Part II: Symmetry-protected (SPT) order

1. The AKLT model

a) Motivation

AFM Heisenberg chain: $H = \sum_i \vec{S}_i \cdot \vec{S}_{i+1}$

$\vec{S}_i$: spin S operator

What is the physics (in the ground state)?

- S half-integer:

  Lieb-Schultz-Mattis theorem ('61):

  H is symm. breaking or gapless

  ($\Leftrightarrow$ there is more than one ok state w/ low energy)

- S integer:

  Haldane ('83) [$\to$ Nobel prize!]

  H has unique g.s. w/ gap above!

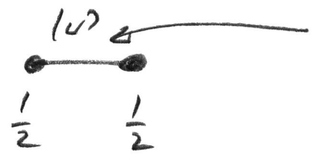
  argument via mapping to "non-linear sigma model"

  "Haldane gap"

Can we prove Haldane gap on microscopic level?

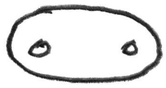
$\to$ AKLT model (Aff., K., L., T. '87)

b) The construction



$$|\omega\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle - |1\rangle|0\rangle\right)$$

singlet state: $(u \otimes u)|\omega\rangle = |\omega\rangle$
$\forall u \in SU(2)$

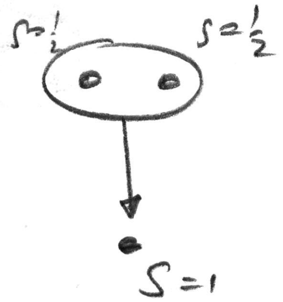
 : $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$:

antisym. space $\left\{ |s=0, s_z=0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle - |1\rangle|0\rangle\right) \right.$

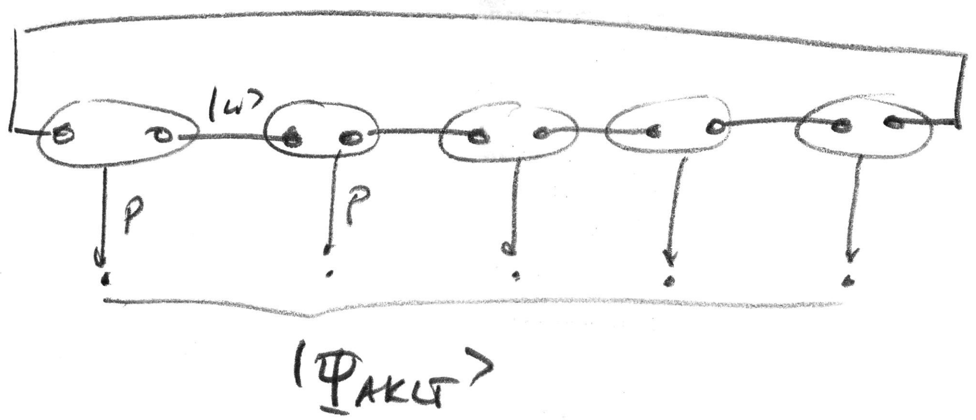
sym. space
$$\left\{ \begin{array}{l} |s=1, s_z=-1\rangle = |0\rangle|0\rangle \\ |s=1, s_z=0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle + |1\rangle|0\rangle\right) \\ |s=1, s_z=-1\rangle = |1\rangle|1\rangle \end{array} \right.$$

Define P to be <u>projection</u> onto $S=1$ space ($=$ sym. space):

$$P = |{+1}\rangle\langle 00| + |0\rangle \frac{|01\rangle + |10\rangle}{\sqrt{2}} + |{-1}\rangle\langle 11|$$



AKLT state: arrange $\circ\!\xrightarrow{\;|\omega\rangle\;}\!\circ$ on a ring and project pairs



$$|\Psi_{AKLT}\rangle = P^{\otimes N} \; |\omega\rangle^{\otimes N}$$

tensor product in different partitions!

What properties does Ψ have?

<u>$u \in SU(2)$:</u> $\quad |\omega\rangle = (u \otimes u)|\omega\rangle$

$$\Rightarrow \quad |\omega\rangle^{\otimes N} = \underbrace{u \otimes \ldots \otimes u}_{2N} \, |\omega\rangle^{\otimes N}$$

$$u \in \exp[i \cdot \vec{\phi} \cdot \vec{S}\,] \qquad \phi: 0 \ldots 4\pi, \quad S = 1/2$$

$$\Downarrow$$

$$R_u^{S=1} = \exp[i \vec{\phi} \cdot \vec{S}\,] \qquad S = 1$$

2π - periodic

$$P(u \otimes u) = R_u^{S=1} \, P$$

$$\Rightarrow \quad P^{\otimes N} |\omega\rangle^{\otimes N} = P^{\otimes N} (u \otimes \ldots \otimes u) |\omega\rangle^{\otimes N}$$

$$= \left(R_u^{S=1}\right)^{\otimes N} (u \otimes \ldots \otimes u) \, |\omega\rangle^{\otimes N} = \left(R_u^{S=1}\right)^{\otimes N} |\omega\rangle^{\otimes N}$$

$$R(\vec{\phi})^{\otimes N} |\psi_{AKLT}\rangle = |\psi_{AKLT}\rangle$$

for any rotation $R(\vec{\phi}) = \exp[i\vec{\phi}\cdot\vec{S}]$

$\Rightarrow |\psi_{AKLT}\rangle$ is $SU(2)$-invariant

(or better: $SO(3)$-invariant: $\vec{\phi}$ is 2π-periodic)

<u>Key point</u> ($\to$ later): $|\psi_{AKLT}\rangle$ is <u>symmetric</u>

because it is <u>constructed</u> from <u>symmetric objects</u> ($|\omega\rangle$, P).

($\to$ Should work for any symmetry!)

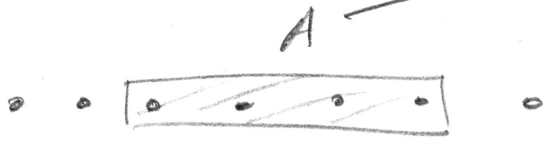
c) The AKLT Hamiltonian

$SO(3)$-invariant wavefunction

$\to$ is it g. state of $SO(3)$ inv. Hamiltonian?

[Remark: Not <u>any</u> state is g.s. of local Hamiltonian:

$\to$ require area law:



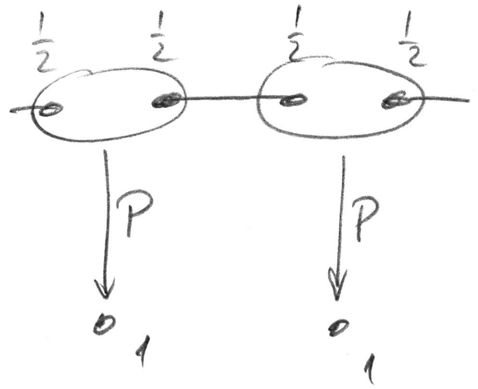
ent. of A w/ rest scales like <u>boundary</u>

AKLT state satisfies the area law:

cut 2 ent. pairs!

... In fact, all states w/ area law are approx. of this form (w/ some $|\omega\rangle$ & P). [Hastings, Vazirani et al.]

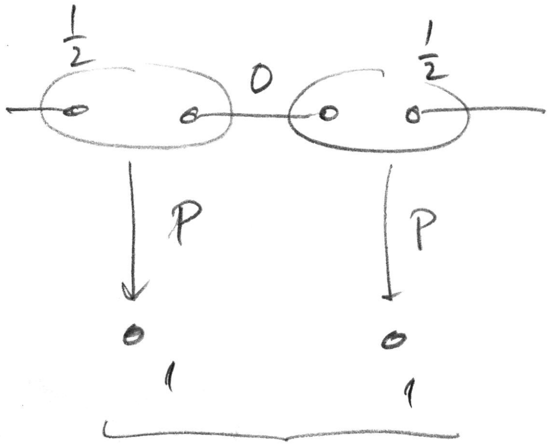
Consider 2 sites of AKLT chain:



$$\underbrace{\tfrac{1}{2}\otimes\tfrac{1}{2}}_{0\oplus 1}\otimes\underbrace{\tfrac{1}{2}\otimes\tfrac{1}{2}}_{0\oplus 1} = 0\oplus 1\oplus 1\oplus 0\oplus 1\oplus 2 = 2\times 0\oplus 3\times 1\oplus 2$$

$$1\otimes 1 = 0\oplus 1\oplus 2 \quad \checkmark$$

But: $\underset{|\omega\rangle}{\circ\!\!-\!\!-\!\!\circ}$ has spin $=0$!



$$\tfrac{1}{2}\otimes 0\otimes\tfrac{1}{2} = 0\oplus 1$$

$\downarrow$ P preserves spin!

$$1\otimes 1 = 0\oplus 1\oplus \cancel{2}$$

spin 2 cannot appear!

2-site state lives in $S=0$ & $S=1$ space!

Construct $h_{i,i+1} = \Pi_{S=2}$

Proj. on $S=2$ subspace (sym. space) of 2 sites.

$$\left.\begin{array}{l} h_{i,i+1} \geq 0 \\ h_{i,i+1} |\psi_{AKLT}\rangle = 0 \end{array}\right\} \ |\psi_{AKLT}\rangle \text{ is \underline{ground state} of } h_{i,i+1}.$$

Build:

$$H = \sum_i \underbrace{h_{i,i+1}}_{\geq 0} \Rightarrow H \geq 0$$

$$H|\psi_{AKLT}\rangle = \sum_i \underbrace{h_{i,i+1}|\psi_{AKLT}\rangle}_{=0} = 0$$

$\Rightarrow$ $|\psi_{AKLT}\rangle$ is <u>ground state</u> of H!

$H = \sum h_{i,i+1}$ is $SU(2)$ invariant by construction

Explicitly: $h_{i,i+1} = \underbrace{\frac{1}{2} \vec{S}_i \cdot \vec{S}_{i+1}}_{\text{Heisenberg}} + \frac{1}{6} (\vec{S}_i \cdot \vec{S}_{i+1})^2 + \underbrace{\frac{1}{3}}_{\text{irrelevant}}$

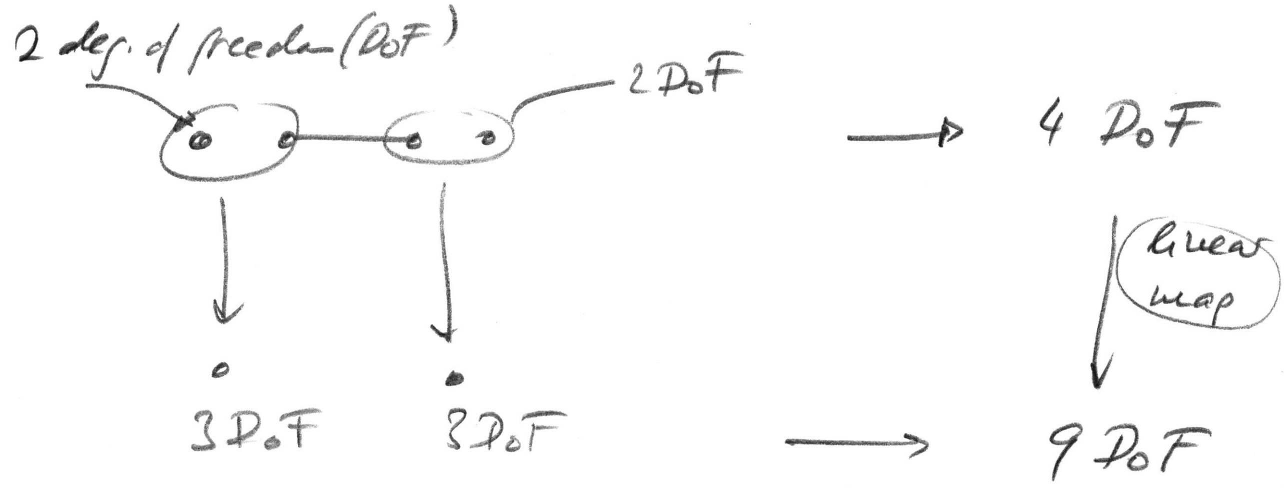
$\rightarrow$ (good) approx. to Heisenberg Ham.

Can show:

- $|\psi_{AKLT}\rangle$ is unique gr. state of H
- H has a gap above ground state

→ Rigorous proof of Haldane gap for a related model.

<u>Note on Hamiltonian</u>: Can be understood w/out SU(2) rep. theory:



→ only a 4 dim. space of the phys. spins reached.

⇒ a $\underbrace{\text{5-dim. space } \mathcal{S}}_{\equiv \text{ the } S=2 \text{ space}}$ <u>cannot</u> be reached.

⇒ $h_{i,i+1} = \Pi_{\mathcal{S}}$.

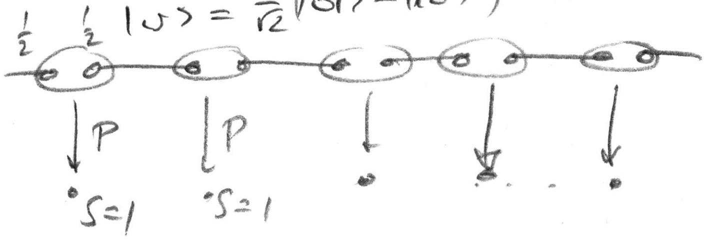
This works for <u>any</u> $|\omega\rangle$ & P — if necessary, consider <u>more sites</u>!

▷ End lecture 4!

Last lecture:

AKLT model:

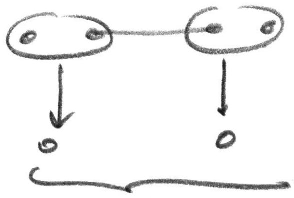
$|\omega\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$



$|\psi_{AKLT}\rangle$

- $|\psi_{AKLT}\rangle$ rot. invariant

- "parent Hamiltonian:



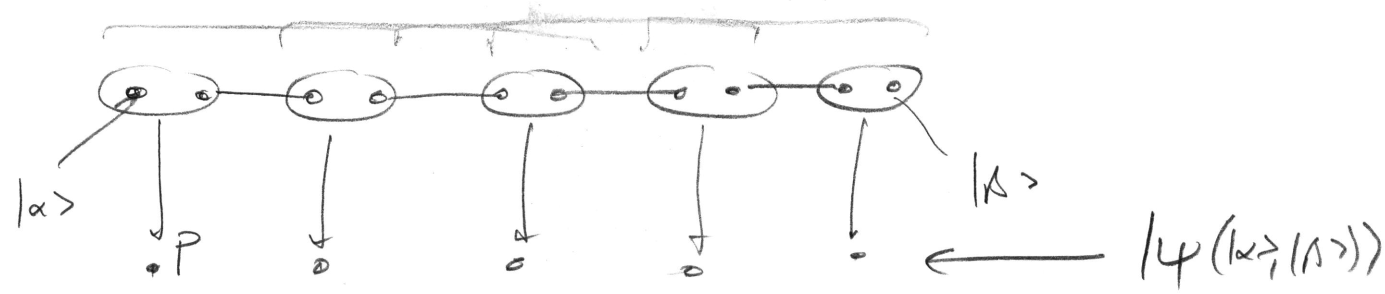
no spin 2 $\Rightarrow$ $h_{i,i+1} = \Pi_{S=2}$

- unique G.S., gap above

d) Fractionalization & edge modes

Consider system on open boundaries:

$$H = \sum_{i=1}^{N-1} h_{i,i+1}$$



Left/right end condition $|\alpha\rangle, |\beta\rangle \in \mathbb{C}^2$

$\Rightarrow$ any $|\psi(|\alpha\rangle,|\beta\rangle)\rangle$ is a ground state.

(& can show: these are all)

AKLT chain exhibits $S=1/2$ excitations at the edge:

These are <u>not</u> elementary (local) excitations of a spin chain:

$$\text{excit} \approx \sum f_x S_x^+ |\psi\rangle \rightarrow$$ changes spin by <u>integer</u> value!

The integer spin constituents of the system <u>fractionalize</u> at the edge!

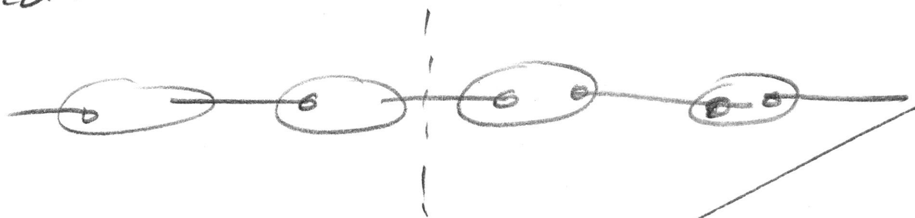
As long as we <u>keep $SU(2)$ symmetry</u>,

we cannot distinguish diff. gnd states locally

→ "symmetry protection" of edge modes.

(cf. SSH model: there fermionic parity as symmetry)

~~Related:~~



~~ent. betw. left & right carried by one $S=1/2$ particle:~~

~~Ent. spectrum looks like half-int. spin, although~~

~~we have int. spin syst. → fractionalization in ES.~~

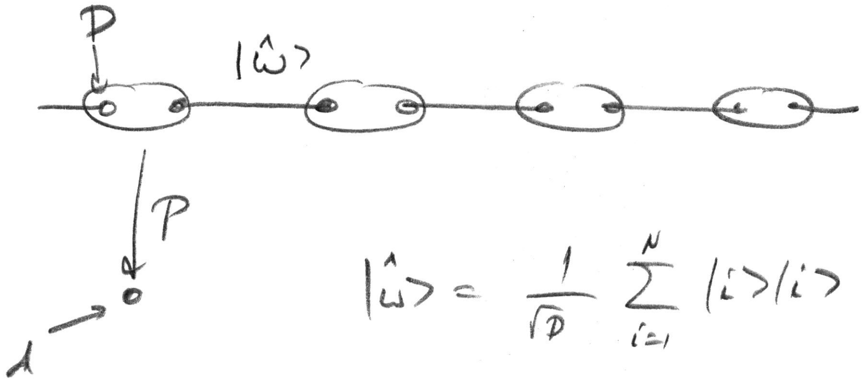
~~(→ more in next lecture!)~~

Note: similar spin-$\frac{1}{2}$ structure in ent. spectrum

→ coming next!

e) Matrix Product States (MPS) & symmetries

General ansatz for triv. 1D states:



$|\psi\rangle$

$$|\hat\omega\rangle = \frac{1}{\sqrt{D}} \sum_{i=1}^{N} |i\rangle|i\rangle$$

$P : \mathbb{C}^D \otimes \mathbb{C}^D \to \mathbb{C}^d$ arbitrary

[Note: Before: $|\omega\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) = \left(Y \otimes \mathbb{1}\right)|\hat\omega\rangle$, $Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\to (P \otimes \dots \otimes P)(|\omega\rangle \otimes \dots) =$

$(P \otimes \dots \otimes P)(Y \otimes \mathbb{1} \otimes Y \otimes \mathbb{1} \otimes \dots)(|\hat\omega\rangle \otimes \dots)$

$= \underbrace{\left(P(\mathbb{1} \otimes Y)\right)}_{\text{new } P\,!} \otimes \left(P(\mathbb{1} \otimes Y)\right) \otimes \dots \quad (|\hat\omega\rangle \otimes \dots)$]

"Matrix Product States" (MPS; why? → later!)

- MPS are g.states of local parent Hams (cf. construction last lecture!)
- For almost all P, the g.s. is unique & H has a gap.

Conversely:

$$H = \sum h_i \text{ has gap } \xrightarrow{\text{Hastings}} \text{ g.s. has area law}$$

Hastings, Vatisani et al. $\searrow$ $\qquad\qquad$ $\swarrow$ Verstraete & Cirac

g.s. well approx. by MPS

$\Rightarrow$ MPS right framework to study 1D gapped phases.

H has symmetry: $[H, u_g^{\otimes N}] = 0$

$\Rightarrow$ g. state $|\psi\rangle$ has symmetry: $|\psi\rangle = u_g^{\otimes N}|\psi\rangle$

<u>Fundamental theorem of MPS:</u>

$$|\psi\rangle = u_g^{\otimes N}|\psi\rangle \Leftrightarrow \exists \overset{\text{unique!}}{V_g} : u_g P = P(\overline{V_g} \otimes V_g)$$

(Note: "$\Leftarrow$" trivial: $(\overline{V_g} \otimes V_g)|\hat{\omega}\rangle = |\hat{\omega}\rangle$

$$\frac{1}{\sqrt{D}} \sum (\overline{V_g})_{ij} |i\rangle\langle j| \otimes (V_g)_{i'j'} |i'\rangle\langle j'| \, |k\rangle|k\rangle \qquad \delta_{jk}\delta_{j'k}$$

$$= \frac{1}{\sqrt{D}} \sum \underbrace{(\overline{V_g})_{ik} (V_g)_{i'k}}_{=(V_g V_g^\dagger)_{i'i} = \delta_{i'i}} |i\rangle|i'\rangle = \frac{1}{\sqrt{D}} \sum |i\rangle|i\rangle \;)$$

1) Classification of phases under symmetries (SPT phases)

$$[H, U_g^{\otimes N}] = 0 \Rightarrow |\psi\rangle = U_g^{\otimes N}|\psi\rangle \Rightarrow \exists V_g : U_g P = P\, V_g \otimes \overline{V_g}$$

P establishes an isomorphism

$$U_g \cong V_g \otimes \overline{V_g}$$

e.g. SO(3)

U_g is a representation of $g \in G$: $U_g U_h = U_{gh}$

$$V_g V_h \otimes \overline{V}_g \overline{V}_h = V_{gh} \otimes \overline{V_{gh}} \quad :$$

Is V_g also a representation?

All we can say is $\quad V_g V_h = e^{i\omega(g,h)} V_{gh}$!

(In fact, the phase of V_g is not even well-defined)

Are there non-trivial examples?

SO(3): half-integer spin rotations; e.g. $S=1/2$:

$$R_{\vec{\phi}} = \exp[i \cdot \vec{\phi} \cdot \vec{S}]$$

e.g. π-rotation about z:

$$R_{z,\pi} = \exp\left[i\pi \begin{pmatrix} 1/2 & \\ & -1/2 \end{pmatrix} \right] = \begin{pmatrix} i & \\ & -i \end{pmatrix} = i \cdot \sigma_z$$

$$R_{z,0} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

$$R_{z,\pi} \cdot R_{z,\pi} = i^2 \sigma_z^2 = -\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \overset{\omega(\pi,\pi)}{e^{i\pi}} R_{z,0}$$

But... the <u>Phase</u> of V_g is arbitrary – we could define

$R_{z,\pi} := \sigma_z \Rightarrow$ problem goes away.

$V_g \sim e^{i\phi_g} V_g$: <u>equivalence relation</u> on $\omega(g,h)$!

Look for quantities <u>invariant</u> under equiv. relation:

$$R_{z,\pi} R_{x,\pi} R^{\dagger}_{z,\pi} R^{\dagger}_{x,\pi} = \sigma_z \sigma_x \sigma_z \sigma_x = \underline{\underline{-\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}}}$$

phases cancel

phases cancel

<u>But</u>: As SO(3)-rotations, this is the identity:

$\boxed{\text{show}}$

$\Rightarrow$ <u>non-trivial phase</u> (-1)!

half-int. spin reps. are <u>non-trivial proj.</u>
<u>representations</u> of SO(3)!

<u>Claim:</u> Under SO(3), int. vs. half-int. spin define diff. phases!

<u>Proof:</u> V_g only int. or only half-int.

$$\Rightarrow U_g P = P\,\underbrace{(V_g \otimes \bar{V}_g)}_{\text{integer spin}}$$ has generally solutions P.

$S=1$ rotation (or other int.)

The solutions can be continuously changed.

both int. or ½ int. — or gen. ~~…~~ same w

For $U_g P^1 = P^1(V_g^1 \otimes \bar{V}_g^1)$, $U_g P^2 = P^2(V_g^2 \otimes \bar{V}_g^2)$:

$V_g^1 \oplus V_g^2 =: V_g$ is <u>still</u> proj. rep.:

Can construct smooth deformation

$$U_g P(\theta) = P(\theta)\left((V_g^1 \oplus V_g^2) \otimes (\overline{V_g^1 \oplus V_g^2})\right)$$

which <u>connects</u> P^1 & P^2: <u>same phase!</u>

On the other hand: V_g^1 int. spin, V_g^2 ½-int.

– or gen.: inequiv. ω –

$V_g^1 \oplus V_g^2$ is <u>not even proj. rep.</u>:

$$(V_g^1 \oplus V_g^2)(V_h^1 \oplus V_h^2) = e^{i\omega_1(g,h)} V_{gh}^1 \oplus e^{i\omega_2(g,h)} V_{gh}^2$$

↯ ⇒ no symmetric interpolation!

Classes of proj. reps – e.g. int. vs half-int. spin for SO(3) – classify the SPT phases!

<u>Note:</u>

- If we drop sym., all states are in same phase
- $V_g = 1$ (D=1) trivial phase

  ⇒ ½-int. V_g is <u>non-trivial phase</u>

  w/out local order parameter & no sym. breaking
- Entanglement $|\omega\rangle$ transforms as V_g:

  $V_g \otimes \bar{V}_g |\omega\rangle = |\omega\rangle$

⇒ int. vs. half-int. multiplet structure in ent. spectrum