Canonical formulation of an alternative model of linearised massive gravity

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Introduction

To begin the discussion regarding N degrees of freedom systems with constraints let's consider the functional $\mathcal{S}$:

$$\mathcal{S}\left[q_i\left(t\right)\right] = \int_{t_1}^{t_2} dt\ L\left(q_i\left(t\right), \dot{q}_i\left(t\right)\right),$$

which is called an action, and contains all the information about the dynamics of the system.

The path taken by the system is that for which the action is stationary ($\delta\mathcal{S} = 0$) under small system configuration changes (δq).

$\delta\mathcal{S} = 0$ leads us to the Euler-Lagrange equations:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0.$$

Condition for regular Lagrangians

Explicitly, we obtain

$$\ddot{q}_j \underbrace{\frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i}}_{W^{ij}} = \frac{\partial L}{\partial q_i} - \dot{q}_j \frac{\partial^2 L}{\partial q_j \partial \dot{q}_i}.$$

From the above equations we can only determine the accelerations in terms of positions and speeds if $W = det W^{ij} \neq 0$.

Hamiltonian formulation for systems with regular Lagrangians

In addition to the Lagrangian formulation of classical mechanics, there is a formulation due to Hamilton.

The transition to the Hamiltonian formulation of the mechanics is given by the Legendre transformation

$$H\left(p\left(t\right),q\left(t\right)\right)=\left[p^{i}\left(t\right)\dot{q}_{i}\left(t\right)-L\left(q\left(t\right),\dot{q}\left(t\right)\right)\right]\Big|_{p^{i}=\frac{\partial L}{\partial \dot{q}_{i}}}.$$

The variation of H gives us

$$\delta H=\dot{q}_{i}\delta p^{i}-\frac{\partial L}{\partial q_{i}}\delta q_{i}.$$

From where we can see that H is independent of the speeds.

Hamiltonian formulation for systems with regular Lagrangians

We have $\frac{\partial p^i}{\partial \dot{q}_j} = W^{ij}$, from where we see that $W \neq 0$ is necessary to obtain $\dot{q}_i$ as a function of p^i.

If we write the action as:

$$\mathcal{S}\left[q_i\left(t\right)\right] = \int_{t_1}^{t_2} dt \;\; \left[p^i \dot{q}_i - H\left(q_i\left(t\right), p^i\left(t\right)\right)\right],$$

we can find Hamilton's equations, making variations of the action. Then, we have:

$$\dot{q}_i = \frac{\partial H}{\partial p^i},$$

$$\dot{p}^i = -\frac{\partial H}{\partial q_i}.$$

Hamiltonian formulation for systems with regular Lagrangians

Let's recall that for two functions of the phase space, the Poisson bracket is defined as

$$\{A\left(p,q\right), B\left(p,q\right)\} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p^i} - \frac{\partial A}{\partial p^i}\frac{\partial B}{\partial q_i},$$

and

$$\frac{d}{dt}A = \{A, H\} + \frac{\partial A}{\partial t}.$$

With what we have seen we can find the relevant information from our system.

Hamiltonian Formulation of Singular Lagrangians

Condition for the singularity of the Lagrangian

The Lagrangian is said to be singular if $W = 0$.

The singularity of the Hessian is equivalent to the non-invertibility of $p^i = \frac{\partial L}{\partial \dot{q}_i}$, this means that we have variables that are not independent. This results in the existence of relations that connect the positions variables and the momentum variables

$$\phi_r\left(q, p\right) = 0,\ \ r = 1, ..., R.$$

These conditions are called the primary constraints.

To take into account the constraints, we attach these to the theory via Lagrange multipliers. Here the multipliers act as new dynamical variables.

Hamiltonian Formulation of Singular Lagrangians

Then we come to the action

$$\mathcal{S} = \int dt \left[p^i(t)\dot{q}_i(t) - H(p(t), q(t)) - \lambda^r(t)\phi_r(p(t), q(t)) \right].$$

By making variations of the action we obtain

$$\begin{aligned} \dot{p}^i &= -\frac{\partial H}{\partial q_i} - \lambda^r \frac{\partial \phi_r}{\partial q_i}, \\ \dot{q}_i &= \frac{\partial H}{\partial p^i} + \lambda^r \frac{\partial \phi_r}{\partial p^i}, \\ \phi_r &= 0. \end{aligned}$$

We can rewrite $\dot{p}^i$ and $\dot{q}_i$ in the following way

$$\begin{aligned} \dot{p}^i &= \left[\{p^i, H\} + \lambda^r \{p^i, \phi_r\} \right] \Big|_{\Gamma_P}, \\ \dot{q}_i &= \left[\{q_i, H\} + \lambda^r \{q_i, \phi_r\} \right] \Big|_{\Gamma_P}. \end{aligned}$$

Hamiltonian Formulation of Singular Lagrangians

Then, since the Hamiltonian is not uniquely determined, we can go over to another Hamiltonian

$$\tilde{H} = H + \lambda_r \phi_r.$$

Using this Hamiltonian the equations of motion take their ordinary form similar to the case in which there are no constraints

$$\begin{aligned} \dot{F} &= \left[\{F, \tilde{H}\} - \{F, \lambda^r\}\phi_r \right] \Big|_{\Gamma_P} \\ &\approx \{F, \tilde{H}\}. \end{aligned}$$

To have a consistent system we need the time derivatives of the constraints ϕ_r to be zero

$$\dot{\phi}_n = \{\phi_n, \tilde{H}\} = \{\phi_n, H\} + \lambda_r\{\phi_n, \phi_r\} \approx 0.$$

The following may happen:

- We get an expression for λ_r.
- New relationships emerge between momenta variables and positions variables that are independent of the primary constraints (secondary constraints). When secondary constraints arise, the process is repeated until no new constraints appear.

Hamiltonian Formulation of Singular Lagrangians

The distinction between primary and secondary constraints is of little importance in the final form of the Hamiltonian formulation.

A different classification of constraints, and more generally, of functions defined in the phase space, plays a central role.

This is the concept of first-class and second-class functions.

- A function $F(q, p)$ is said to be First Class if: $\{F, \varphi_r\} \approx 0$ with all the constraints.
- F will be Second Class otherwise.

Second-class constraints give rise to a matrix with entries

$$\mathcal{C}_{mn} = \{\chi_m, \chi_n\},$$

whose determinant is not zero even weakly.

Then we can define the Dirac bracket for two functions of the phase space as

$$\{A, B\}^* = \{A, B\} - \{A, \chi_m\}\, \mathcal{C}^{mn}\, \{\chi_n, B\},$$

where $\mathcal{C}^{mn}$ is the inverse of $\mathcal{C}_{mn}$.

Generalization to infinite degrees of freedom

Generalization to infinite degrees of freedom

Now the Lagrangian is a function of the fields and their derivatives with respect to time.

$$L = L\left[\varphi\left(x\right), \dot{\varphi}\left(x\right)\right],$$

L can be expressed as:

$$L = \int d^{D-1}x \; \mathcal{L}.$$

$\mathcal{L}$ is a function of φ and its first derivatives, $\mathcal{L}\left(\varphi, \partial_\mu \varphi\right)$.

Generalization to infinite degrees of freedom

Then, the action is

$$\mathcal{S}\left[\varphi\right] = \int d^D x \ \mathcal{L}.$$

From this we get the Euler-Lagrange equations for the fields

$$\frac{\partial \mathcal{L}}{\partial \varphi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \varphi_a\right)} = 0.$$

We define the conjugated momentum of the fields as:

$$\pi^a\left(x\right) = \frac{\delta L}{\delta \dot{\varphi}_a\left(x\right)} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_a\left(x\right)}.$$

Generalization to infinite degrees of freedom

The Hamiltonian is:

$$H = \int d\boldsymbol{x}\; \mathcal{H} = \int d\boldsymbol{x} \left(\pi^a(x)\,\dot{\varphi}_a(x) - \mathcal{L}\right)\Big|_{\pi^a = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_a}}.$$

The Poisson brackets for two functional of the phase space become

$$\{A(x), B(y)\} = \int d\boldsymbol{z} \left(\frac{\delta A(x)}{\delta \varphi_a(z)}\frac{\delta B(y)}{\delta \pi^a(z)} - \frac{\delta A(x)}{\delta \pi^a(z)}\frac{\delta B(y)}{\delta \varphi_a(z)}\right).$$

The fundamental Poisson brackets are defined as

$$\{\varphi_a(x), \varphi_b(y)\} = 0 = \{\pi^a(x), \pi^b(y)\},$$

$$\{\varphi_a(x), \pi^b(y)\} = \delta_a{}^b \delta^{(D-1)}(\boldsymbol{x} - \boldsymbol{y}).$$

Generalization to infinite degrees of freedom

The Dirac brackets for fields are:

$$\{A(x), B(y)\}^* = \{A(x), B(y)\} + \\ - \int d\boldsymbol{x'} d\boldsymbol{y'} \{A(x), \chi_s(x')\} \mathcal{C}^{ss'}(x', y') \{\chi_{s'}(y'), B(y)\},$$

where $\mathcal{C}^{ss'}(x', y')$ is the inverse of the Dirac matrix such that:

$$\int d\boldsymbol{z}\, \mathcal{C}_{ss''}(x, z) \mathcal{C}^{s''s'}(z, x') = \delta_{ss'} \delta(\boldsymbol{x} - \boldsymbol{x'}).$$

With this we go from a system of N degrees of freedom to a system with infinite degrees of freedom.

Example: Proca theory

Proca theory

The Proca action is:

$$\mathcal{S} = -\int d^4x \left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu\right].$$

Where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Then, separating the terms into space and time, we get

$$\mathcal{S} = \frac{1}{4}\int d^4x \left(2F_{0i}F_{0i} - F_{ij}F_{ij} + 2m^2 A_0 A_0 - 2m^2 A_i A_i\right).$$

The generalized momenta are

$$\pi^i(\boldsymbol{x}) = \frac{\delta L}{\delta(\partial_0 A_i(\boldsymbol{x}))} = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_i(\boldsymbol{x}))} = F_{0i}(\boldsymbol{x}) = -E^i(\boldsymbol{x}),$$

$$\pi^0(\boldsymbol{x}) = \frac{\delta L}{\delta(\partial_0 A_0(\boldsymbol{x}))} = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_0(\boldsymbol{x}))} = 0.$$

Proca theory

The Hamiltonian with the coupled constraint is

$$\tilde{H} = \frac{1}{4}\int d^3x \left(2\pi^i\pi^i + 4\pi^i\partial_i A_0 + F_{ij}F_{ij} - 2m^2 A_0 A_0 + 2m^2 A_i A_i + \lambda\ \phi_1\right),$$

where $\phi_1 = \pi^0 \approx 0$.

We define

$$\{\pi^\mu(\boldsymbol{x}), \pi^\nu(\boldsymbol{y})\} = 0\,,\ \{A_\mu(\boldsymbol{x}), A_\nu(\boldsymbol{y})\} = 0\,,\ \{A_\nu(\boldsymbol{x}), \pi^\mu(\boldsymbol{y})\} = \delta^\mu_\nu\,\delta^3(\boldsymbol{x}-\boldsymbol{y}).$$

Preserving the constraint in time we obtain a secondary constraint

$$\phi_2 = \partial_i\pi^i + m^2 A_0 \approx 0$$

Then, preserving the secondary constraint in time, we are able to find the Lagrange multiplier

$$\lambda = -m^2\partial_i A_i$$

Proca theory

The Dirac matrix is

$$\mathcal{C}_{ss'}(\boldsymbol{x}, \boldsymbol{y}) = \begin{pmatrix} 0 & -m^2\delta^3(\boldsymbol{x}-\boldsymbol{y}) \\ m^2\delta^3(\boldsymbol{x}-\boldsymbol{y}) & 0 \end{pmatrix}.$$

Its inverse is

$$\mathcal{C}^{ss'}(\boldsymbol{x}, \boldsymbol{y}) = \begin{pmatrix} 0 & \frac{\delta^3(\boldsymbol{x}-\boldsymbol{y})}{m^2} \\ -\frac{\delta^3(\boldsymbol{x}-\boldsymbol{y})}{m^2} & 0 \end{pmatrix}.$$

Proca theory

Then the Dirac's brackets are

$$\{A_\mu(\boldsymbol{x}), \pi^\nu(\boldsymbol{y})\}^* \quad = \quad (\delta^\nu{}_\mu - \delta^0{}_\mu \delta^\nu{}_0)\delta^3(\boldsymbol{x} - \boldsymbol{y}),$$

$$\{A_\mu(\boldsymbol{x}), A_\nu(\boldsymbol{y})\}^* = -\frac{1}{m^2}\left(\delta^0{}_\mu \delta^i{}_\nu + \delta^0{}_\nu \delta^i{}_\mu\right)\partial_i\left(\delta^3(\boldsymbol{x} - \boldsymbol{y})\right),$$

$$\{\pi^\mu(\boldsymbol{x}), \pi^\nu(\boldsymbol{y})\}^* = 0.$$

The theory is described by the Hamiltonian

$$H = \frac{1}{4}\int d^3x(2\pi^i\pi^i + 4\pi^i\partial_i A_0 + F_{ij}F_{ij} - 2m^2 A_0 A_0 + 2m^2 A_i A_i).$$

Dirac method for the Morand and Solodukhin action

Study of the kinematics of the model of Morand and Solodukhin

The Morand and Solodukhin action is

$$\mathcal{S}\left[h, B\right] = \int d^4x \left[-\frac{m_1}{2}(h_{\mu\nu}h^{\nu\mu} - h_{\mu}{}^{\mu}h_{\nu}{}^{\nu}) + \right.$$
$$\left. -\frac{m_2}{2}\left(B_{\alpha\beta,\sigma}B^{\alpha\beta,\sigma} - 2B^{\alpha}{}_{\beta,\alpha}B^{\sigma\beta}{}_{,\sigma}\right) + B_{\alpha\beta,}{}^{\mu}\partial_\rho h_{\mu\sigma}\varepsilon^{\sigma\rho\alpha\beta} \right]$$

$$B_{\alpha\beta,\mu} = -B_{\beta\alpha,\mu}$$

Making variations of the action we obtain the following equations

$$m_1\left(h^{\mu\nu} - \eta^{\mu\nu}h\right) + \varepsilon^{\nu\rho\alpha\beta}\partial_\rho B_{\alpha\beta,}{}^{\mu} = 0 \tag{1}$$

$$m_2\left(B^{\alpha\beta,\sigma} - \eta^{\alpha\sigma}B^{\gamma\beta}{}_{,\gamma} + \eta^{\beta\sigma}B^{\gamma\alpha}{}_{,\gamma}\right) - \varepsilon^{\nu\rho\alpha\beta}\partial_\rho h^{\sigma}{}_{\nu} = 0 \tag{2}$$

Study of the kinematics of the model of Morand and Solodukhin

By studying the equations of motion we obtain:

$h = 0$, $\partial_\nu h^{\nu\mu} = 0$, $h_{\mu\nu} = h_{\nu\mu}$, $B^{\gamma\beta}{}_{,\gamma} = 0$, $\partial_\sigma B^{\alpha\beta,\sigma} = 0$, $\partial_\alpha B^{\alpha\beta,\sigma} = 0$, $\varepsilon_{\alpha\beta\sigma\theta} B^{\alpha\beta,\sigma} = 0$

Substituting the above in (2) and (3) we have left

$$m_1 h^{\mu\nu} + \varepsilon^{\nu\rho\alpha\beta} \partial_\rho B_{\alpha\beta,}{}^{\mu} = 0, \tag{3}$$

$$m_2 B^{\alpha\beta,\sigma} - \varepsilon^{\nu\rho\alpha\beta} \partial_\rho h^{\sigma}{}_{\nu} = 0. \tag{4}$$

From this we see how to represent $h_{\mu\nu}$ in terms of $B_{\alpha\beta,\sigma}$ and vice versa.

Study of the kinematics of the model of Morand and Solodukhin

Substituting (4) in (5) we have left

$$\left(\Box - m^2\right) B^{\alpha\beta}{}_{,\sigma} = 0,$$

and substituting (5) in (4),

$$\left(\Box - m^2\right) h_{\mu\nu} = 0,$$

where $m^2 = \frac{m_1 m_2}{2}$ y $\Box = \partial^\mu \partial_\mu$.

Canonical formulation of the Morand and Solodukhin action

Separating the Morand and Solodukhin action in space and time, we obtain

$$\begin{aligned}\mathcal{S} = \int d^4x \, \big[& h_{00} \left(-m_1 h_{ii} + \varepsilon^{ijk}\partial_i B_{jk,0}\right) - h_{l0} \left(-m_1 h_{0l} + \varepsilon^{ijk}\partial_i B_{jk,l}\right) + \\ & + B_{0i,0} \left(-2m_2 B_{ji,j} + 2\varepsilon^{ijk}\partial_j h_{0k}\right) + B_{0i,j} \left(m_2 B_{0i,j} - m_2 \delta_{ij} B_{0k,k} - 2\varepsilon^{ikl}\partial_k h_{jl}\right) \\ & - \frac{m_1}{2}\left(h_{ij}h_{ji} - h_{ii}h_{jj}\right) - \frac{m_2}{2}\left(B_{ij,k}B_{ijk} - B_{ij,0}B_{ij,0} - 2B_{ik,i}B_{jk,j}\right) + \\ & + \varepsilon^{ijk} B_{ij,0}\partial_0 h_{0k} - \varepsilon^{ijk} B_{ij,l}\partial_0 h_{lk} \big] \, .\end{aligned}$$

Let's define the following: $B_{0i,0} \equiv A_i$, $B_{0i,j} \equiv A_{ij}$, $B_{ij,0} \equiv \varepsilon_{ijk} W_k$, $B_{ij,l} \equiv \varepsilon_{ijk} W_{kl}$, $W_k = \frac{1}{2}\varepsilon_{kij} B_{ij,0}$, $W_{kl} = \frac{1}{2}\varepsilon_{kij} B_{ij,l}$

Canonical formulation of the Morand and Solodukhin action

Substituting the definitions in the action we have

$$\mathcal{S} = \int d^4x \left[h_{00}\left(-m_1 h_{ii} + 2\partial_i W_i\right) - h_{l0}\left(-m_1 h_{0l} + 2\partial_i W il\right) + \right.$$
$$+2A_i\varepsilon_{ijk}\left(-m_2 W_{jk} + \partial_j h_{0k}\right) + A_{ij}\left(-m_2\delta_{ij}A_{kk} + m_2 A_{ij} - 2\varepsilon_{ikl}\partial_k h_{jl}\right) +$$
$$\left. -\frac{m_1}{2}\left(h_{ij}h_{ji} - h_{ii}h_{jj}\right) - m_2\left(W_{ij}W_{ij} - W_i W_i\right) + 2W_i\partial_0 h_{0i} - 2W_{ji}\partial_0 h_{ij}\right].$$

Then we can establish the following:

$\pi^{ij} = -2W_{ji}$, $\pi^i = 2W_i$, $\{h_{0i}, \pi^j\} = \delta_i{}^j\delta^3(\boldsymbol{x} - \boldsymbol{y})$,
$\{h_{ij}, \pi^{kl}\} = \delta_i{}^k\delta_j{}^l\delta^3(\boldsymbol{x} - \boldsymbol{y})$

Canonical formulation of the Morand and Solodukhin action

On substitution, we obtain

$$\mathcal{S} = \int d^4x \left[h_{00}\left(-m_1 h_{ii} + \partial_i \pi^i\right) + h_{i0}\left(m_1 h_{0i} + \partial_j \pi^{ij}\right) \right.$$
$$+ A_i \varepsilon_{ijk}\left((\partial_j h_{0k} - \partial_k h_{0j}) - m_2 \pi^{jk}\right) + A_{ij}\left(-m_2 \delta_{ij} A_{kk} + m_2 A_{ij} - 2\varepsilon_{ikl}\partial_k h_{jl}\right)$$
$$\left. - \frac{m_1}{2}\left(h_{ij}h_{ji} - h_{ii}h_{jj}\right) - \frac{m_2}{4}\left(\pi^{ij}\pi^{ji} - \pi^i \pi^i\right) + \pi^i \partial_0 h_{0i} + \pi^{ij}\partial_0 h_{ij} \right].$$

A_{ij} has no dynamics and is associated with a constraint that allows us to determine it. So,

$$A_{ij} = \frac{1}{2m_2}\left(\delta_{ij}\varepsilon_{lmn}\partial_l h_{mn} + 2\varepsilon_{ikl}\partial_k h_{jl}\right)$$

Canonical formulation of the Morand and Solodukhin action

Substituting in the action and rearranging we have:

$$
\begin{aligned}
\mathcal{S} = \int d^4x \, \Big[& h_{00}\left(-m_1 h_{ii} + \partial_i \pi^i\right) + h_{i0}\left(m_1 h_{0i} + \partial_j \pi^{ij}\right) + \\
& + A_i \varepsilon_{ijk}\left(\partial_j h_{0k} - \partial_k h_{0j} - m_2 \pi^{jk}\right) + \frac{1}{2m_2}\delta_{ij}\varepsilon_{lmn}\varepsilon_{ijk}\partial_l h_{mn}\partial_i h_{jk} + \\
& - \frac{1}{2m_2}\left(\partial_k h_{jl}\partial_k h_{jl}\right)\left(\partial_k h_{jl}\partial_l h_{jk}\right) - \frac{m_1}{2}\left(h_{ij}h_{ji} - h_{ii}h_{jj}\right) + \\
& - \frac{m_2}{4}\left(\pi^{ij}\pi^{ji} - \pi^i \pi^i\right) + \pi^i \partial_0 h_{0i} + \pi^{ij}\partial_0 h_{ij} \Big] .
\end{aligned}
$$

Canonical formulation of the Morand and Solodukhin action

We see that the Hamiltonian is

$$
\begin{aligned}
H \quad = \quad & \int d^3x \left[\frac{m_1}{2} \left(h_{ij} h_{ji} - h_{ii} h_{jj} \right) + \frac{m_2}{4} \left(\pi^{ij} \pi^{ji} - \pi^i \pi^i \right) + \right. \\
& - \frac{1}{2m_2} \delta_{ij} \varepsilon_{lmn} \varepsilon_{ijk} \partial_l h_{mn} \partial_i h_{jk} + \frac{1}{2m_2} \left(\partial_k h_{jl} \partial_k h_{jl} \right) \left(\partial_k h_{jl} \partial_l h_{jk} \right) + \\
& \left. + h_{00} \varphi + h_{i0} \varphi^i + A_i \psi^i \right],
\end{aligned}
$$

with

$$
\varphi = m_1 h_{ii} - \partial_i \pi^i, \; \varphi^i = - \left(m_1 h_{0i} + \partial_j \pi^{ij} \right), \; \psi^i = \varepsilon_{ijk} \left(m_2 \pi^{jk} - 2 \partial_j h_{0k} \right)
$$

Canonical formulation of the Morand and Solodukhin action

By preserving constraints in time, we get new constraints

$$\boxed{\chi = \frac{m_2}{2}\pi^{ii}} \qquad \boxed{\chi^i = \frac{m_2}{2}\pi^i + \partial_j h_{ji} - \partial_i h_{jj}} \qquad \boxed{\xi^i = m_2 \varepsilon_{ijk} h_{jk}}$$

By preserving again we can find Lagrange multipliers

$$\boxed{h_{00} = \frac{2}{3} h_{kk} + \frac{2}{3}\frac{1}{m_1 m_2}\varepsilon_{kni}\varepsilon_{kl'm'}\partial_{l'}\partial_n h_{im'}}$$

$$\boxed{h_{i0} = \frac{1}{m_1}\left(\partial_i \pi^{kk} - \partial_j \pi^{ij}\right)}$$

$$\boxed{A_i = \varepsilon_{ijk}\left(\frac{1}{4}\pi^{jk} - \frac{1}{2m_2}\partial_k h_{j0}\right)}$$

canonical formulation of the Morand and Solodukhin action

To count the physical degrees of freedom of the theory we can use the following expression

$$2 \times \begin{pmatrix} \text{Number of physical} \\ \text{degrees of freedom} \end{pmatrix} = \begin{pmatrix} \text{total number of} \\ \text{canonical variables} \end{pmatrix} + \\ - \begin{pmatrix} \text{number of original} \\ \text{second-class constraints} \end{pmatrix}$$

canonical formulation of the Morand and Solodukhin action

After finding the Dirac matrix and its inverse, we are able to calculate the Dirac brackets

$$\{h_{0i}(\boldsymbol{x}), \pi^{j}(\boldsymbol{y})\}^{*} =$$
$$\frac{1}{m^2}\left(\frac{1}{3}\partial_i\partial_j p_{kk}^{(m)} - \frac{1}{3}\partial_i\partial_k p_{kj}^{(m)} - \frac{1}{2}\partial_j\partial_k p_{ki}^{(m)} + \frac{1}{2}\partial_k\partial_k p_{ij}^{(m)}\right)\delta^3(\boldsymbol{x}-\boldsymbol{y})$$

$$\{h_{0i}(\boldsymbol{x}), h_{kl}(\boldsymbol{y})\}^{*} = \frac{1}{m_1}\left(\frac{1}{2}\partial_l p_{ik}^{(m)} + \frac{1}{2}\partial_k p_{il}^{(m)} - \frac{1}{3}\partial_i p_{kl}^{(m)}\right)\delta^3(\boldsymbol{x}-\boldsymbol{y})$$

$$\{h_{ij}(\boldsymbol{x}), \pi^{kl}(\boldsymbol{y})\}^{*} = \left(\frac{1}{2}\delta_{il} p_{jk}^{(m)} + \frac{1}{2}\delta_{jl} p_{ik}^{(m)} - \frac{1}{3}\delta_{kl} p_{ij}^{(m)}\right))\delta^3(\boldsymbol{x}-\boldsymbol{y})$$

$$\{\pi^{ij}(\boldsymbol{x}), \pi^{k}(\boldsymbol{y})\}^{*} =$$
$$\frac{2}{m_2}\left(\partial_k p_{ij}^{(m)} + \frac{1}{3}\delta_{ij}\left(\partial_l p_{lk}^{(m)} - \partial_k p_{ll}^{(m)}\right) - \frac{1}{2}\partial_j p_{ik}^{(m)} - \frac{1}{2}\delta_{kj}\partial_l p_{li}^{(m)}\right)\delta^3(\boldsymbol{x}-\boldsymbol{y})$$

canonical formulation of the Morand and Solodukhin action

$$\boxed{\{h_{0i}(\boldsymbol{x}),\pi^{kl}(\boldsymbol{y})\}^* = 0} \quad \boxed{\{h_{ij}(\boldsymbol{x}),\pi^{k}(\boldsymbol{y})\}^* = 0}$$

$$\boxed{\{h_{ij}(\boldsymbol{x}),h_{kl}(\boldsymbol{y})\}^* = 0} \quad \boxed{\{\pi^{ij}(\boldsymbol{x}),\pi^{kl}(\boldsymbol{y})\}^* = 0}$$

$$\boxed{\{\pi^{i}(\boldsymbol{x}),\pi^{j}(\boldsymbol{y})\}^* = 0} \quad \boxed{\{h_{0i}(\boldsymbol{x}),h_{0j}(\boldsymbol{y})\}^* = 0}$$

The Hamiltonian of the theory is

$$H = \int d^3x \left[\frac{m_1}{2}\left(h_{ij}h_{ji} - h_{ii}h_{jj}\right) + \frac{m_2}{4}\left(\pi^{ij}\pi^{ji} - \pi^{i}\pi^{i}\right) + A_{ij}\varepsilon_{ikl}\partial_k h_{jl}\right]$$

canonical formulation of the Morand and Solodukhin action

The time derivatives of the phase space variables are

$$\dot{h}_{ij} \quad = \quad \frac{m_2}{2}\pi^{ij} + \partial_j h_{0i}$$

$$\dot{h}_{0i} = \partial_k h_{ik}$$

$$\dot{\pi}^{ij} \quad = \quad -\left(m_1 h_{ij} + 2\varepsilon_{jrl}\partial_r A_{li}\right)$$

$$\dot{\pi}^{i} \quad = \quad m_1 h_{0i} + 2\varepsilon_{ikr}\partial_k A_r$$

Conclusions

Conclusions

In this work we carried out the canonical formulation in flat space-time of an alternative model for linearised massive gravity in (3+1) dimensions proposed by Morand and Solodukhin.

We found that the field $h_{\mu\nu}$ has 5 independent components, like the field $B_{\alpha\beta,\sigma}$.Then these fields describe a theory with 5 degrees of freedom, so it is a good candidate for a massive theory of spin-2.

When applying the Dirac method to the theory, we found that all constraints were second class, so there are no generators for gauge transformations. Therefore this theory is not gauge invariant.

Conclusions

We found that the theory is written in terms of the Dirac brackets and by the Hamiltonian

$$H = \int d^3x \left[\frac{m_1}{2} \left(h_{ij} h_{ji} - h_{ii} h_{jj} \right) + \frac{m_2}{4} \left(\pi^{ij} \pi^{ji} - \pi^{i} \pi^{i} \right) + A_{ij} \varepsilon_{ikl} \partial_k h_{jl} \right]$$

Using the obtained algebra in terms of the Dirac brackets and this Hamiltonian, we saw that the covariant equations of motion found by the Lagrangian formulation are reproduced as expected.