Dirichlet boundary value problem for fractional monogenic functions

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Abstract

We solve the Dirichlet boundary value problem over distinguishing domains for Clifford fractional-monogenic functions in $\mathbb{R}^n$ (Riemann-Liouville sense).

Fractional calculus

$(D_{a^+}^{\alpha} f)(x)$ is the fractional Riemann-Liouville derivative of order $\alpha > 0$

$$(D_{a^+}^{\alpha} f)(x) = \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt \quad (1)$$

where $n = [\alpha]+1$. The Riemann-Liouville fractional integral of order $\alpha > 0$ is given by (2)

$$(I_{a^+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} \quad (2)$$

$I_{a^+}^{\alpha}(L_1)$ is the class of functions f represented by (2) of a summable function; $f = I_{a^+}^{\alpha}\varphi$ where $\varphi \in L_1(a,b)$.

Fractional Clifford Analysis

$\{e_1, ..., e_d\}$ is the standard basis of Euclidean vector space in $\mathbb{R}^d$.

$\mathbb{R}_{0,d}$ is the associated real Clifford algebra with the following multiplication rules:

$$\begin{cases} e_0 := 1, \\ e_i e_j + e_j e_i = -2\delta_{ij}, \quad i,j = 1, ..., d \end{cases}$$

Vector space $\mathbb{R}_{0,d}$ is given by the set $\{e_A : A \subseteq \{1, ..., d\}\}$ with $e_A = e_{\alpha_1 \alpha_2 \ldots \alpha_r}$ where $1 \leq \alpha_1 < \cdots < \alpha_r \leq d$, $0 \leq r \leq d$. Consider $f : \Omega \subset \mathbb{R}^{1+d} \to \mathbb{R}_{0,d}$ whose representation is given by

$$f = \sum_A e_A f_A$$

where f_A are real valued functions. Clifford fractional (Riemann-Liouville) Cauchy-Riemann operator is defined by (3)

$$D_+^{\alpha} = D_+^{(\alpha_0, \ldots, \alpha_d)} = \sum_{j=0}^{d} e_j D_{x_j^+}^{\frac{1+\alpha_j}{2}} \quad (3)$$

This operator is known as the fractional Dirac operator.

Definition

A $\mathbb{R}_{0,d}$-valued function f is called **left fractional monogenic if it satisfies $D_+^{\alpha} f = 0$ on Ω.**

Theorem 1

Consider the Clifford valued function f given by $f(x_0, ..., x_d) = \sum_A e_A f_A(x_0, ..., x_d)$ where $f \in AC^2(\Omega)$ and

$$\begin{aligned} f_A(x_0, x_1, ..., x_d) &= f_A(x_0, x_1, ..., x_d) \\ &= \cdots = f_A(x_0, x_1, ..., x_d) \end{aligned}$$

(The red on each component means the respective direction of the derivative). Then these statements are true:

- $D_{x_j^+}^{\frac{1+\alpha_j}{2}} \left[D_{x_j^+}^{\frac{1+\alpha_j}{2}} f_A \right] = D_{x_j^+}^{1+\alpha_j} f_A, \ \forall\ j = 0, 1, ..., d$
- $D_{x_j^+}^{\frac{1+\alpha_j}{2}} \left[D_{x_i^+}^{\frac{1+\alpha_i}{2}} f_A \right] = D_{x_i^+}^{\frac{1+\alpha_i}{2}} \left[D_{x_j^+}^{\frac{1+\alpha_j}{2}} f_A \right]$, $\forall i \neq j \in \{0, 1, ..., d\}$

By Theorem 1 and the multiplication rules, we obtain the fractional Laplace operator:

$$D_+^{\alpha}(\overline{D_+^{\alpha}} f) = \Delta_+^{\alpha} = \sum_{i=0}^{d} D_{x_i^+}^{1+\alpha_i}$$

From this factorization we have that if $f = \sum_A e_A f_A$ is a fractional monogenic function, then $\Delta_+^{\alpha} = 0$ and so f_A is a fractional monogenic function.

Dirichlet boundary value problem in $\mathbb{R}^{n+1}$ for fractional monogenic functions

Key idea: Let $\Omega \in \mathbb{R}^{n+1}$ be a domain that can be decomposed into fibres with small parts, which carry certain properties. Consider the Euclidean space $\mathbb{R}^{n+1}$ and choose $1 \leq \mu \leq n+1$ and choose μ indices $k_1, ..., k_\mu$.

The Fibre: the $\mu-$dimensional fibres in the $k_1, ..., k_\mu-$directions in a given bounded domain Ω are the intersections of Ω and the $\mu-$ dimensional planes.

$$\Pi = \{x = (x_0, ..., x_n) : x_{k_1} = c_1, ..., x_{k_\mu} = c_\mu\}$$

Distinguishing part: Ω can be decomposed into $\mu-$dimensional fibres in the $k_1, ..., k_\mu-$directions if there exists a $(1+n-\mu)-$dimensional part $S_{k_1 \ldots k_\mu}$ of $\partial\Omega$ with certain properties.

The subset $S_{k_1 \ldots k_\mu}$ is called the **distinguishing part** for the corresponding decomposition of Ω.

Theorem 2

- Consider $\mathbb{R}_{0,n}-$fractional monogenic functions u.
- Under conditions given by Theorem 1, the following Dirichlet boundary value problem is solved:
  - Let $S_{k_1 \ldots k_\mu}$ be the distinguishing part of the certain decomposition of Ω $(1 \leq \mu \leq n+1)$.
  - Knowing the boundary values of 2^{n-1} components u_A on the whole boundary, the rest of the 2^{n-1} can be recovered from:
  - their values on the $n-$dimensional distinguishing part S_0 of the boundary, for instance, the $(n-1)-$dimensional distinguishing part S_B, where B represents the $n-1$ complementary directions of the fibre.

Following a reduction process, the final reduction of the distinguishing part until we get the component u_0 is completely determined by its value at the point $S_{0123\cdots n}$ of the boundary.

Example

Decomposition of $\Omega \subset \mathbb{R}^3$ in distinguishing parts: A particular case is the cylindrical domain $\Omega \subset \mathbb{R}^3$. It can be decomposed in:

- The lower covering surface, a distinguishing part in the x_0-direction representing the $1-$dimensional fibre S_0.
- S_{01}, the distinguishing part in the x_0, x_1-directions (the $2-$dimensional fibre).
- S_{012}, the distinguishin part x_0, x_1, x_2-directions (the $3-$dimensional fibre).

$$\begin{bmatrix} D_{x_0^+}^{\frac{1+\alpha_0}{2}} u_0 & -D_{x_1^+}^{\frac{1+\alpha_1}{2}} u_1 & -D_{x_2^+}^{\frac{1+\alpha_2}{2}} u_2 \\ D_{x_1^+}^{\frac{1+\alpha_1}{2}} u_0 & D_{x_0^+}^{\frac{1+\alpha_0}{2}} u_1 & D_{x_2^+}^{\frac{1+\alpha_2}{2}} u_{12} \\ D_{x_2^+}^{\frac{1+\alpha_2}{2}} u_0 & D_{x_0^+}^{\frac{1+\alpha_0}{2}} u_2 & -D_{x_1^+}^{\frac{1+\alpha_1}{2}} u_{12} \\ -D_{x_2^+}^{\frac{1+\alpha_2}{2}} u_1 & D_{x_1^+}^{\frac{1+\alpha_1}{2}} u_2 & D_{x_0^+}^{\frac{1+\alpha_0}{2}} u_{12} \end{bmatrix} = \mathbf{0}$$

This is the Cauchy-Riemann system written in matrix form and it is integrable by the Theorem 2. Consider $p_0 = D_{x_1^+}^{\frac{1+\alpha_1}{2}} u_1 + D_{x_2^+}^{\frac{1+\alpha_2}{2}} u_2$, $p_1 = -D_{x_0^+}^{\frac{1+\alpha_0}{2}} u_1 - D_{x_2^+}^{\frac{1+\alpha_2}{2}} u_{12}$ and $p_2 = D_{x_0^+}^{\frac{1+\alpha_0}{2}} u_2 + D_{x_1^+}^{\frac{1+\alpha_1}{2}} u_{12}$. This system is compatible:

$$\begin{aligned} & D_{x_1^+}^{\frac{1+\alpha_1}{2}} p_0 - D_{x_0^+}^{\frac{1+\alpha_0}{2}} p_1 = D_{x_0^+}^{\frac{1+\alpha_0}{2}} D_{x_2^+}^{\frac{1+\alpha_2}{2}} u_{12} + D_{x_0^+}^{1+\alpha_0} u_1 + \\ & D_{x_1^+}^{\frac{1+\alpha_1}{2}} D_{x_2^+}^{\frac{1+\alpha_2}{2}} u_2 = -D_{x_2^+}^{1+\alpha_2} u_1 + D_{x_1^+}^{\frac{1+\alpha_1}{2}} D_{x_2^+}^{\frac{1+\alpha_2}{2}} u_2 + D_{x_0^+}^{\frac{1+\alpha_0}{2}} D_{x_2^+}^{\frac{1+\alpha_2}{2}} u_{12} \\ & \qquad = -D_{x_2^+}^{\frac{1+\alpha_2}{2}} \left[D_{x_2^+}^{\frac{1+\alpha_2}{2}} u_1 - D_{x_1^+}^{\frac{1+\alpha_1}{2}} u_2 - D_{x_0^+}^{\frac{1+\alpha_0}{2}} u_{12} \right] = 0 \end{aligned}$$

Case I

If $u_1 = (x - x_2)^{\alpha_2 - 1}$ and $u_2 = (x - x_1)^{\alpha_1}$ with $\alpha_0 = 3, \alpha_1 = 5$, $\alpha_2 = 4$, $x_0 = 1.5$, $x_1 = 3$ and $x_2 = 2$, Figure 1 shows u_1, u_2 and u_{12}.

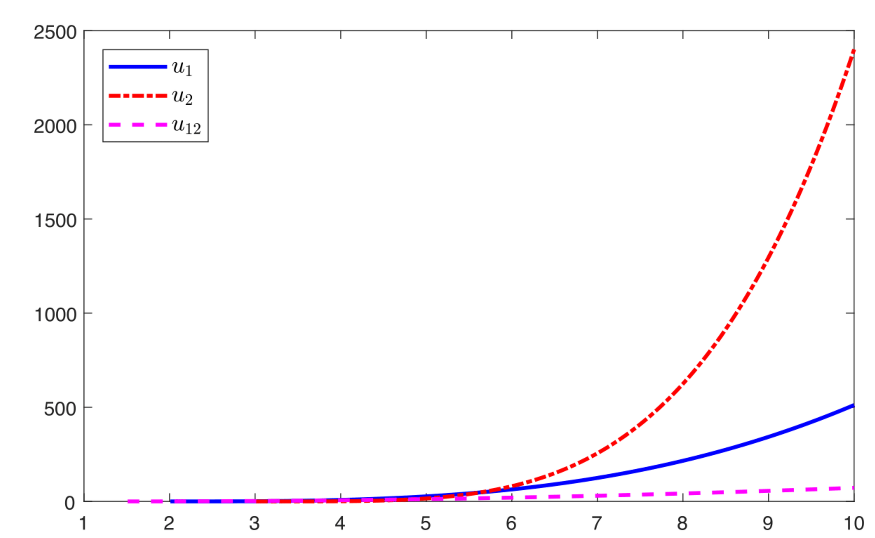

Figure 1: Explicit solutions, case I

Case II

If now $u_1 = (x - x_2)^{2\alpha_2 - 1}$, $u_2 = (x - x_1)^{2\alpha_1 - 1}$ with the same values for α_0, α_1, α_2, x_0, x_1 and x_2, then Figure 2 shows u_1, u_2 and u_{12}.

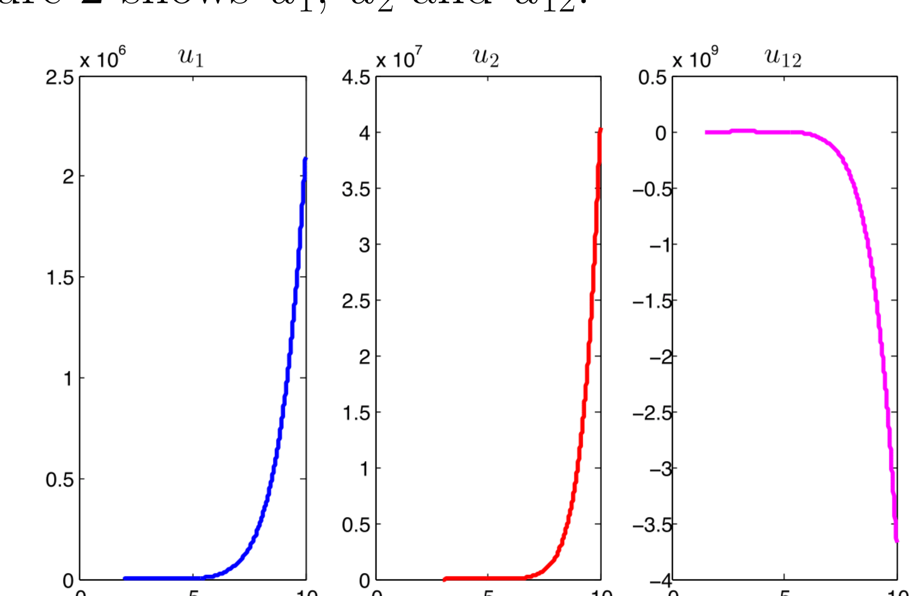

Figure 2: Explicit solutions, case II