotopy classes

We consider $N = 2$ indistinguishable particles living in a D-dimensional space. Their positions are denoted by $x_i, i = 1, 2, x_i \in \mathbb{R}^D$. The space of configurations of this system is

$$X_2^{(D)} = \left(\mathbb{R}^{2D} - I\right)/\mathcal{S}_2 \tag{1}$$

where I is the set of hyperplanes where the two particles coincide, and $\mathcal{S}_2$ is the permutation group of two objects. We also consider the space

$$Y_2^{(D)} = \mathbb{R}^{2D} - I \tag{2}$$

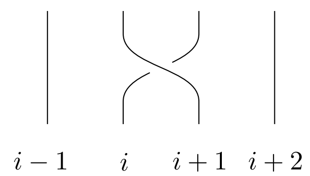
Propose simple drawings to show that:

- $Y_2^{(1)}$ is not connected
- $Y_2^{(3)}$ is simply connected and $X_2^{(3)}$ is doubly connected
- $Y_2^{(2)}$ is multiply connected.

Ex. 2: Abelian anyons and one-dimensional representations of the Braid group

Generically, the transformation of a quantum state upon exchange of anyons can be described by the action of an operator ρ on that quantum state. The set of possible outcomes is determined by considering that ρ is a representation of the braid group $\mathcal{B}_N$. As seen in the lecture, $\mathcal{B}_N$ is generated by the set of σ_i's, where σ_i is the operator which braids the strand i *below* the strand $i + 1$:

σ_i :



a) Graphically show that $\sigma_j\sigma_{j+1}\sigma_j = \sigma_{j+1}\sigma_j\sigma_{j+1}$

b) Consider the one-dimensional representation ρ of $\mathcal{B}_N$: $\rho(\sigma_j) = e^{i\theta_j}$. Show that the $N-1$ phases θ_j have to be equal.

c) Using the previous result, can you predict what are the possible outcomes of the braiding of abelian anyons?

Ex.3: Permutation group

a) The permutation group $\mathcal{S}_N$ can be formed by the elements s_i that exchange particles i and $i+1$. Write the algebra formed by the s_i. What is the difference with the one formed by the σ_i?

b) Using the result of Ex.1, comment on the relevance of $\mathcal{S}_N$ for the description of systems of N quantum particles in $D = 3$.

b) Consider the one-dimensional representation of the s_i algebra and explain why one can find only fermionic or bosonic representations.

Ex.4: Expansion of the solutions of a Fuchsian equation

We consider Euler's hypergeometric differential equation

$$\left(z(1-z)\frac{d^2}{dz^2} + [\gamma - (\alpha + \beta + 1)z]\, \frac{d}{dz} - \alpha\beta \right) f(z) = 0, \tag{3}$$

Eq. (3) is a linear ordinary differential equation (ODE) of Fuchsian type with singularities $z_1 = 0$, $z_2 = 1$ and $z_3 = \infty$. To study the monodromy of its solution, we need to write down an expansion around its singularities, which is the purpose of this exercise.

a) Consider the function $f^{(a)}(z) = z^a \sum_{n=0}^{\infty} c_n(a) z^n$, where $c_0(a) = 1.0$ and $c_n(a) \in \mathbb{C}$. Fuch's theorem guarantees the existence of a solution of this form if $z_1 = 0$ is a regular singularity. Determine the possible values of a and a recurrence relation for the $c_n(a)$ coefficients.

b) We consider the monodromy associated to a small loop γ_0 around zero. How do the solutions of Eq. (3) transform under γ_0?

b) How would you verify that $z_2 = 1$ is a regular singularity? Determine the monodromy associated to a small loop γ_1 around $z_2 = 1$.