Bogota 2018: Conformal bootstrap and topological order

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Contents

Introduction

Conformal Field Theory (CFT) are mass-less quantum field theory where the scale-invariant symmetry adds to the Poincare symmetry. The main goal of these lectures is to provide rudiments of Conformal boostrap approach and the properties of CFTs that are behind its connection with topological states of matter,

Lecture 1, Configuration space of indistinguishable particles: multiply connected spaces, permutation and braiding group

Introduction

The conformal bootstrap approach is an approach that is alternative to a Lagrangian approach to quantum field theories: therefore introducing CFT is more difficult as the concepts used are not familiar to students and they can be quite abstract. In the first lecture we want to give one the main idea behind the relation between CFT and tolopogical statess. More precisely, the symmetry function of the conformal algebra, the conformal blocks, form the representations of the braiding group $\mathcal{B}_N$. This property has been the one that inspired the connections between CFTs and topological states, in particular the prediction of non-Abelian statistics in fQH.

In the first lecture we remind how to formulate correctly the problem of the statitics of spinless quantum particle in two dimensions and we introduce to the mathematical structures emerging from this study. There will be no CFT, but rigorous mathematical results of complex analysis.

Quantum mechanics on multiply connected spaces and the emergence of anyonic statistics

The topology of the space on which the quantum particles live has very deep consequences. We refer the student to the seminal papers [1], [2] and [3].

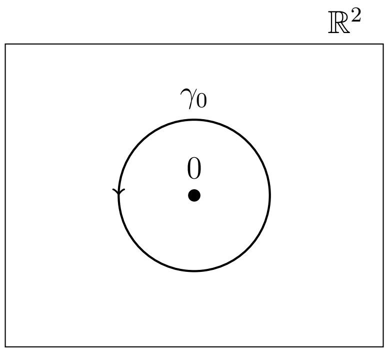
A simple model: particle in magnetic field

To illustrate this let start by briefly recalling the most famous example of a quantum effect with topological origin: the magnetic Aharonov-Bohm effect [4, 5], that has been experimentally observed [6]. Consider a plane pierced at the origin by a magnetic flux Φ_0. The vector potential $\vec{A}$, that can be taken as:

$$\vec{A} = \frac{\Phi_0}{2\pi}\left(\frac{y}{r^2}, -\frac{x}{r^2}\right), \tag{0.1}$$

such that :

$$\int_{\gamma_0} \vec{A}\, d\vec{l} = \int \vec{\nabla} \times \vec{A}\, d\vec{S} = \Phi_0 \int \delta^2(x)\, d\vec{S} = \Phi_0. \tag{0.2}$$



Now consider a charged particle living in the region $X_1 = \mathbb{R}^2 - \{0\}$. The vector potential appears therefore in the covariant derivative $\vec{D}$ an therefore in the Hamiltonian:

$$\vec{D} = \vec{\nabla} - i\vec{A}, \quad H = \vec{D}^2, \tag{0.3}$$

where all the constants $(\hbar, e, c = 1)$ has been set to one. The $\vec{A}$ cannot vanish (or "gauged away"), even when the magnetic field vanishes everywhere in the particle configuration space X_1. In topological terms, this can be expressed by saying that X_1 is a multi-connected topological space as γ_0 cannot be contracted to a point (see below). The Hamiltonian contains therefore a term which is topological in nature. In order to solve the Schroedinger equation:

$$H\Psi(x) = E\Psi(x) \quad \Psi(0) = 0, \tag{0.4}$$

where $\Psi(x)$ is the uni-valued wavefunction, a convenient way is to eliminate the vector potential in H by attaching a (Dirac) tail to $\Psi(x)$:

$$\tilde{\Psi}(x) = e^{i\int_\gamma \vec{A}d\vec{l}}\Psi(x), \tag{0.5}$$

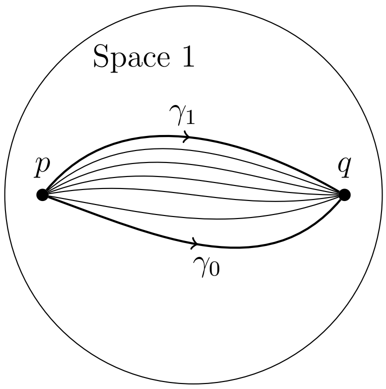
where γ is a path connecting an (arbitrary chosen) point x_0 to x. The problem (0.4) is now written as:

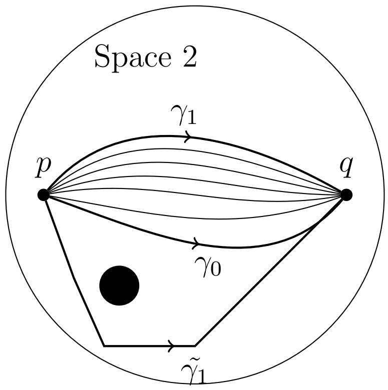
$$\vec{D}^2\tilde{\Psi}(x) = E\tilde{\Psi}(x), \quad \tilde{\Psi}(xe^{2i\pi\theta}) = e^{i\Phi_0}\tilde{\Psi}(x), \quad \tilde{\Psi}(x) = 0 \tag{0.6}$$

So one eliminates the topological term in the Hamiltonian at the price of having a wave-function defined on a surface with a branch cut (we will return on this later).

Notions of topology

Let us recall some basic notions of topology, in particular the multiply connected space and the homotopy classes. A (topological) space is multiply connected if any two points can be connected (connected) by a continuous path and there exist paths that connect two same marked points but are not *homotopic* . Two paths γ_1, γ_2 are homotopic, if they can continuously deform one into another. The relation of homotopy fixes an equivalent relation: we will note as $[\gamma]$ the class of equivalence of a path γ. Below, on the left, an example of a simply connected space: any couple of curves γ_0 and γ_1, connecting two points p and q, can be deformed one into another continuously.





The Space 2 is an example of a double-connected. The curves γ_1 and $\tilde{\gamma_1}$ are not homotopic. Given a topological space X, and the set of closed paths in X that start and end at a same point x_0, one can define a product $\gamma_1\gamma_2 = \gamma_3$ between two paths by gluying them. This product is consistent with the homotopy classes, $\gamma_1\gamma_2 = \gamma_3, \rightarrow [\gamma_1][\gamma_2] = [\gamma_3]$. The associated group is called *fundamental group*, $\pi_1(X_N)$. For instance $\pi_1(\mathbb{R}^2 - 0) = \mathbb{Z}$: this means that the element of the groups, and therefore the homotopy classes $[\gamma]$ are associated to integers, $[\gamma_n] \rightarrow n$, where γ_n is a loop that winds the origin n times. One has $[\gamma_n][\gamma_m] = [\gamma_{n+m}]$.

Statistics of indistinguishable particles

There is a very well known argument, that can still be found in many text books. Consider N indistinguishable quantum particles living in a D dimensional space. The particles positions are denoted by x_i, $i = 1, \cdots N$, $x_i \in \mathbb{R}^D$, with x_i being a vector with D components. First one considers the many-body wave-function Ψ as a function of the N positions x_i, $\Psi(x_1, \cdots, x_N)$. Note that, choosing as configuration space $Y_N = \mathbb{R}^{DN}$ (a point in this space is associated to the set $\{x_i\}$), one is artificially labeling the particles. The exchange operator $P_{j,k}$ acts on this Hilbert space as:

$$P_{j,k}\Psi(x_1, \cdots, x_j, \cdots, x_k, \cdots, x_N) = \Psi(x_1, \cdots, x_k, \cdots, x_j, \cdots, x_N) \tag{0.7}$$

When the particles are indistinguishable, their interaction should not depend on their label, i.e. the operator P commutes with the Hamiltonian H, $[H, P] = 0$. Therefore the two operators have a common space of eigenstates. As $P^2 = 1$, the possible eigenvalue can be only 1 (bosons) or -1 (Fermions). This argument, that does not depend on the dimension of the space, fails to predict the existence of anyonic statistics in $D = 2$. The point is that we are implicitly assuming that the exchange operation is well defined. On the other hand, we can expect that when non-trivial topological effects are present, this assumption is in general not true. For instance, multi-valued wavefunction can enter into the game and the operation of exchanging two variables becomes ambiguous. Or, in $D = 1$ the exchange of two particles is also not well defined: one cannot exchange particle k and particle j without scattering with other particles. For a discussion of the failure of the above argument in the 1D case, see [7].

The problem of N indistinguishable particles has been put on more solid ground by considering as space of configurations the space X_N:

$$X_N = \frac{\mathbb{R}^{DN} - \{\text{coinciding points}\}}{\mathcal{S}_N}, \tag{0.8}$$

where $\mathcal{S}_N$ is the permutation group of N objects. The above notation a point $q \in X_N$ is the equivalence class $q = [\{x_i\}]$ formed by the configurations $\{x_i, \cdots, x_N\}$ of points that are related one to the other by permutations of the particle labels. The many-body wavefunction Ψ is now an (univalued) function defined on this space $\Psi = \Psi(q)$. The crucial observation is that X_N is, for $D > 1$, a multiply-connected space (see Ex. 1).

It is very convenient to use a Feynmann path-integral approach and study the evolution of the wavefunction $\Psi(q, t)$ during the time $t \in [0, 1]$. We denote as $x_j(t)$ the wordline of the $j-$ particle. The set of worldlines $\{x_j(t)\}$, $j = 1, \cdots, N$ form a loop starting and ending at the same point q in the configuration space X_N. From now on, we will be not interested in the dynamics of the particles (i.e. the precise form of the functions $x_i(t)$), but we will focus on the homotopy of the loops.

The Feynman sum over all the possible loops can be cast then into a sum over all the path belonging to the same homotopy class and then a sum over the different classes,

$$\begin{aligned}
\Psi\left(q, t=1\right) &= \sum_{A=\text{Homotopy class}} \chi(A) \sum_{\text{All loops in A}} e^{i\text{Action[loop in } A]}\,\Psi\left(q, t=0\right) \\
&= \sum_{A\in\pi_1(X_N)} \chi(A) \sum_{\text{loops}\in A} e^{i\text{Action[loop} \in A]}\,\Psi\left(q, t=0\right) \\
&= \sum_{A\in\pi_1(X_N)} \chi(A)\Psi_A\left(q, t=1\right),
\end{aligned} \tag{0.9}$$

where we assume Ψ to be normalized. In the above formula we used the freedom to associate different weights $\chi(A)$ to each homotopy class: the addition of these weights corresponds to adding a topological term (so a term that depends only on the topology) to the action. Every function $\Psi_A\left(q, t=1\right)$ has been obtained from the same *local* dynamics, and therefore one expects any local observable O to give the same measure, in particular $\langle\Psi_A|O|\Psi_A\rangle = \langle\Psi|O|\Psi\rangle$ for each A. As the physical measures are invariant under the action of the group $\pi_1(X_N)$, the weights χ_A should then be taken from its scalar *unitary* *representation* [1]. In analogy of what we have seen for the case of a particle in a localized magnetic flux, we can eliminate the topological term in the action by attaching a tail to the function $\Psi(q,t)$ and study the topology free evolution of multi-valued (in X_N) wavefunction $\tilde{\Psi}(q,t)$. Consider an initial point q_0 (arbitrary) in X_N and B a the homotopy class of braidings from q_0 to q. We define

$$\tilde{\Psi}_C(q,t) = \chi(C)\Psi(q,t), \quad C \in \pi_1(X_N). \tag{0.10}$$

Note that the function $\tilde{\Psi}_C(q,t)$ is an uni-valued function in the *covering space* $\tilde{X}_N$. It satisfies

$$\tilde{\Psi}_{CA}(q,t) = \chi(C)\tilde{\Psi}_A(q,t), \quad C, A \in \pi_1(X_N). \tag{0.11}$$

$D \geq 3$

For $D \geq 3$, one can show that:

$$\mathcal{S}_N = \pi_1(X_N), \quad \text{for} \quad D \geq 3. \tag{0.12}$$

In this case the different topological sectors A are in one-to-one correspondence with the different permutations P of N objects. We can associate $\Psi_A(q,t) \rightarrow \Psi\left(\{x_{P(i)}\}, t\right)$. The unitary scalar representations of $\mathcal{S}_N$ are nothing else that the bosonic and fermionic statistics under exchange of two particles (see Ex 2), and the above formula becomes the more familiar symmetrization or antisymetrization formula:

$$\Psi\left(\{x_i(t)\}\right) \propto \sum_{P \in \mathcal{S}_N} \chi(P)\Psi\left(\{x_{P(i)}\}, t\right) \quad \chi(P) = 1 \text{ or } \chi(P) = \mathrm{sign}(P) \tag{0.13}$$

$D = 2$

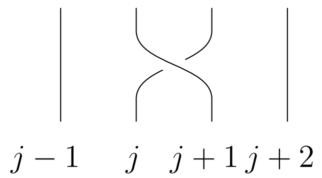
In $D = 2$, the situation is much more rich as the fundamental group $\pi_i\left(X_N\right)$, called the braid group $\mathcal{B}_N$,

$$\mathcal{B}_N = \pi_1\left(X_N\right), \tag{0.14}$$

is infinite dimensional.

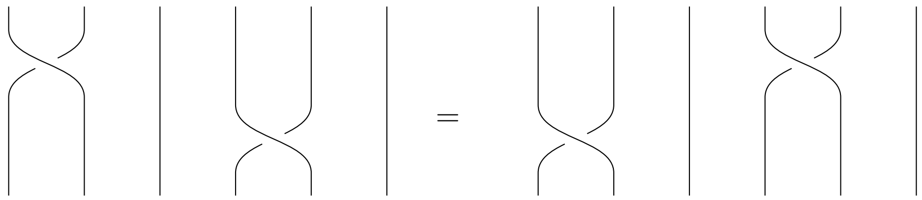
Artin has proved that the braiding group $\mathcal{B}_N$ is isomorph to the group generated by the elements σ_i, that braids the string i *below* $i+1$, as in this figure:

σ_j :

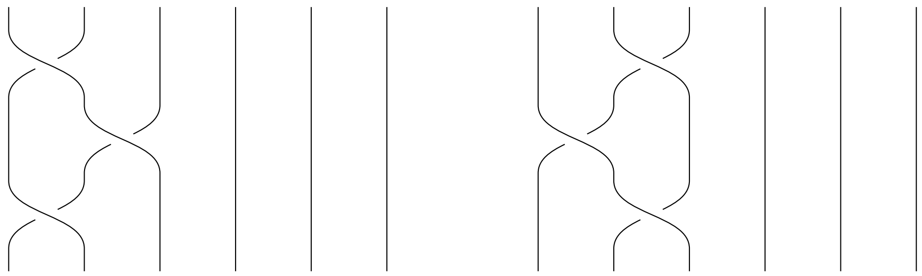


They form the following algebra,

[1] We remember that an unitary $d-$ dimensional representation of a group G (with elements g) is defined by a $d-$ dimensional complex vector space with an Hermitian inner product (the representation space) and by the set unitary $d \times d$ matrices $\rho(g)$, acting on this vector space, such that $\rho(g)\rho(h) = \rho(gh)$, $g, h \in G$



$$\sigma_l \sigma_k = \sigma_k \sigma_l \ (k \neq l \pm 1)$$



$$\sigma_l \sigma_{l+1} \sigma_l = \sigma_{l+1} \sigma_l \sigma_{l+1}$$

There are no further relations among the σ_i's. In particular note that $\sigma_i^2 \neq 1$.

One-dimensional representation of $\mathcal{B}_N$: the abelian anyons

Let us us consider first the case of one-dimensional unitary representation of the braiding group. In this case $\chi(\sigma_i)$ is just a phase (see Ex 1):

$$\chi(\sigma_i) = e^{i\theta}, \quad \theta \in [0, 2\pi[\tag{0.15}$$

. Consider a general element $A \in \mathbb{B}_N$ obtained by a given sequence of σ_i. Then:

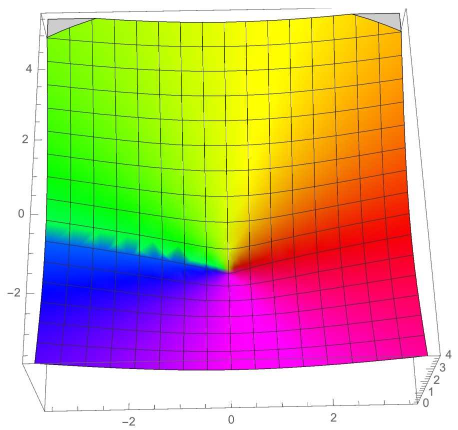
$$A = \sigma_k \sigma_l^{-1} \cdots \sigma_j, \quad \chi(A) = e^{i\theta(N_+ - N_-)}, \tag{0.16}$$

where N_+ (N_-) is the number of braiding σ_k (σ_k^{-1}) where the particle k passes below (above).

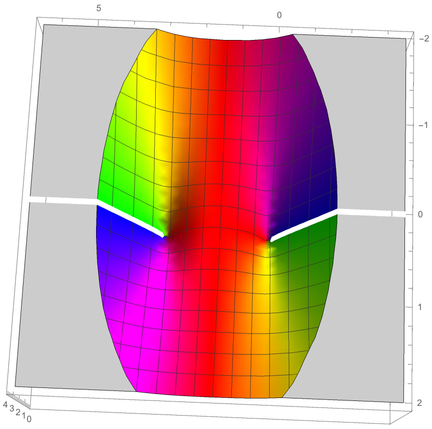
We want to give some examples of functions that satisfy (0.11). These functions can be easily found in the realm of the complex analysis. Consider for instance the function:

$$f(z) = z^{\theta/\pi}, \quad f(ze^{2\pi i}) = e^{2i\pi\theta} f(z). \tag{0.17}$$

The point $z = 0$ represents a branch point: the function is discontinuous when one does an analytic continuation of the function following a loop around $z = 0$. This can be seen in the following plot,



$f(z) = z^{2/3}$



$f(z) = z^{2/3}(1-z)^{2/3}$

The plot on the left shows the function $f = z^{2/3}$: the surface is given by the absolute value $|f(x+iy)|$, while the colors shows its phase. The branch cut is visible by the discontinuity of the green/blue colors. The $z^{2/3}(1-z)^{2/3}$ is represented on the plot on the right.

The behavior of a function going around a singularity is called the **monodromy**. Consider the representations of $\mathbb{B}_2$. It is easy to very that the function:

$$\Psi(\tilde{q}, t) = (z_1(t) - z_2(t))^{\theta/\pi} \tag{0.18}$$

satisfies (0.11).

For general $\mathbb{B}_2$ one can verify that the function:

$$\Psi_{\rm qhL}(z_1, \cdots, \cdots, z_N) = \prod_{i<j} (z_i - z_j)^{\theta/\pi}, \tag{0.19}$$

satisfies (0.11) with (0.16).

So we have learned that the functions with non-trivial monodromy can be related to the representation of the braiding group. These functions are solutions of a particular class of differential equations, the Fuchs differential equation, defined below. For instance the function (0.19),as a function of $z_1 = z$ is solution of the following ordinary differential equation:

$$\left(\frac{d}{dz} + \sum_{j=2}^{N} \frac{\theta}{z - z_j} \right) \Psi_{\rm qhL}(z) = 0 \tag{0.20}$$

that is of Fuchsian type.

Fuchsian differential equations

Consider a linear ordinary differential equation (ODE) in the complex domain $\mathbb{P} = \mathbb{C} \cup \infty$:

$$\left(\frac{d^n}{dz^n} + y_1(z) \frac{d^{n-1}}{dz^{n-1}} + \cdots + y_n(z) \right) f(z) = 0$$

where $y_l(z)$ are analytic in $\mathbb{P}\backslash\{z_2, z_3, \cdots, z_N\}$. The point z_i are said to be a *regular singularity* if it is bounded by an algebraic function, that is

$$\exists M \in \mathbb{N}: \lim_{z \to z_i} |z - z_i|^M |y(z)| = 0 \tag{0.21}$$

A Fuchsian equation is an equation where all singular points are regular singular points. An equation is of Fuchsian type if and only if the function $y_j(z)$ satisf the following property,

$$\left(\prod_{i=1}^{p}(z - z_i)^j\right) y_j(z) = \sum_{l=0}^{K_j} c_l\, z^K, \quad c_l \in \mathbb{C}, \quad K_j \leq j(p-1) \tag{0.22}$$

Let us illustrate with very simple examples the difference between a Fuchsian and a non-Fuchsian equation. The following ODE,

$$\left(\frac{d}{dz} + \frac{1}{z^2}\right) f(z) = 0, \quad \text{Solution:} \quad f(z) = e^{1/z}, \tag{0.23}$$

is not of Fuchs type as it presents an essential singularity at $z = 0$. On the other hand, you can easily check that the following equation is Fuchsian,

$$\left(\frac{d}{dz} + \frac{\alpha}{z}\right) f(z) = 0, \qquad \text{Solution:} \quad f(z) = z^\alpha$$

and indeed the solution has a regular singularity.

Now, let us consider a very important example of Fuchsian equation, the Euler's hypergeometric differential equation:

$$\left(z(1-z)\frac{d^2}{dz^2} + [\gamma - (\alpha + \beta + 1)z]\,\frac{d}{dz} - \alpha\beta\right) f(z) = 0, \tag{0.24}$$

This is a Fuchsian equation of order 2 with three singularities, $z_1 = 0$, $z_2 = 1$ and $z_3 = \infty$. We can find the small z expansion of the solution by using a precedure called the Frobenious method (see Ex 4).

The bi-dimensional space of solution is spanned by the functions $f^{(1)}(z)$ and $f^{(2)}(z)$ that admit the following expansions:

$$\begin{aligned} f^{(1)}(z) &= {}_2F_1\,(\alpha, \beta; \gamma; z) = \sum_{l=0}^{\infty} \frac{(\alpha)_l (\beta)_l}{l!(\gamma)_l} z^l \\ f^{(2)}(z) &= z^{1-\gamma}{}_2F_1\,(\beta + 1 - \gamma, \alpha - \gamma + 1; 2 - \gamma; z)\,, \end{aligned} \tag{0.25}$$

where we used the notation $(x)_l = (x)(x+1)\cdots(x+l-1)$. The convergence radius of the above expansion is $|z| < 1$: the problem is how to analytically continue the above solutions over all the complex plane $\mathbb{C}$. As we explain now, this in turn is equivalent to study the monodromy properties of the solutions (0.25). Let us study the monodromy associated to a small loop γ_0 around zero. In this case we can remain in the region of the complex plane where the (0.25) is convergent. This means that $\gamma_0 : f^{(1)}(z) \to f^{(1)}(z), \quad f^{(2)}(z) \to e^{2i\pi(1-\gamma))} f^{(2)}(z)$. Using a matrix 2×2,

$$\gamma_0 : (f^{(1)}(z), f^{(2)}(z)) \to (f^{(1)}(z), f^{(2)}(z)) M_0\,, \qquad M_0 = \begin{pmatrix} 1 & 0 \\ 0 & e^{2i\pi(1-\gamma)} \end{pmatrix}. \tag{0.26}$$

Now, what about the monodromy around a loop γ_1 going around the other singularity, $z = 1$? It is clear that one cannot use anymore the (0.25) as the loop passes in regions where the sums are not convergent. Let explain what happens when one does a monodromy transformation around $z = 1$. As we found the expansion of the solution around $z = 0$, we could equivalently find the expansion of the solution around $z = 1$ (see Ex.4), obtaining

$$\begin{aligned} f^{(3)}(z) &= {}_2F_1\left(\alpha, \beta; 1 + \alpha + \beta - \gamma; 1 - z\right) \\ f^{(4)}(z) &= (1 - z)^{\gamma-\alpha-\beta}{}_2F_1\left(\gamma - \alpha, \gamma - \beta; 1 + \gamma - \alpha - \beta; 1 - z\right). \end{aligned} \tag{0.27}$$

Being solution of the same second-order differential equation, the functions $f^{(1)}(z)$ and $f^{(2)}(z)$ can be written in as a linear combination of $f^{(4)}(z)$ and $f^{(4)}(z)$, $f^{(1)}(z) \to f^{(1)}(z) + \cdots f^{(1)}(z)$, $\quad f^{(2)}(z) \to \cdots f^{(1)}(z) + \cdots f^{(2)}(z)$. In terms of these functions, which are convergent in a radius $|z - 1| < 1$, the monodromy transformation around γ_1 is diagonal, $\gamma_1 : f^{(3)}(z) \to f^{(3)}(z)$, $\quad f^{(4)}(z) \to e^{2i\pi(\gamma-\alpha-\beta))}f^{(4)}(z)$. So one finds that the functions $f^{(1)}(z)$ and $f^{(2)}(z)$ mix under a monodromy around γ_1. Using the linear relation between the basis $\left(f^{(1)}(z), f^{(2)}(z)\right)$ and the basis $\left(f^{(3)}(z), f^{(3)}(z)\right)$ (see Ex 4), one finds:

$$\gamma_1 : (f^{(1)}(z), f^{(2)}(z)) \to (f^{(1)}(z), f^{(2)}(z))M_1\,, \qquad M_1 = \begin{pmatrix} \frac{1-e_1}{e_1(1-e_2e_3)} & \frac{e_1e_2e_3-1}{e_1(1-e_2e_3)} \\ \frac{e_2-1}{1-e_2e_3} & \frac{e_2(e_3-1)}{1-e_2e_3} \end{pmatrix} \tag{0.28}$$

where we set

$$e_1 = e^{2\pi\sqrt{-1}(\beta-\gamma)}\,,\ e_2 = e^{2\pi\sqrt{-1}(-\beta)}\,,\ e_3 = e^{2\pi\sqrt{-1}(\gamma-\alpha)}\,. \tag{0.29}$$

The linear relation between solutions allows to analytically continue the solutions on all the domain $\mathbb{P}\backslash\{0, 1, \infty\}$.

Ising non-abelian anyons

In the next lectures we will introduce the conformal block of the Virasoro algebra. We will show that conformal blocks of the *Rational* Conformal field theories satisfy Fuchsian equations. For instance, consider four fields $\Phi_{qh}(z_i)$, at positions z_i whose cross ratio is:

$$z = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_2 - z_3)}. \tag{0.30}$$

The cross ratio has these properties under permutation of indixes:

	z	$1-z$	$\frac{1}{z}$	$\frac{z}{z-1}$	$1-\frac{1}{z}$	$\frac{1}{1-z}$
permutations	id	(13)	(23)	(12)	(123)	(132)
	(13)(24)	(24)	(14)	(34)	(243)	(234)
	(12)(34)	(1234)	(1342)	(1324)	(134)	(143)
	(23)(14)	(1432)	(1243)	(1423)	(142)	(124)

(0.31)

The conformal blocks of four quasi-hole fields takes the form:

$$\left\langle \prod_{i=1}^{4} \Phi_{qh}(z_i) \right\rangle = (z_1 - z_4)^{-\frac{1}{8}}(z_2 - z_3)^{-\frac{1}{8}}z^{-\frac{3}{8}}(1 - z)^{-\frac{3}{8}}f\left(z\right), \tag{0.32}$$

where $f(z)$ satifies the (0.24) with $\alpha = 5/4, \beta = 3/4, \gamma = 3/2$. One has therefore an espace of two solutions, $f^{(1)}(z)$ and $f^{(2)}(z)$. One can chose the following basis of functions:

$$\Psi_1 = \left\langle \prod_{i=1}^{4} \Phi_{qh}(z_i) \right\rangle_{(1)} = (z_1 - z_2)^{-\frac{1}{8}} (z_3 - z_4)^{-\frac{1}{8}} \sqrt{(1-z)^{1/4} + (1-z)^{-1/4}} \tag{0.33}$$

$$\Psi_2 = \left\langle \prod_{i=1}^{4} \Phi_{qh}(z_i) \right\rangle_{(2)} = (z_1 - z_2)^{-\frac{1}{8}} (z_3 - z_4)^{-\frac{1}{8}} \sqrt{(1-z)^{1/4} - (1-z)^{-1/4}} \tag{0.34}$$

One can now study the behavior of the vector of wave-functions (Ψ_1, Ψ_2) under the elements of the braiding group. For instance the element σ_1^2, corresponds to make z_1 going around z_2, so $(z_1 - z_2) \to (z_1 - z_2)e^{i2\pi}$. This gives:

$$\sigma_1^2 : (\Psi_1, \Psi_2) \to (\Psi_1, \Psi_2)\rho(\sigma_1^2) \ , \qquad \rho(\sigma_1^2) = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} \tag{0.35}$$

The element σ_2^2 corresponds to make z_2 going around z_3. We can use the above result once we have permuted the indexes (23). In the above table one can see that for the cross ration this means $z \to 1 - z$. Writing the Ψ_1 and Ψ_2 in the new cross ratio, and doing $(z_2 - z_3) \to (z_2 - z_3)e^{i2\pi}$, one can verify that:

$$\sigma_2^2 : (\Psi_1, \Psi_2) \to (\Psi_1, \Psi_2)\rho(\sigma_2^2) \ , \qquad \rho(\sigma_1^2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{0.36}$$

One can see that the above wavefunction realizes a $2-$ dimensional unitarian representation of the brais group B_4.

Exercises

Ex. 1

Consider $N = 2$ particles living in $\mathbb{R}^D$, and the two spaces, $Y_2^{(D)} = \left(\mathbb{R}^{2D} - I\right)$ and $X_2^{(D)} = \left(\mathbb{R}^{2D} - I\right)/\mathcal{S}_2$. Argue by simple drawings that:

- $Y_2^{(1)}$ is not connected
- $Y_2^{(3)}$ is simply-connected and $X_2^{(3)}$ is double connected
- $Y_2^{(2)}$ is multiply connected.

Ex. 2

Consider an unitary one dimensional representation of the group (**??**). Set $\rho(\sigma_j) = e^{i\theta_j}$. Show that the $N - 1$ phases θ_j have to be equal, $\theta_j = \theta$.

Ex.3

The permutation group $\mathcal{S}_N$ can be formed by the elements s_i that exchange particles i and $i+1$. Write the algebra formed by the s_i. What is the difference with the one formed by the σ_i? Consider the one-dimensional representation of the s_i algebra and explain why one can find only fermionic or bosonic representation.

Ex.4

Consider a function $f(z) = z^a\,(1 + c_1(a)z + \cdots)$ that satisfies the ODE (0.24). Verify that the (0.24) is of Fuchsian type and find the values of a and of the constant c_1. Do the same starting from $f(z) = (1-z)^b\,(1 + d_1(b)(1-z) + \cdots)$

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