Lecture three: the magic of dimension $D = 2$

We consider now the complex $\mathcal{C}$ with an Euclidean flat metric. The x_1 and x_2 are the plane coordinates. It is convenient to introduce the complex coordinates

$$z = x_1 + ix_2 \quad \bar{z} = x_1 - ix_2.$$

The conformal transformations, in their infinitesimal form, take the form (see Ex 1 of the previous lecture)

$$z = z + \alpha(z), \quad \bar{z} = \bar{z} + \bar{\alpha}(\bar{z}) \tag{0.1}$$

Since the action of the conformal group factorizes into the action on the holomorphic and anti-holomorphic sector, we can assume the variables z and $\bar{z}$ as independent.

Consider to expand $\alpha(z)$ around the origin:

$$\alpha(z) = \sum_{n\geq 1}^{\infty} \alpha_n z^{n+1} \tag{0.2}$$

where α_n are complex numbers.The $\alpha(z)$ and $\bar{\alpha}$ are functions that are analytic on a finite region D containing the origin and zero elsewhere, see chapter 2 of [1].

Quantum fields transformations

Consider the space of functions $f(z)$ defined on the complex plane $\mathbb{C}$. The transformation $z \to z + \alpha(z)$ correspond to a reparametrization of $f(z)$, $f(z) \to \tilde{f}(z) = f(z + \alpha(z))$. One has

$$\delta f(z) = \tilde{\phi}(z) - \phi(z) = \sum_{n\leq 1} \alpha_n z^{n+1} \partial_z f(z) \tag{0.3}$$

The operators $l_n = z^{n+1}\partial_z$, satisfy the Witt algebra:

$$[l_n, l_m] = (n-m) l_{n+m}, \quad [\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m}, \tag{0.4}$$

where we add the anti-holomorphic transformation $\bar{\alpha}(\bar{z})$, $\bar{l}_n = \bar{z}^{n+1}\partial_{\bar{z}}$. Note that for $n = -1, 0, 1$ we find back the generators of the transformations we have seen in the previous lecture:

$$\begin{aligned} &\text{Translations:} \quad l_{-1}, \bar{l}\_1 \\ &\text{Rotations:} \quad i(l_0 - \bar{l}_0) \\ &\text{Dilatations:} \quad l_0 + \bar{l}_0 \\ &\text{Special Conformal:} \quad l_1, \bar{l}_1 \end{aligned} \tag{0.5}$$

The $f(z)$ are deterministic function. What we are really interested in, is the transformation of *quantum* field $\Phi(z)$. We recall that a quantum field can be thought as a random field $\Phi(z) = \Phi[\phi(z)]$ generated by a given probability $\mathcal{P}[\phi]$. For instance, in a free boson theory, the (primary) fields$\Phi_a(z, \bar{z}) = e^{a\phi(z,\bar{z})}$ where $\psi(z, \bar{z})$ is a scalar field generated from a distribution $\mathcal{P}[\phi] = e^{\int dz d\bar{z}\, \partial_z\phi \partial_{\bar{z}}\phi}$. So, when we give properties concerning the fields $\Phi(z)$ of a quantum field theory, we are always thinking about the properties of the correlation function (expectation values) $\langle\Phi(z)\cdots..\rangle$ containing this field. In mathematical jargon, we are giving its *weak* properties. One can show that, under an infinitesimal transformation (0.2), aroun:

$$\delta_\alpha \Phi(z) = \sum_n \alpha_n L_n \Phi(z) \tag{0.6}$$

where the generators L_n satisfy the celebrated **Virasoro** algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,m}, \tag{0.7}$$

So, with respect to the Witt algebra, there is one additional term (in algebra jargon, it is called the central extension). The constant c (that you can in general consider complex) is called the central charge and it is a crucial parameter. Note that for $n = -1, 0, 1$ the Virasoro algebra is equal to the Witt one, and the parameter c does not play any role. This is related to the fact that these L_{-1}, L_0, L_1 are the generators of the *global* conformal group:

$$z \to \frac{az+b}{cz+d}, \quad (a,b,c,d) \in \mathbb{C}, ab-cd = 1. \tag{0.8}$$

(note that you find the $(2+1)(2+2)/2 = 6$ parameters studied in the previous lecture) It is an old result of complex analysis that the global conformal group is the greatest group that is an automorphism of $\mathbb{C}\cup\infty$, i.e. maps sphere into spheres. The central charge is a conformal symmetry anomaly and originates from the fact that a general conformal map, e.g. a mapping of the plane to a cylinder of radius R, introduces a macroscopic length into the system. The central charge c can be shown to be proportional to the Casimir energy.

In CFT, we call primary fields $\Phi_\Delta(z)$, of dimension Δ, the fields that, under conformal map $z \to g(z)$ transform as

$$\text{Primary fields:} \quad \Phi(z) \to \left(\frac{dg(z)}{dz}\right)^\Delta \Phi\left(g(z)\right), \tag{0.9}$$

that, roughly speaking, generalize the behavior of the scaling fields under a global dilation. The primary operator with scaling dimension Δ satisfies therefore:

$$L_0 \Phi_\Delta = \Delta \Phi_\Delta \tag{0.10}$$

The stress energy tensor an Ward identities

The conformal symmetry implies the existence of two non-vanishing components of the stress-energy, an holomorphic $T(z)$ and anti-holomorphic $\bar{T}(\bar{z})$ field. The Virasoro operators L_n defined above are defined from the Laurent series of the stress-energy tensor $T(z)$

$$T(z)\Phi(0) \equiv \sum_n \frac{1}{z^{n+2}} L_n \Phi(0) \tag{0.11}$$

The conformal Ward identities associated to the Noether current $T(z)$ take the form [1]:

$$\left\langle T(z) \prod_{i=1}^N \Phi(z_i) \right\rangle = \sum_{i=1}^N \left(\frac{\Delta_i}{z-z_i} + \frac{1}{z-z_i}\partial_{z_i} \right) \left\langle \prod_{i=1}^N \Phi(z_i) \right\rangle \tag{0.12}$$

From the asymptotic behavior of $T(z)$, $\lim_{z\to\infty} T(z) \sim 1/z^4$ the 0.12 implies the following three identities:

$$\sum_{j=0}^{N} \partial_{z_j} \left\langle \prod_{i=1}^{N} \Phi_{\alpha_i}(z_i) \right\rangle = 0,$$
$$\sum_{j=0}^{N} (\Delta_j + z_j \partial_{z_j}) \left\langle \prod_{i=1}^{N} \Phi_{\alpha_i}(z_i) \right\rangle = 0,$$
$$\sum_{j=0}^{N} \Big(2z_j\Delta_j + 2z_j\partial_{z_j} \Big) \left\langle \prod_{i=1}^{N} \Phi_{\alpha_i}(z_i) \right\rangle = 0 \tag{0.13}$$

These identities express the invariance of the CFT correlation function under the global conformal transformation. The stress energy tensor is not a primary operator and transforms as:

$$T(z) = \left(\frac{dg(z)}{z}\right)^2 T\left(g(z)\right) + c\{g, z\}, \quad \{g, z\} = \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2. \tag{0.14}$$

Note that the Scwhartzian derivative $\{g, z\}$ vanishes for global conformal transformation, consistently with what we have said about the intepretation of the central charge.

CFT Hilbert space: representation Virasoro algebra

The Hilbert space $\mathcal{H}$ construction in quantum field theory starts from the foliation of the space-time (in these lectures we are always assuming Euclidean metric, but in the following we call one space direction as time). Each leaf of the foliation is associated to the Hilbert space. On one of this leaf, one typically define an reference $|\text{in}\rangle >$ state, the QFT vacuum, that is then propagated on the others leafs by the generators of the symmetry. In a (massive) QFT, with Poincare' symmetry, the foliation is done by hyperplanes at constant t. The $|\text{in}\rangle$ state is associate to the $t = -\infty$ hyperplane that is successively propagated by the time translation generators, the Hamiltonian. Note that the space of linear operators acting on $|\text{in}\rangle$ is not isomorphic to the space $\mathcal{H}$, i.e. it has not the same dimension. To understand this, let discretize each space direction in M sites. A state $|A\rangle \in \mathcal{H}$, corresponding to a certain slice of time, it is a vector with M components and the space of linear operator acting on it are M^2 matrices, as one can define a local operator acting on each point M. In a CFT theory, the presence among the symmetry generators of the dilation operators, make convenient to construct the Hilbert space by radial quantization, that means that the foliation of the space is done by concentric hyperspheres. The state $|\text{in}\rangle$ defined on each hypersphere is then evoluted by using the dilation generators, that plays the role of the Hamiltonian. So, one can see that the state $\text{in}\rangle$ is associated to a point, the origin. This is at the basis of the operator-state correspondence in CFT. Assume $|0\rangle$ is the vacuum of the theory, then

$$|\Delta\rangle = \lim_{z\to 0} \Phi_\Delta(z)|0\rangle \tag{0.15}$$

Brief review of angular momentum theory

The theory of angular momentum in quantum mechanics is the study of the representation of the Lie algebra sl_2, that is the Lie algebra of the group of rotations $SU(2)$. We briefly review here the algebraic construction of its representations. This is a (relatively) simple example where we can apply an approach which is analogous to the one we will apply for Virasoro algebra later.

The Lie generators J^+, J^- and J^z, are the infinitesimal generators of the $SU(2)$ rotations and they satisfy the following Lie algebra:

$$\left[J^+, J^-\right] = \mp 2J^z, \quad \left[J^+, J^z\right] = J^+, \quad \left[J^-, J^z\right] = J^- \tag{0.16}$$

The J^2 operator, defined by:

$$J^2 = \frac{1}{2}\left(J^+J^- + J^-J^+\right) + (J^z)^2 \tag{0.17}$$

commutes with the algebra. We can then define a vectorial space $j, m\rangle$ of eigenvectors of J^2, J^z indexed by the two eigenvalues $(j(j+1), m)$,

$$J^2|j,m\rangle = j(j+1)|j,m\rangle, \quad J^z|j,m\rangle = m|j,m\rangle \tag{0.18}$$

Note that, at this stage, the fact that we we $j(j+1)$ as eigenvalue of J^2 is purely a convenient parametrization. In the eigenspace with eigenvalue j, we have from the commuation relation:

$$J^+|j,m\rangle = c^+(m)|j,m+1\rangle, \quad J^-|j,m\rangle = c^-(m)|j,m-1\rangle \tag{0.19}$$

The fact that

$$(J^\pm)^\dagger \to J^\mp, \tag{0.20}$$

implies that that the $J^+J^-|$ and J^-J^+ matrix elements:

$$\langle j,m|J^+J^-|j,m\rangle = ||J^-|j,m\rangle||, \quad \langle j,m|J^-J^+|j,m\rangle = ||J^+|j,m\rangle||, \tag{0.21}$$

are norms. Imposing the inner product to be positive definite, one obtains the condition:

$$\langle j,m|J^+J^- + J^-J^+|j,m\rangle = 2\langle j,m|J^2 - (J^z)^2\,|j,m\rangle = j(j+1) - m^2 \geq 0 \tag{0.22}$$

So the above condition fixes upper (m_{max}) and lower (m_{min}) for the possible values of m. This in turn implies that $J^+|j,m_{max}\rangle = 0$ and $J^-|j,m_{min}\rangle = 0$. It is straightforward by using $J^-J^+ = J^2 - (J^z)^2 + J^z$ and $J^+J^- = J^2 - (J^z)^2 - J^z$ to derive $m_{max} = -m_{min} = j$ and $j = n/2$ with n non-integer.

Verma modules

We can back now to the Virasoro algebra. We apply the same approach as seen above. Noting as $|\Delta\rangle$ the highest weight state with conformal dimension Δ, $L_0|\Delta\rangle = \Delta\,|\Delta\rangle$, the descendant states $L_{-n_1}\cdots L_{-n_N}|\Delta\rangle$, with $n_1 \geq n_2 \geq \cdots \geq n_N$, form a basis of the Verma module, $\mathcal{V}_\Delta$ with dimension Δ. We indicate a general element of this basis with $L_{-Y}\,\Phi_\Delta(0)$, where $Y = \Big(n_1, \cdots, n_N\Big)$ is a Young diagram that has N non-zero parts. One has,

$$L_0\,|L_{-Y}\Delta\rangle = \Big(\Delta + |Y|\Big)\,|L_{-Y}\Delta\rangle, \tag{0.23}$$

where $|Y| = \sum_{i=1}^N n_i$ is the number of cells in the Young diagram Y.

Inner product

We can define an inner product in the Hilbert space by impose that :

$$\lim_{z\to\infty} z^{2\Delta_1}\langle\Phi_{\Delta_1}(z)\Phi_{\Delta_2}(0)\rangle = \langle\Delta_1|\Delta_2\rangle = \delta_{\Delta_1,\Delta_2} \tag{0.24}$$

where $\langle\Delta_1$ is the (dual) outer state at the $z=\infty$. You see from the above relation that states at different Δ are orthogonal. Consider the field:

$$L_n\Phi_{\Delta_1}(0) = \frac{1}{2\pi i}\oint_0 d\, z z^{n+1}T(z)\Phi_\Delta(0). \tag{0.25}$$

Using the transformations (0.9) and (0.14), and using $\Phi_\Delta(\infty) = \lim_{z\to\infty}\ z^{2\Delta}\Phi_\Delta(z)$, under the inverse mapping $\omega = g(z) = 1/z$ one obtains:

$$L_n\Phi_{\Delta_1}(\infty) = \frac{1}{2\pi i}\oint_\infty d\, \omega\omega^{-n+1}T(\omega)\Phi_\Delta(\infty) \tag{0.26}$$

That means that:

$$\lim_{z\to\infty}\langle L_{-n}\Phi_{\Delta_1}(\infty)L_{-m}\Phi_{\Delta_2}(0)\rangle = \langle\Phi_{\Delta_1}(\infty)L_nL_{-m}\Phi_{\Delta_2}(0) = \langle L^\dagger_{-n}\Delta_1|L_{-m}\Delta_2\rangle \tag{0.27}$$

So the inner product in the CFt Hilbert space is defined by:

$$L^\dagger_{-n} = L_n \tag{0.28}$$

We have also the following resolution of the identity operator:

$$I = \sum_{\Delta\in\mathcal{S}}\ \sum_{Y,Y'} |L_{-Y}\Delta\rangle\ \mathcal{H}^{-1}_{Y,Y'}\langle L^\dagger_{-Y'}\Delta|, \tag{0.29}$$

The matrix $\mathcal{H}_{L,L'}(\Delta)$ is the Gram matrix whose entries are the scalar products of the descendants , $\mathcal{H}_{Y,Y'}(\Delta) \equiv \langle L_{-Y}\Delta|L'_{-Y'}\Delta\rangle$. These scalar products are computed by using the Virasoro commutation relations. The Gram matrix is block-diagonal, $\mathcal{H}_{Y,Y'}(\Delta) = \text{diag}(1, H^{(1)}_{Y,Y'}(\Delta), H^{(2)}_{Y,Y'}(\Delta), \cdots)$ where the $H^{(l)}_{Y,Y'}$ are obtained by the scalar products of the descendants at level $|Y| = |Y'| = l$,

Restricting to level $|Y| \le 2$, and on the basis $\{|\Delta\rangle, L_{-1}|\Delta\rangle, L_{-2}|\Delta\rangle, L^2_{-1}|\Delta\rangle\}$, the matrix $\mathcal{H}$ reads:

$$\mathcal{H}_{Y,Y'}(\Delta)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2\Delta & 0 & 0 \\ 0 & 0 & \frac{c}{2}+4\Delta & 6\Delta \\ 0 & 0 & 6\Delta & 4\Delta(4\Delta+1) \end{pmatrix} \tag{0.30}$$

Already at this point there are very important consequences of conformal invariance. For instance one can see that, a unitary CFT, whose inner product is positive definite, $\Delta \ge 0$, as:

$$\langle L^\dagger_{-1}\Delta|L_{-1}\Delta\rangle \ge 0 \to \langle L^\dagger_{-1}\Delta|L_{-1}\Delta\rangle = \langle\Delta|L_1L_{-1}\Delta\rangle = 2\Delta \ge 0 \tag{0.31}$$

0.1 Exercises

0.1.1 Ex 1.

Consider $z \in \mathbb{C} \cup \infty$ and the conformal map:

$$\omega = g(z) = \frac{L}{2\pi} \ln(z) \tag{0.32}$$

Describe the space where ω lives and show where the axis $Re[z] = 0$, $Re[1] = 0$, $Im[z] = 0$ are mapped. Using (0.9), compute the two point correlation function on the ω space.

0.1.2 Ex 2.

Using the Virasoro commutation relations, rederive the matrix of the inner product (0.30)

References

[1] V. S. Dotsenko, "Série de cours sur la théorie conforme, https://cel.archives-ouvertes.fr/cel-00092929." 2006.

[2] S. Kanno, Y. Matsuo and S. Shiba, *Analysis of correlation functions in Toda theory and AGT-$\mathcal{W}$ relation for SU(3) quiver*, *Phys. Rev.* **D82** (2010) 066009, [1007.0601].