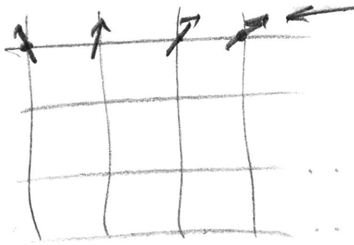
Topological order in quantum spin systems

1. Quantum spin systems



spin, e.g. spin $\frac{1}{2}$: $\mathbb{C}^2$ (or $\mathbb{C}^d$)

basis $|0\rangle = |\uparrow\rangle$, $|1\rangle = |\downarrow\rangle$

state of one spin : $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$

basis of N spins: $\underbrace{|i_1 i_2 \dots i_N\rangle}_{2^N \text{ basis states}}$, $i_1 = 0,1;\ i_2 = 0,1, \dots$

state of N spins: $|\psi\rangle = \sum c_{i_1 \dots i_N} |i_1 \dots i_N\rangle$

Physics described by local interactions

$(\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$H = \sum_{\langle ij \rangle} h_{ij}$$

local terms, e.g.

Define

$h_{ij} = \sigma_z^i \otimes \sigma_z^j$

$h_{ij} = \vec{\sigma}^i \cdot \vec{\sigma}^j = \sigma_x^i \cdot \sigma_x^j + \dots$

Our goal (for now): understand phys. at low temp.

$\Rightarrow$ ground state $H|\psi_0\rangle = E_0|\psi_0\rangle$.

Why Q. spin systems [orally]:

* tight binding

(* Hubbard)

* discretize cont. theories

2. Mean-field theory

What diff. states/phases can such a q. spin syste take?

Example: TFIM

$$H = -\sum_{\langle ij\rangle} \sigma_z^i \sigma_z^j + h \sum_i \sigma_x^i$$

par. in z: $\uparrow\uparrow, \downarrow\downarrow$

align in +x: $\frac{|0\rangle + |1\rangle}{\sqrt{2}} =: |+\rangle$

large h:

$|\psi\rangle \approx |++\ldots+\rangle$

(non-magn.)

small h:

$|\psi\rangle = \begin{cases} |00\ldots0\rangle \\ |11\ldots1\rangle \end{cases}$

(magn.)

Can we class. this more generally?

Landau: Symmetries!

Here: $\uparrow \leftrightarrow \downarrow$ is sym.: $[\sigma_x^{\otimes N}, H] = 0$

How do ground states behave w.r.t. sym.?

$\sigma_x^{\otimes N} |+\ldots+\rangle = |+\ldots+\rangle$

eigenstate

symmetry preserved

$\sigma_x^{\otimes N} |0\ldots0\rangle = |1\ldots1\rangle$

$\sigma_x^{\otimes N} |1\ldots1\rangle = |0\ldots0\rangle$

no eigenstate

symmetry broken

How can we detect this?

$\rightarrow$ Order parameter σ_z^i : $\sigma_z \sigma_x = -\sigma_x \sigma_z$

idea: all spins behave the same

$$O := \frac{1}{N} \sum_i \sigma_z^i \; : \; O \sigma_x^{\otimes N} = -\sigma_x^{\otimes N} O$$

$$|\psi\rangle = \sigma_x^{\otimes N} |\psi\rangle \Rightarrow \langle\psi|O|\psi\rangle = \langle\psi|\underbrace{\sigma_x^{\otimes N} O \sigma_x^{\otimes N}}_{= -O}|\psi\rangle = -\langle\psi|O|\psi\rangle$$

$$\Rightarrow \langle\psi|O|\psi\rangle = 0$$

i.e.: $|\psi\rangle$ sym. $\Rightarrow \langle\psi|O|\psi\rangle = 0$

$\langle\psi|O|\psi\rangle \neq 0 \Rightarrow |\psi\rangle$ breaks sym.

$\langle +\dots+|O|+\dots+\rangle = 0$ | $\langle 0\dots0|O|0\dots0\rangle = +1$

$\langle 1\dots1|O|1\dots1\rangle = -1$

Order par.:
- identifies phase
- labels ground states
- distinguishes g.s. from local property
- counts ground states:

  $\#GS = \#$ possible values of σ_z.

What about states like $|\psi\rangle = \alpha|0\dots0\rangle + \beta|1\dots1\rangle$ ("cat-like states")

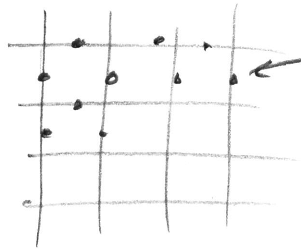
$H' = H + \varepsilon \cdot O \Rightarrow$ split $|0\dots0\rangle$ & $|1\dots1\rangle$

$\rightarrow$ only $|0\dots0\rangle$ & $|1\dots1\rangle$ stable against pert!

Good class. memory, but bad quantum memory.

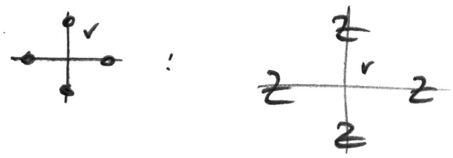
3. The Toric Code model

Shorthand: $X \equiv \sigma_x$; $Z \equiv \sigma_z$.



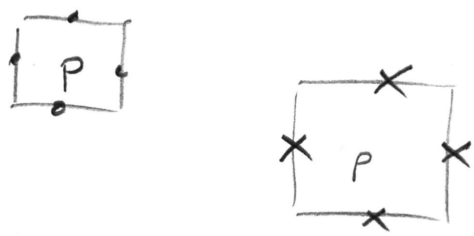
spin $\frac{1}{2}$: $\mathbb{C}^2$ w/ basis $\{|0\rangle, |1\rangle\}$.

For each vertex v,



$$\underbrace{A_v := \bigotimes_{i \in v} Z^i}_{\text{eigenvals } \pm 1}$$

For each plaquette p

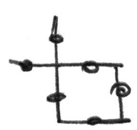


$$\underbrace{B_p := \bigotimes_{i \in p} X^i}_{\text{eigenvals } \pm 1}$$

Define Toric Code Ham

$$H_{TC} = -\sum_{\text{all } v} A_v - \sum_{\text{all } p} B_p$$

Note:
$[A_v, A_{v'}] = 0$
$[B_p, B_{p'}] = 0$
$[A_v, B_p] = 0$



What are the ground states of this model?

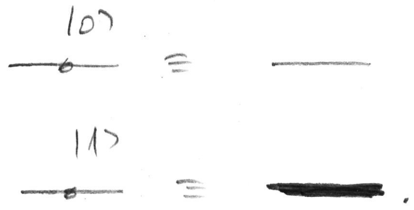
Define: $|\psi\rangle$ min. en. of each local term

Claim: Model is frustration free: The ground states $|\psi\rangle$ satisfy $A_v|\psi\rangle = +|\psi\rangle \; \forall v$; $B_p|\psi\rangle = +|\psi\rangle \; \forall p$.

Obs: Since all A_v & B_p commute, we can analyze G. space term by term – each A_v & B_p gives new constr. on G.S.!

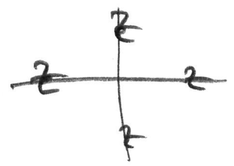
Start w/ the A_v.

string picture:



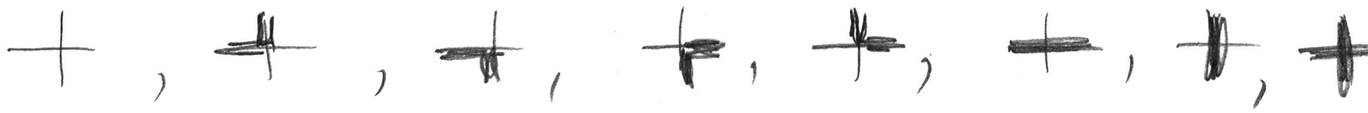
What are +1-eigenstates of A_v?

$$Z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{matrix} \leftarrow \text{no string} \\ \leftarrow \text{string} \end{matrix}$$



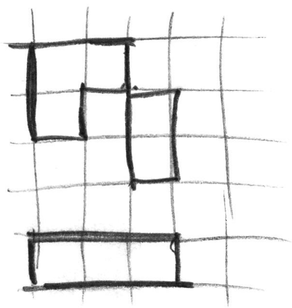
$(-1)^{\#\text{strings}}$

$\underline{\underline{+1}}$: # strings $\underline{\underline{\text{even}}}$



all closed loops!

Joint +1 eigenstate of all A_v: all closed loop patterns!



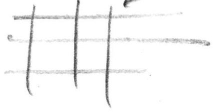
etc.

Restrict analysis to joint +1 space of A_v = space of all loop configs, & now add B_p's.

end of 1st lecture

Lecture II :

Wrap-up Lecture I :

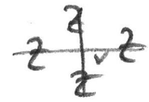
- Q. spin systems:  $|0\rangle, |1\rangle$ $\qquad H = \sum h_i$

- Landau theory:

  sym. breaking $\Rightarrow$ local order parameter
  - identify phase
  - count g. states
  - label g. states

- Toric Code model:

$|0\rangle, |1\rangle$ 

$A_v =$ 

$B_p =$ 

all commuting!

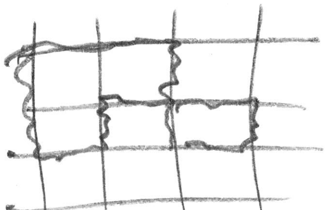
$$H = -\sum_v A_v - \sum_p B_p$$

string picture: $|0\rangle$ = 

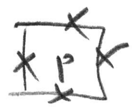
$|1\rangle$ = 

$A_v \equiv +1$: strings form closed loops

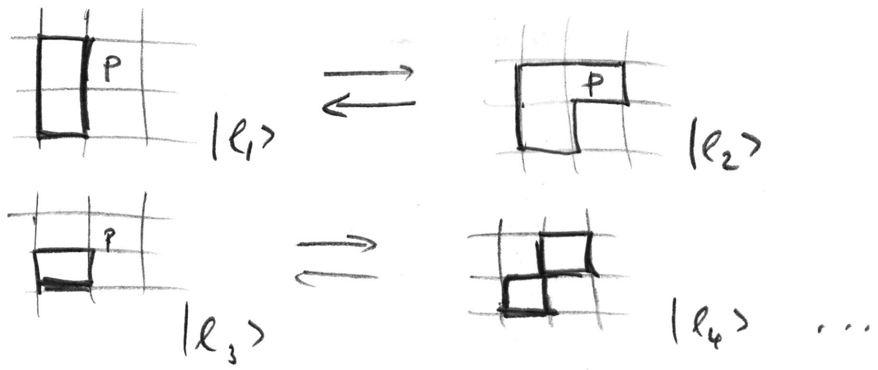
All $A_v = +1$:
all loop patterns



... now add B_p's.

What is effect of B_p? B_p =  : <u>flip</u> all spins!

B_p:



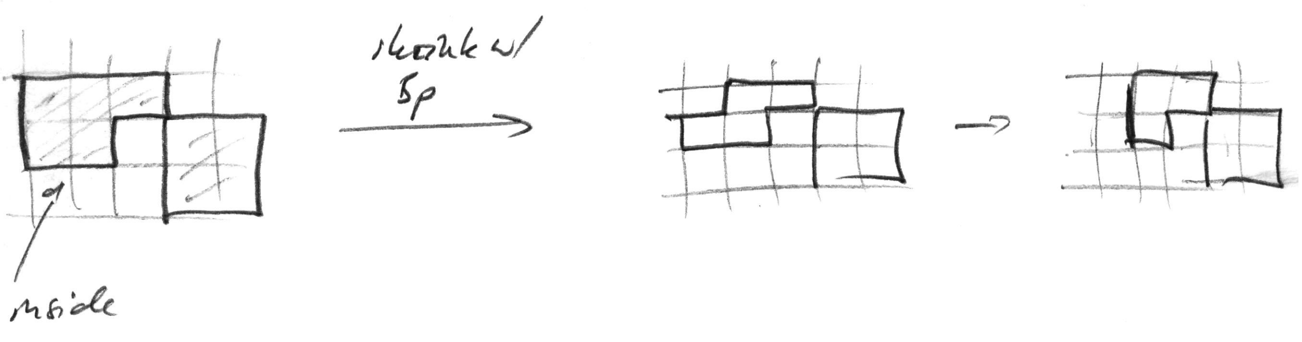
+1 - eigenstate of B_p: all loop patterns coupled by B_p must appear w/ <u>equal weight</u>:

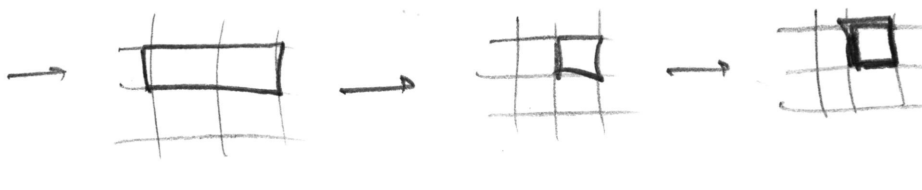
$$|\ell_1\rangle + |\ell_2\rangle, \quad |\ell_3\rangle + |\ell_4\rangle, \ldots$$

Joint +1-eigenstate of <u>all</u> B_p:

All loop patterns related by <u>sequence</u> of B_ps must have same weight.

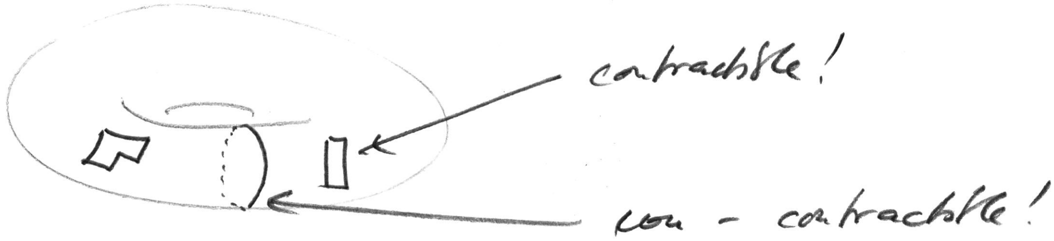
<u>What patterns are coupled?</u>

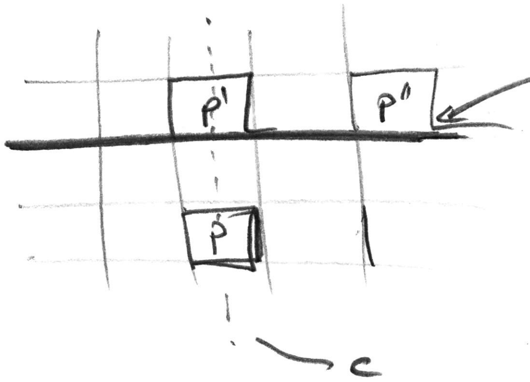




All loops can be removed by contracting them to a point

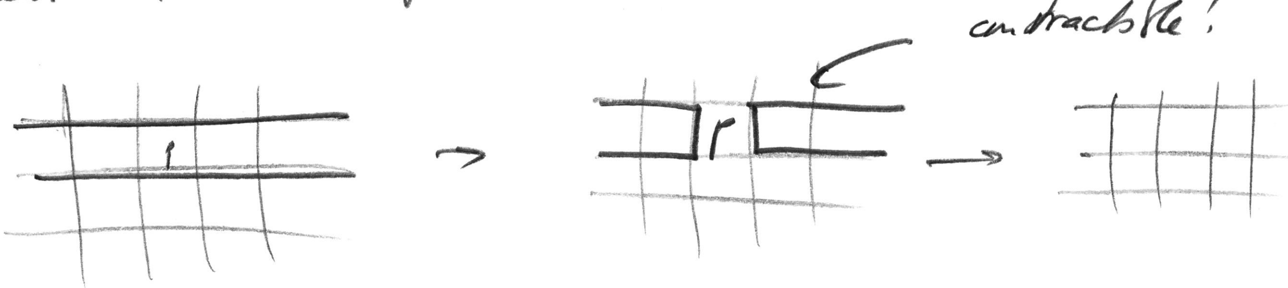
- if they can be contr. to a point!



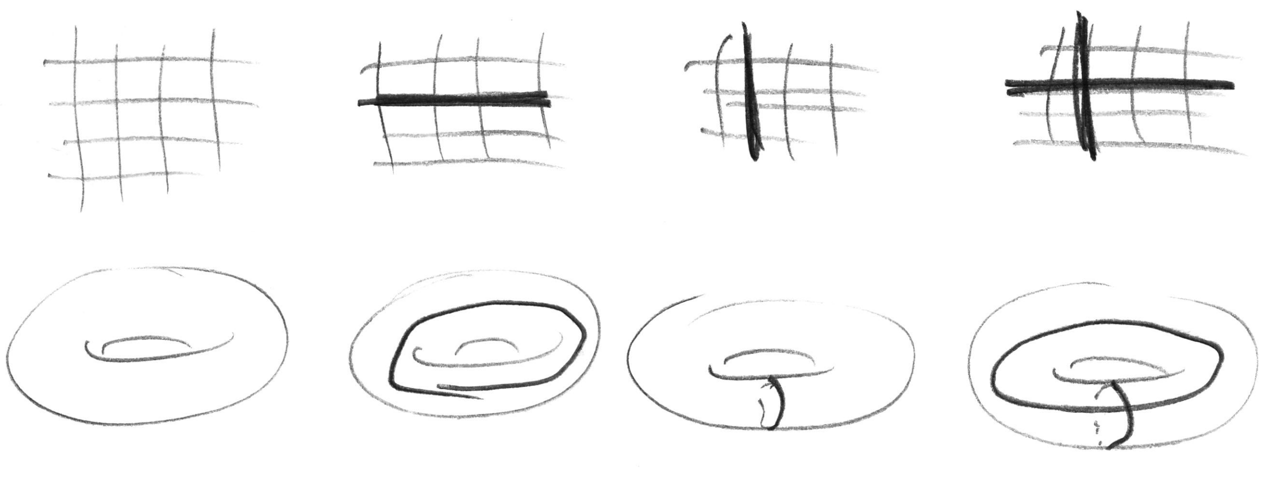


cannot be removed by B_p:
any B_p can chg. # loops
crossing cut c only by 0 or ±2!

What about 2 loops?



Any loop config can be mapped to one of:

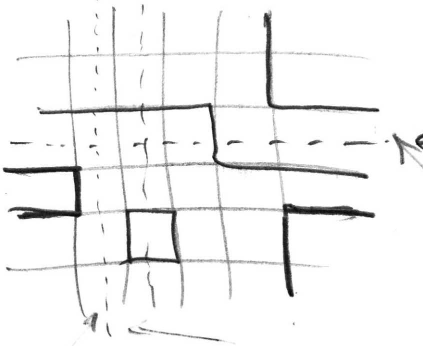


There are 4 ground states:

$$|\psi_{h,v}\rangle = \sum |\text{loops}\rangle$$

$h,v=0,1$

$\sum$ over all loop patterns with $\left\{ \begin{matrix} 0: \text{even} \\ 1: \text{odd} \end{matrix} \right\}$ loops in horizontal (h) / vertical (v) direction around the torus.



$v = \#$crossings mod 2

indep. of position!

$h = \#$crossings mod 2

Is it clear that these are indep. G.S.?

Yes: Define $\hat{Z}_h = \bigotimes_i Z^i$, and $\hat{Z}_v$ analog.

$i \leftarrow$ along vert cut.

$$[\hat{Z}_h, H] = 0, \quad [\hat{Z}_v, H] = 0, \quad [\hat{Z}_h, \hat{Z}_v] = 0$$

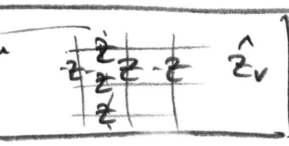
$\Longrightarrow$ eigenvals ± 1 of $\hat{Z}_h$ / $\hat{Z}_v$ equal $|\psi_{h,v}\rangle$,

and label ground states of H!

4. The ground space of the Toric Code

$|\psi_{h,v}\rangle$ span ground space $\Rightarrow$ gen. G.S. $|\psi\rangle = \sum_{h,v} \alpha_{h,v} |\psi_{h,v}\rangle$

- $\hat{Z}_h = \bigotimes_{i \uparrow \text{vert.}} Z^i$ & $\hat{Z}_v = \bigotimes_{i \uparrow \text{horiz.}} Z^i$ allow to label G.S.

Pic: 

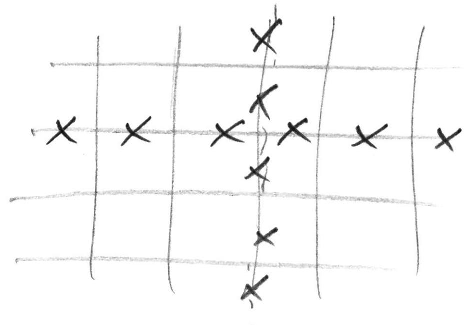
Note: All $\hat{Z}_h$ are related by multiplication by B_p explain!

$\Rightarrow \hat{Z}_h' |\psi\rangle = \hat{Z}_h \underbrace{B_p |\psi\rangle}_{+1\text{-eigenstate}} = \hat{Z}_h |\psi\rangle$
($|\psi\rangle$: any G.S.)

$\Rightarrow$ any $\hat{Z}_h$ does the same job!

- How to map betw. G.S.?
- Define $\hat{X}_h = \bigotimes_{i \text{ hor.}} X^i$

$\hat{X}_v = \bigotimes_{i \text{ ver.}} X^i$



$\hat{X}_h / \hat{X}_v$ add a loop in hor./vert. direction!

$$\begin{tikzcd}
|\psi_{0,0}\rangle \arrow[r, shift left, "\hat{X}_h"] \arrow[d, shift right, "\hat{X}_v"'] & |\psi_{1,0}\rangle \arrow[l, shift left] \arrow[d, shift right] \\
|\psi_{0,1}\rangle \arrow[r, shift left, "\hat{X}_h"] \arrow[u, shift right] & |\psi_{1,1}\rangle \arrow[l, shift left] \arrow[u, shift right, "\hat{X}_v"']
\end{tikzcd}$$

Again: Diff. $\hat{X}_h, \hat{X}_v$ related by mult. w/ B_p. (10)

$$\hat{Z}_h |\psi_{h,v}\rangle = (-1)^h |\psi_{h,v}\rangle$$

$$\hat{X}_h |\psi_{h,v}\rangle = |\psi_{h\oplus 1,v}\rangle$$

$\times \ell^2$

act like Pauli matrices on h.

.. and the same for $\hat{X}_v, \hat{Z}_v$ & v.

$|\psi_{h,v}\rangle \equiv |h,v\rangle \leftarrow$ state of 2 two-level systems (~~qudits~~) (qubits)...

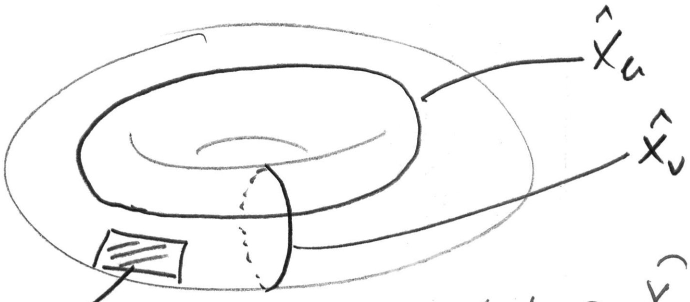
.. with Paulis $\hat{X}_h, \hat{Z}_h$ & $\hat{X}_v, \hat{Z}_v$.

(Note: $\underbrace{\hat{X}_h \hat{Z}_h = -\hat{Z}_h \hat{X}_h, \quad \hat{X}_v \hat{Z}_v = -\hat{Z}_v \hat{X}_v, \quad \hat{X}_h \hat{Z}_v = \ldots = 0}_{\text{Pauli algebra of 2 qubits.}}$)

Degenerate ground space = <u>order</u>.

$\rightarrow$ Is there some <u>order parameter</u>?

$$|\psi_{l,v}\rangle = (\hat{X}_h)^l (\hat{X}_v)^v |\psi_{0,0}\rangle$$



draw O first $\Rightarrow$ $\hat{X}_h, \hat{X}_v$ placed accordingly!

$$\langle\psi_{l,v}|O|\psi_{l,v}\rangle = \langle\psi_{0,0}|(\hat{X}_h)^l (\hat{X}_v)^v O (\hat{X}_h)^l (\hat{X}_v)^v |\psi_{0,0}\rangle$$

comm.

$$= \langle\psi_{0,0}|O|\psi_{0,0}\rangle$$

$\Rightarrow$ any local O gives same value in each $|\psi_{l,v}\rangle$

$\Rightarrow$ $|\psi_{l,v}\rangle$ locally indist.!

But... is this enough? cf. $|0\ldots0\rangle \pm |1\ldots1\rangle$?

$\Rightarrow$ Need this for any ground state!

Any G.S. can be reached by evolving w/ $\hat{X}_0, \hat{Z}_0$:

e.g. $\exp[i\phi \hat{X}_h] = \cos\phi\, \mathbb{1} + i\sin\phi\, \hat{X}_h$

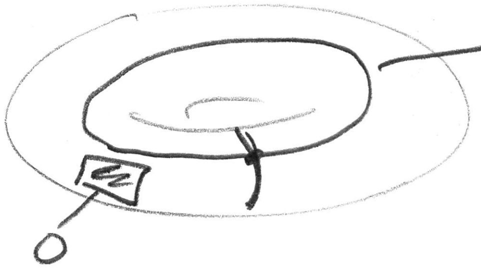
$(\hat{X}_h)^2 = 1$

$$\Rightarrow \exp[i\phi \hat{X}_h]|\psi_{00}\rangle = \cos\phi\, |\psi_{00}\rangle + i\sin\phi\, |\psi_{10}\rangle$$

By combining $\hat{X}_0$ & $\hat{Z}_0$, taking products, sums, etc., & evolving w/ $\exp[i\ldots]$, any rotation can be reached!

$$\Rightarrow \underbrace{\exp[i\phi\, f(\hat{X}_u, \hat{Z}_u, \hat{X}_v, \hat{Z}_v)]}_{\text{unitary}} |\psi_{0,0}\rangle = \text{any G.S. } |\psi\rangle$$

unitary acts here!



comm.

$$\langle\psi|O|\psi\rangle = \langle\psi_{00}|\mathcal{U}^{\dagger}\, O\, \mathcal{U}|\psi_{00}\rangle = \langle\psi_{00}|O|\psi_{00}\rangle$$

$\Rightarrow$ no local order parameter exists

$\Rightarrow$ ground states all locally indistinguishable

($\to$ use as q. memory: stable!)

$\Rightarrow$ any G.S. can be transformed into any other by acting along 2 loops

(cf. sym.-br.: 0D vs. 2D)

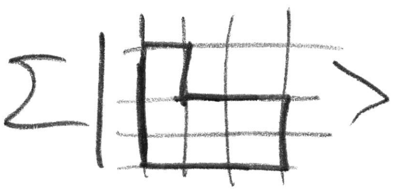
[$\Rightarrow$ G.S. can be distinguished by loop operators

$\triangleright$ end lecture 2 $\to$ for L3, put summary of above

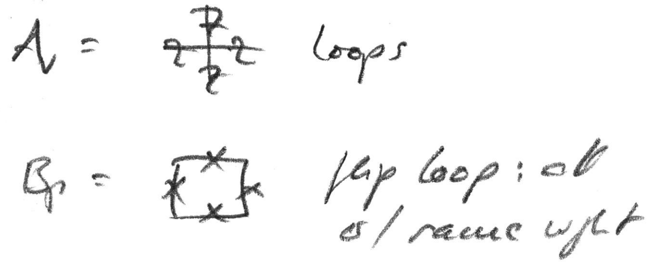
Lecture III

Review Lecture I + II:

TC model:

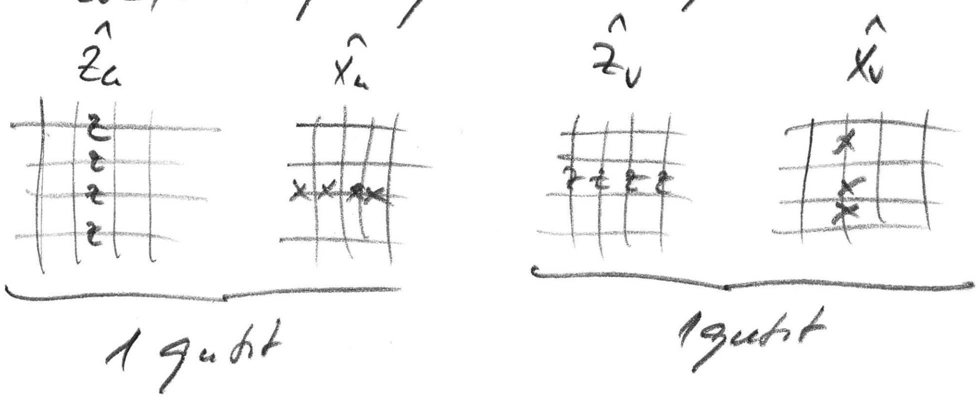


$$H = -\sum A_v - \sum B_p$$



4 GS

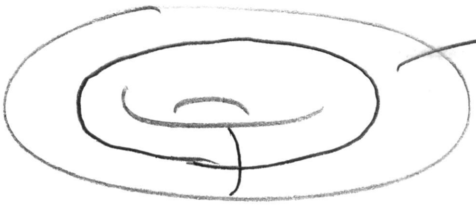
even/odd parity around cyl. H/V



- Any trafo by acting along 2 loops:
  → no local order par
- GS identified & transformed by <u>loop operators</u>

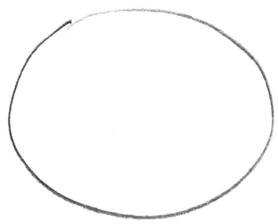
  (cf. Ising model: 0+2 vs. 1+1)
- ground states locally indist.
  → stable against pert. ⇒ Q. Memory
  (Intuition: env. <u>learns</u> state!)

Ground space degeneracy $\Longleftrightarrow$ non-contractible loops around torus



2 loops $\Rightarrow$ 2^2 G.S.

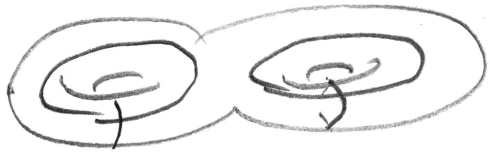
Modification of TC on sphere:



no non-contr. loops

$\Rightarrow$ one G.S.

... or on genus 2:



$\rightarrow$ 4 loops

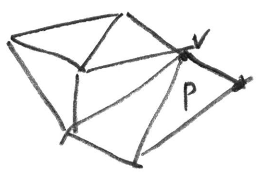
$\Rightarrow$ $2^4 = 4^2$ G.States!

$\Rightarrow$ G.S. degen dep. on topology!

$\Rightarrow$ incomp. w/ local order parameter!

$\Rightarrow$ global property of system crucial!

[Rem: How to modify?



$A_v = \prod_v Z$

$B_p = \prod_p X$

] $\leftarrow$ Cecile!

5) Anyonic excitations

Excitations = eigenstates above G.S.

Elementary excitations:

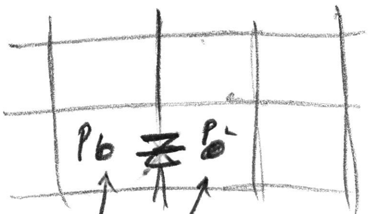
↙ lowest lying eigenstates ⟷ localized objects ↘

typ. ≠, but here: comm. Ham: equal

How to find/construct? — Start from G.S.

$H = \sum \otimes$ Pauli's : (anti)comm. w/ Pauli's

⟶ try to apply Paulis to G.S.



B_{p_1}, B_{p_2} anti-com. w X:

$$B_{p_1}\left(Z|\psi\rangle\right) = -Z B_{p_1}|\psi\rangle = -\left(Z|\psi\rangle\right)$$

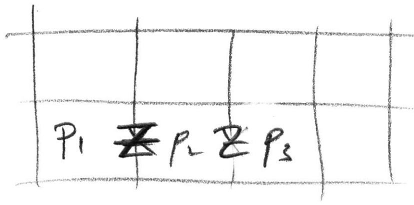
& same for B_{p_2}.

⟶ two terms in H violated;

energy $2 \times (+2)$.

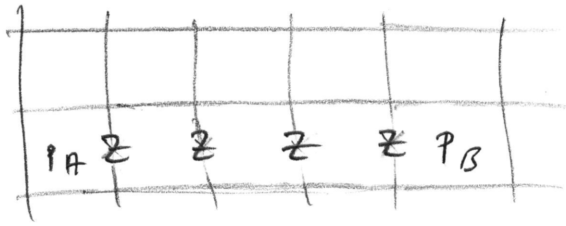
Is this one excitation? Can we maybe "excite" only <u>one</u> B_p?

Try to "fix" B_{p_2}: apply another Z ($Z \otimes Z$ com.!)



<u>Now:</u> -1 on B_{p_1} & B_{p_3}

Continue...



(-1) on B_{p_A} & B_{p_B}.

$\Longrightarrow$ elementary excitations on plaquettes

$\Longrightarrow$ seem to come in <u>pairs</u>

Do they have to come in pairs?

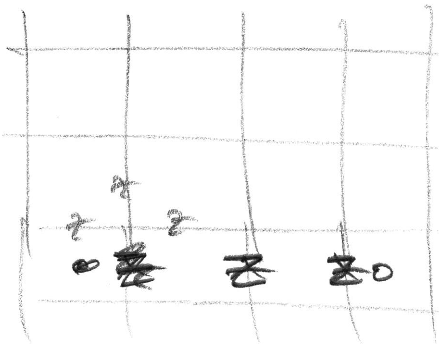
$$\prod_p B_p = 1$$

$$\Rightarrow \underbrace{\Big(\prod_p B_p\Big)}_{-1 \text{ for each plaq. excitation}} |\psi\rangle = |\psi\rangle$$

$\longrightarrow$ even # of plaq. excitations!

Is location of x string relevant?

("Hidden" property of excitation)



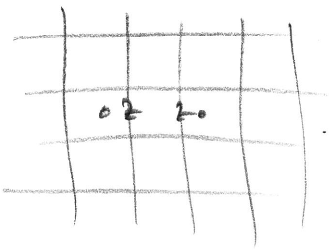
mult. w/ A_v

$\Rightarrow$ diff. string, same state:

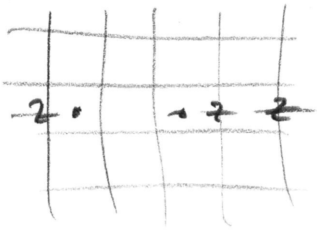
$$\underbrace{A_v (z \dots z)}|\psi\rangle = (z \dots z)|\psi\rangle$$

No ... at least if cont. deformable!

But:



$\longleftrightarrow$



$|\phi_1\rangle$ $\qquad$ $|\phi_2\rangle$

$$z_2 z_3 |\psi\rangle \longrightarrow z_1 z_4 z_5 |\psi\rangle$$

$$\parallel$$

$$z_2 \cdot z_3 \underbrace{(z_1 z_2 z_3 z_4 z_5)}_{\hat{Z}_v} |\psi\rangle$$

$\Rightarrow$ Dep. on $|\psi\rangle$, $|\phi_1\rangle$ & $|\phi_2\rangle$ can differ by phase, or some global property (other $|\psi\rangle$)

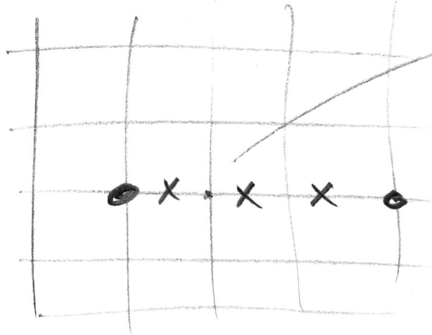
$$\uparrow$$

$$\left[|\psi_{00}\rangle + |\psi_{01}\rangle\right]$$

$$\downarrow$$

$$\left[|\psi_{00}\rangle - |\psi_{01}\rangle\right]$$

Dual excitations: String of X:

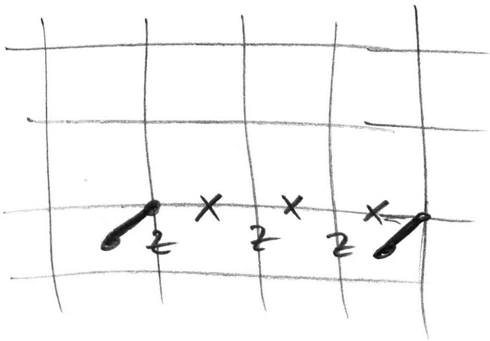


[draw step by step]

... similar properties.

We call them fluxes (Z string) & charges (X string).
(magn. excitat.) (electric excit.)

We can also make a combined excitation:



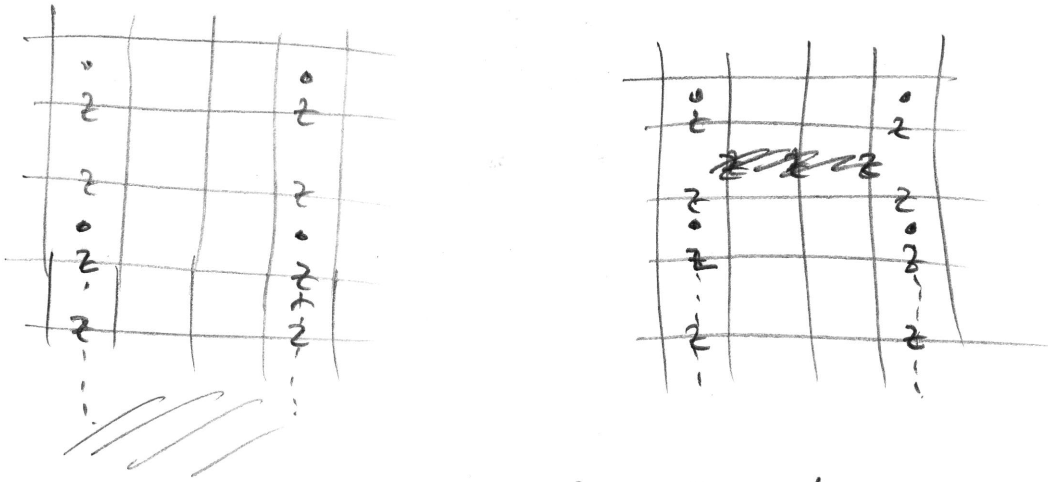
We can create pairs, move them, & annihilate them again by applying X & Z!

Move excit. around torus

$\Rightarrow$ logical operator on Q. space!

What is the statistics of these excit.?

a) Self-statistics:

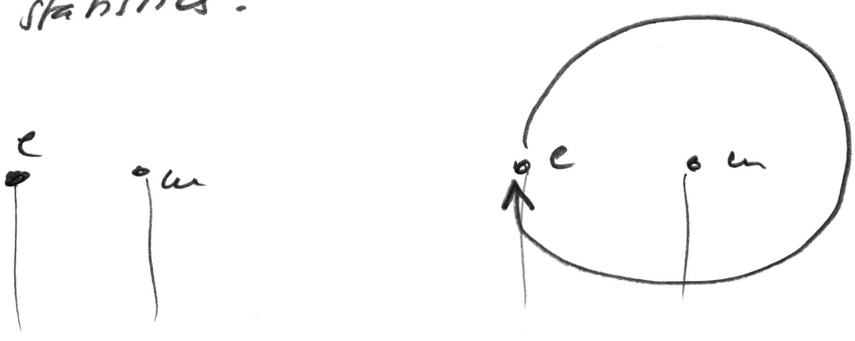


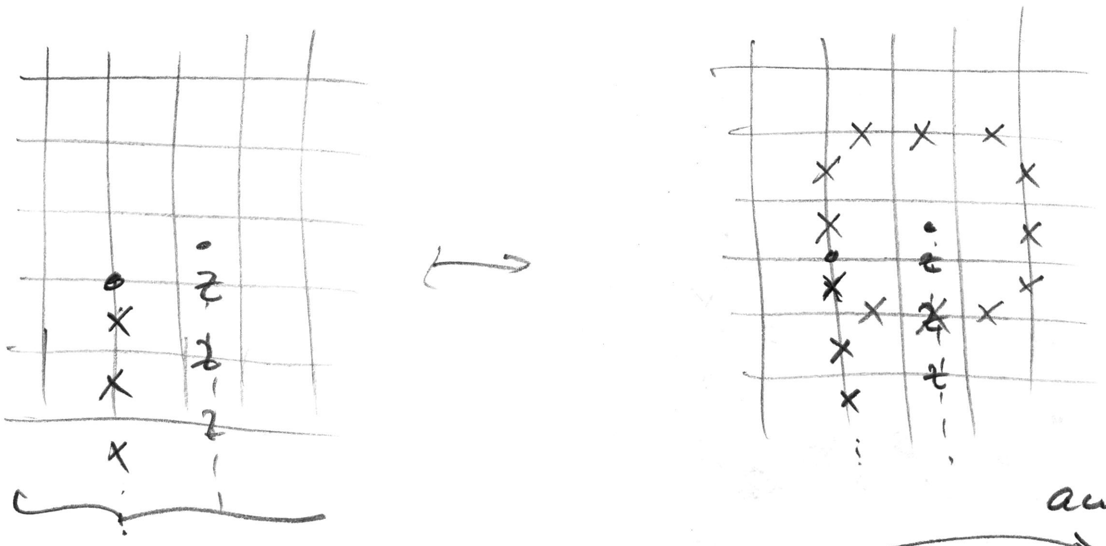
<u>equal</u> : boson!

... same for X!

$\Rightarrow$ fluxes & charges are bosons!

b) Mutual statistics:





$X\dots Z\dots|\psi\rangle$

anti-com.

$X_0 (X\dots)(Z\dots)|\psi\rangle$

$X_0|\psi\rangle = |\psi\rangle$

$= -(X\dots)(Z\dots)X_0|\psi\rangle$

$= -(X\dots)(Z\dots)|\psi\rangle$

$\Rightarrow$ <u>-1 phase</u>

$\Rightarrow$ e & m have a mutual fermionic statistics!

c) <u>The fermion</u>

What about the <u>combined</u> e-m-particle?

$\rightarrow$ (locali:) ferm. self-statistics!

[→ ~~Rückseite~~!]

▷ End lecture III