1 Lecture2: Conformal transformations and quantum fields transformations

1.1 Introduction

We define now the basics about CFT. We will define the conformal transformations in any dimension and in $D = 2$ dimension. In the nect Lecture we will make the conncection with what we have seen in Lecture 1.

1.2 Conformal transformations

Consider the space $\mathbb{R}^D$, and a system of coordinates x^μ, $\mu = 1, \cdots, D$. Henceforth we assume the Euclidean signature where the metric in the space is given by:

$$(ds)^2 = \eta_{\mu,\nu} dx^\mu dx^\nu, \quad \eta_{\mu,\nu} = \text{diag}(1, 1, \cdots, 1), \tag{1.1}$$

where the summation over repeated indexes is implied. Consider a change of coordinates $x^\mu \to (x')^\rho(\{x^\mu\})$. One has:

$$(ds)'^2 = \eta_{\mu,\nu}(dx')^\mu (dx')^\nu = \eta_{\mu,\nu} \frac{\partial (x')^\mu}{\partial x^\rho} \frac{\partial (x')^\nu}{\partial x^\sigma} dx^\mu dx^\nu \tag{1.2}$$

Consider an infinitesimal transformation,

$$(x')^\rho(\{x^\mu\}) = x^\rho + \alpha^\rho(\{x^\mu\}), \quad |\alpha^\rho| << 1, \tag{1.3}$$

which gives:

$$(ds)'^2 = (ds)^2 + \left(\frac{\partial \alpha^\rho(\{x^\mu\})}{\partial x^\sigma} + \frac{\partial \alpha^\sigma(\{x^\mu\})}{\partial x^\rho} \right) dx^\rho dx^\sigma \tag{1.4}$$

We denote $\alpha^\rho_\sigma = \partial \alpha^\rho(\{x^\mu\})/\partial x^\sigma$ and $\alpha = \sum_{\rho=1}^D \partial \alpha^\rho(\{x^\mu\})/\partial x^\rho$. We can write the function $\partial \alpha^\rho(\{x^\mu\})/\partial x^\sigma$ as

$$\alpha^\rho_\sigma = \frac{1}{2}\left(\alpha^\rho_\sigma - \alpha^\sigma_\rho\right) + \frac{2\alpha}{D}\eta_{\rho,\sigma} + \underbrace{\left(\alpha^\rho_\sigma + \alpha^\sigma_\rho\right) - \frac{2\alpha}{D}\eta_{\rho,\sigma}}_{=S_{\rho,\sigma}} \tag{1.5}$$

In the above equation the component $S_{\rho,\sigma}$ corresponds to the traceless symmetric tensor that takes into account of the shear transformations. The conformal transformation are the subset of coordinate transformation for which the shear component vanishes:

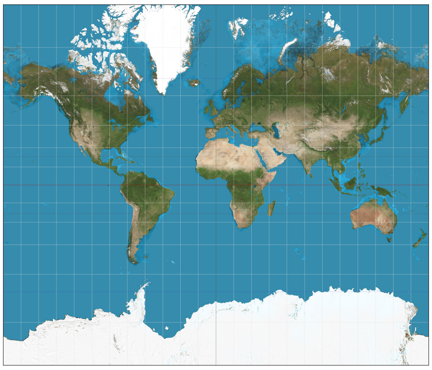
$$\text{Conformal Transformation:} \quad S_{\rho,\sigma} = 0, \tag{1.6}$$

Consider the curves $\gamma^\rho : x^\mu = csts$, $\mu \neq \rho$. A conformal transformation preserves the angles formed by the tangents of this curves at any point. This is the origin of the name conformal (keeps the form). Note also that, for a conformal transformation:

$$(ds)'^2 = \left(1 + \frac{2}{D}\text{Tr}\left(\alpha^\rho_\sigma\right)\right)(ds)^2, \tag{1.7}$$

that means that the effect of the conformal transformation can be thought as a local dilation that vary along the space. Conformal transformation have been known and used in the navigation map since the 1569:

Ex: Mercator map (1569), lines of constant bearing are straight



For $D \geq 3$, the expansions in the coordinates $\{x^\mu\}$ of the functions $\alpha^\rho(\{x^\mu\})$, solutions of (1.6), have only the constant, linear and quadratic term (see Ex 1). The infinitesimal transformations can be written:

$$\begin{aligned}
&\text{Translations:} \quad \alpha^\rho = \delta v^\rho \\
&\text{Rotations:} \quad \alpha^\rho = \delta\omega_{\rho,\nu} x^\nu \quad \omega_{\rho,\nu} : \text{Antisymmetric} \\
&\text{Dilatations:} \quad \alpha^\rho = \delta\lambda x^\rho \\
&\text{Special Conformal:} \quad \alpha^\rho = \delta s^\rho |x|^2 - 2(\delta s^\mu x^\mu) x^\rho
\end{aligned} \tag{1.8}$$

From the above infinitesimal transformation one sees that there are $d + d(d+1)/2 + 1 + d = (d+1)(d+2)/2$ parameters that define the conformal transformation in any dimension.

1.3 Scale invariant models: examples

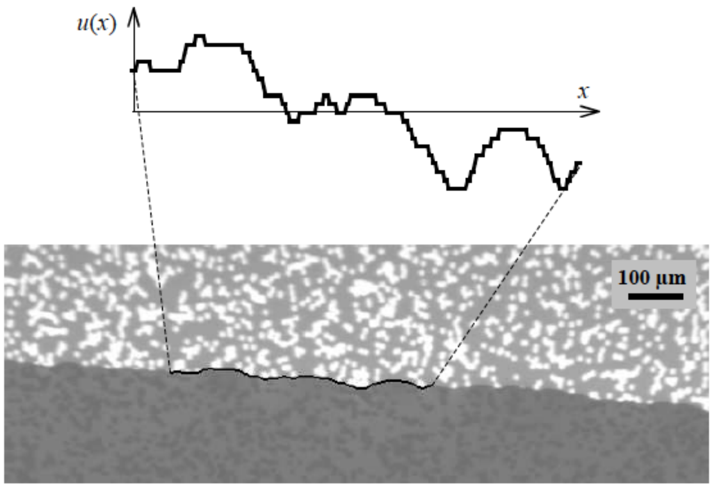
1.3.1 Deterministic self-similarity

;

$$N_a = \text{Nombre pic with size} \geq a$$
$$N_a \sim a^{-D}, \quad D = 2.28 \pm 0.06$$

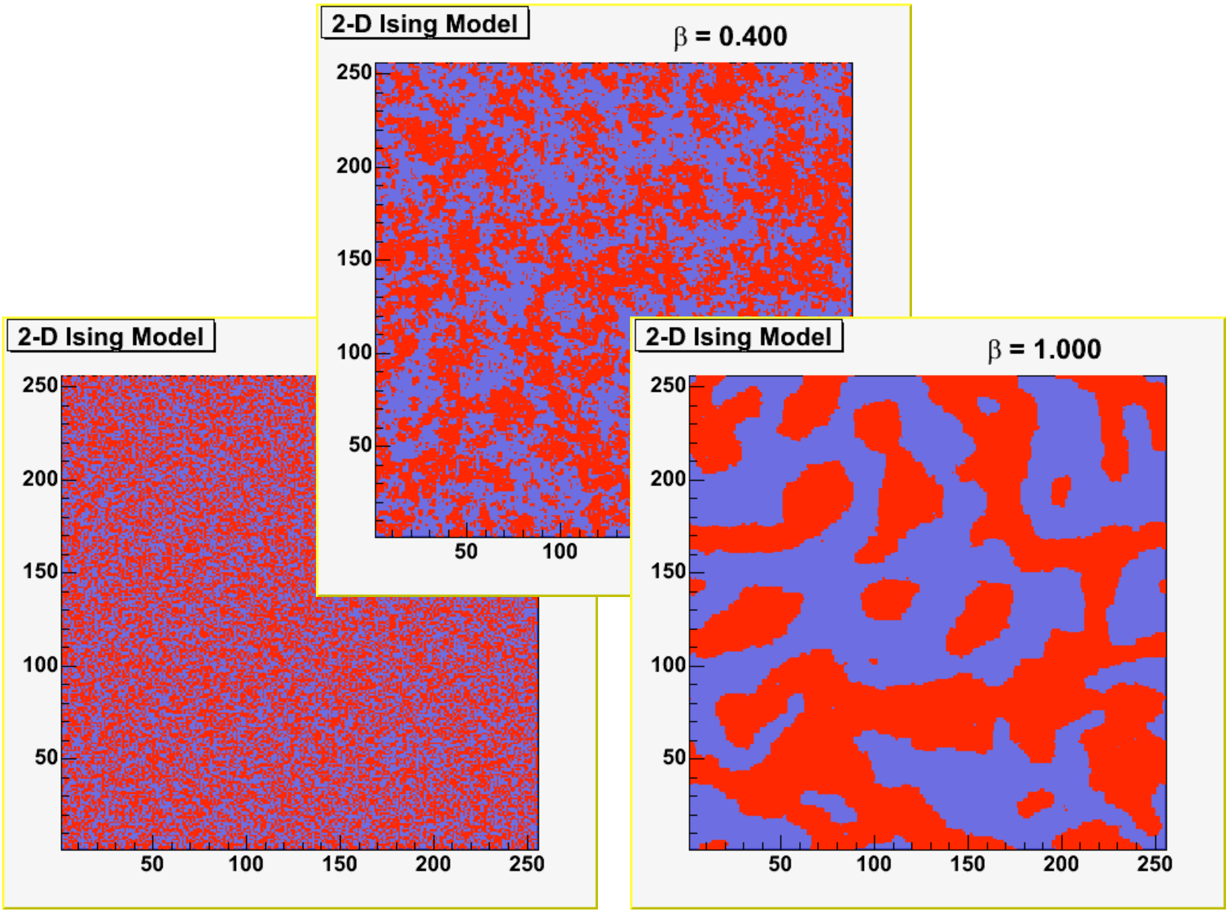
1.3.2 Statistical scale invariance

Water meniscus on a random substrate



$$\prec \int_0^L dx\, u^2(x)/L \succ \sim L^{2\zeta}, \quad \zeta = 0.388 \pm 0.002 \tag{1.9}$$

The process $u(x)$ considered in $x \in [x_0, x_0 + L]$ and the process $\lambda^{-\zeta} u(x)$ in $x \in [(x_0)\lambda, (x_0 + L)\lambda]$ **are the same in law**



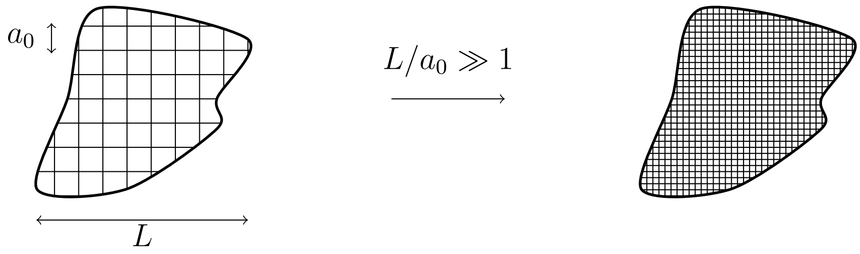
Two-dimensional Ising model:

$$Z = \sum_{\{s_i\}} e^{-\frac{J}{kT}\sum_{<i,j>} s_i s_j} \tag{1.10}$$

1.4 Conformal field theories: their role in critical model

We briefly review what are the physical systems that enjoy conformal invariance, what this means and how we are naturally lead to consider Conformal field theory. I will illustrate this with the physics of critical phenomena. We may think to a lattice spin system at equilibrium, with partition function. Let us call L the linear size of system, and a_0 an ultraviolet cutoff, e.g. the lattice spacing. We suppose that the system has a second order phase transition at a critical temperature $T = T_c$. The correlation length $\xi(T)$, determining the behavior of the thermal fluctuations and the coupling between the system degrees of freedom, diverges at the critical temperature, $\lim_{T\to T_c} \xi(T) \to \infty$. The divergence of the correlation length is understood by observing that the fluctuations become important over all the length scales l, $a_0 << l << L$. Besides the UV and IR cut-offs, the system has not characteristic length and is therefore scale invariant.

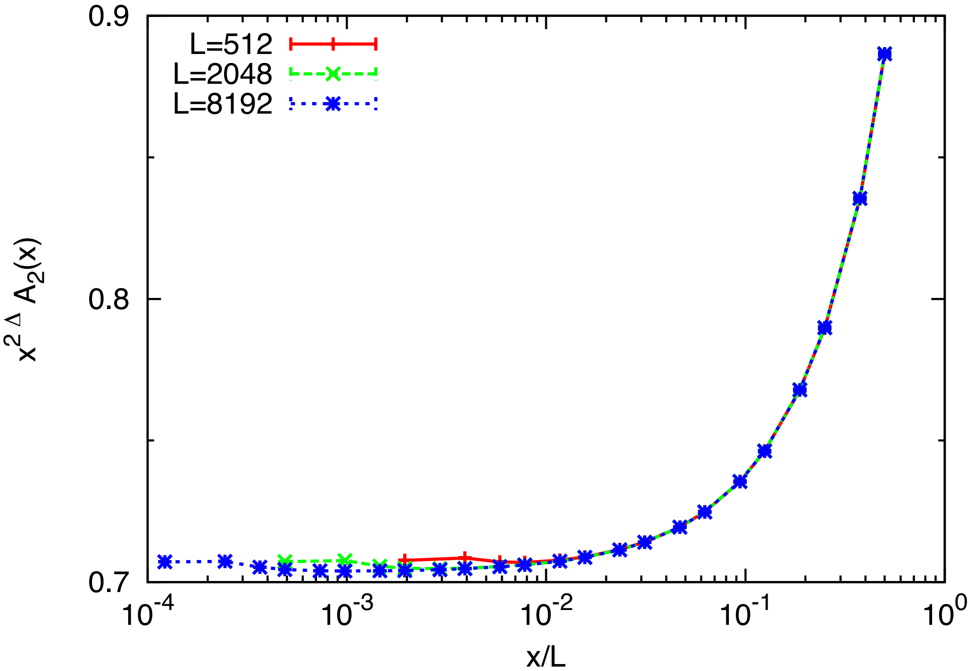
Imagine that we are able to tune the UV cutoff without affecting the global shape of the system, as illustrated below



We want to study the scaling limit $a_0 \to 0$ of the statistical averages $\prec \cdots \succ$ of lattice operators at the critical temperature T_c. The theory of renormalization assumes that there exist operators whose statistical averages, if appropriately renormalized, admit a finite scaling limit. The study of the scaling limit in critical systems is important because it is has universal character. The findings obtained in this limit depend on general features, i.e. on the dimension of space and on the symmetries of the system under consideration, and are relevant to all systems belonging to the same class of universality. The magnetic order parameter $s(x)$ is an example of scaling lattice operator. Consider the limit of the spin spin correlation;

$$\lim_{a_0 \to 0} a_0^{-2\Delta^{\text{phys}}} \prec s(x_1)s(x_2) \succ,$$

where x_i are fixed positions in the domain. The limit has been (rigourously proven) to exist for $\Delta^{\text{phys}} = 1/8$. Below you find real Montecarlo simulations (courtesy of Marco Picco) taken for a lattice of size 8192×8192:



In the scaling limit, we can identify the statistical averages of the operators $a^{-\Delta_\sigma^{\rm phys}}\sigma(x)$ with the correlation functions of fields $\Phi(x)$ in a massless (scale invariant) Euclidean quantum field theory.

$$\lim_{a_0\to 0} a_0^{-2\Delta^{\rm phys}} \prec s(x_1)s(x_2) \succ \propto \langle \Phi(x_1)\Phi(x_2)\rangle = \frac{1}{|x_1-x_2|^{2\Delta^{\rm phys}}} \tag{1.11}$$

1.5 CFT, a first hint of the conformal bootstrap approach

Note at the limit, there is no dependence on any scale. That means that the correlation function $\langle\Phi(x_1)\Phi(x_2)\rangle$ in invariant under dilation. Thus we require that:

$$x \to \tilde{x} = \lambda x : \quad \left\langle \tilde{\Phi}(\tilde{x}_1)\tilde{\Phi}(\tilde{x}_2)\right\rangle = \langle\Phi(x_1)\Phi(x_2)\rangle \tag{1.12}$$

As you can see, that implies that the fields transform under dilation:

$$x' \to \lambda x : \tilde{\Phi}(\tilde{x}_1) = \lambda^{\Delta^{\text{phys}}}\Phi(\lambda x_1) \tag{1.13}$$

So, we have here an example of how a field in a CFT transforms under a dilation: starting from a lattice model, we have "experimentally" argued how a quantum field (more precisely a quasi-primary field) have to behave under a scaling transformation and we impose the invariance of the correlation function.

Now we can reason in an inverse way: we provide the variations of the fields $\delta\Phi$ of a QFT under the infinitesimanl conformal tranformations (1.8). For a scalar fields, we have:

$$\begin{aligned}
&\text{Translations:} \quad \delta_{\text{Transl}}\Phi(\{x^\mu\}) = v^\rho\partial_\rho\Phi(\{x^\mu\}) = \delta v^\rho P_\rho\Phi(\{x^\mu\})\\
&\text{Rotations:} \quad \delta_{\text{Rot}}\Phi(\{x^\mu\}) = \delta\omega_{\rho,\nu}x^\nu\partial_\rho\Phi(\{x^\mu\}) = \delta\omega_{\rho,\nu}\Lambda_{\nu,\rho}\Phi(\{x^\mu\})\\
&\text{Dilatations:} \quad \delta_{\text{Dil}}\Phi(\{x^\mu\}) = \delta\lambda(x^\rho\partial_\rho + \Delta^{\text{phys}})\Phi(\{x^\mu\}) = \delta\lambda\, D\Phi(\{x^\mu\})\\
&\text{Special Conformal:} \quad \left(|x|^2\partial_\rho - 2x^\rho x^\mu\partial_\mu - 2\Delta^{\text{phys}}x^\rho\right)\Phi(\{x^\mu\}) = \delta s^\rho K_\rho\Phi(\{x^\mu\}),
\end{aligned} \tag{1.14}$$

where we have defined the *generators* of the conformal transformations $(P_\rho, \Lambda_{\nu,\rho}, D, K_\rho)$. The generators $P_\rho, \Lambda_{\nu,\rho}$ are the infinitesimal generators of the Poincare' group. We remind that, from the Coleman-Mandula theorem, the only what symmetries that are possible in a massive relativistic theory of interacting particles are direct products of the Poincar group and an internal symmetry group [1]. The conformal group is therefore the generalization of the Coleman-Mandula theorem for mass-less quantum field theories. The algebra reads:

$$\begin{aligned}
&[\Lambda_{\mu,\nu}, \Lambda_{\sigma,\rho}] = \eta_{\nu,\sigma}\Lambda_{\mu,\sigma} - \cdots ..\\
&[P_\mu, \Lambda_{\sigma,\rho}] = \eta_{\mu,\sigma}P_\sigma - \eta_{\mu,\sigma}P_\rho\\
&[P_\mu, P_\rho] = 0\\
&[D, \Lambda_{\sigma,\rho}] = 0\\
&[P_\mu, D] = P_\mu\\
&[K_\mu, D] = K_\mu
\end{aligned} \tag{1.15}$$

We will see in the next lecture that in $D = 2$ the conformal algebra that generalize the above one has an infinite number of generators: it is the **Virasoro** algebra.

Let us assume a correlation function $G(x_1, x_2) = \left\langle \Phi_{\Delta_1^{\text{phys}}}(x_i)\Phi_{\Delta_2^{\text{phys}}}(x_2)\right\rangle$ of a general CFT with two fields with different scaling fields exists. We impose that it is invariant under conformal transformations:

$$\text{Translations:} \quad \delta_{\text{Transl}} G = \left(\partial_{x_1^\mu} + \partial_{x_2^\mu}\right) G = 0 \to G = G(x_1 - x_2)$$
$$\text{Rotations:} \quad \delta_{\text{Rots}} G = \left(x^\nu \partial_\rho - x^\rho \partial_\mu\right) G = 0 \to G = G(|x_1 - x_2|)$$
$$\text{Dilatations:} \quad \delta_{\text{Dil}} G = \left(\Delta_1^{\text{phys}} + \Delta_2^{\text{phys}} + |x_1 - x_2| \partial_{|x_1-x_2|}\right) G = 0 \to$$
$$\to G(|x_1 - x_2|) = \frac{1}{|x_1 - x_2|^{\Delta_1^{\text{phys}} + \Delta_2^{\text{phys}}}} \tag{1.16}$$
$$\text{Special Conformal:} \quad .. \to \Delta_1^{\text{phys}} = \Delta_2^{\text{phys}}, \tag{1.17}$$

So, we have obtained that, in any space dimension $D \geq 3$, the conformal transformations impose the following form to the two

$$\langle \Phi_1(x_1) \Phi_2(x_2) \rangle = \frac{\delta_{\Delta_1^{\text{phys}}, \Delta_2^{\text{phys}}}}{|x - y|^{2\Delta_1^{\text{phys}}}}, \tag{1.18}$$

In an analogous way, one obtains this result for the three-point functions:

$$\langle \Phi_1(x_1) \Phi_2(x_2) \Phi_3(x_3) \rangle = \frac{C_{123}}{|x_1 - x_2|^{\eta_3} |x_2 - x_3|^{\eta_1} |x_3 - x_2|^{\eta_2}} \tag{1.19}$$

with $\eta_1 = \Delta_2^{\text{phys}} + \Delta_3^{\text{phys}} - \Delta_1^{\text{phys}}$, etc. So the conformal invariance fixe the spatial dependence of the two and three point functions. Already this is an highly not trivial achievment of the conformal bootstrap approach!

The values of the scaling dimension Δ^{phys} as well as of the structure constants C_{123}, which are the basic informations to solve a quantum field theory, remains unknown. Usinng an analogy with the study of angular momentum in quantum mechanics, the dimensions Δ^{phys} can be understood as the $SU(2)$ Casimir invariant J^2 while the structure constants C_{123} are the analogous of the ClebschGordan coefficients.

1.6 Exercises

1.6.1 Ex. 1

Consider the case $D = 3$ and find the possible solutions $\alpha^j(x, y, z)$, $j = x, y, z$ for (1.6). Show that all the derivative of type $\partial_z^3 \alpha^j(x, y, z) = 0$, $\partial_z^2 \partial_y \alpha^j(x, y, z) = 0$, vanish. Instead for $D = 2$ shows that all harmonic function $\alpha^j(x, y)$ are solution of for (1.6).

1.6.2 Ex. 2

Consider a sphere of radius 1, on which a point P has spherical coordinate (θ, ϕ), θ and ϕ being respectively the polar and azimuthal angle (measured in radiants). Consider two maps from the sphere to a cylinder which is tangent at the sphere on the equator. On the cylinder we take coordinate x, $x \in [0, 2\pi]$, and $y \in \mathbb{R}$. Consider two coordinates transformations:

$$\text{Map1}: \quad x = \phi, y = \tan(\theta) \tag{1.20}$$
$$\text{Map2}: \quad x = \phi, y = \ln\left[\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right] \tag{1.21}$$

The Map1 is the geometrical projection of a point on a sphere on the cylinder, Map2 is the Mercator map (1569). Show that Map1 is not a conformal map, while Map2 is. Show that in Map 2, there is a scaling factor that vary along the polar angle, $\lambda(\theta) = \cos(\theta)^{-1}$.

1.6.3 Ex. 3

Using (1.17), prove the (1.18).

References

[1] S. Coleman and J. Mandula, *All possible symmetries of the s matrix*, *Phys. Rev.* **159** (Jul, 1967) 1251–1256.