Problem Set 3 (Thursday)

The Large N Planar Limit

Considering the semiclassical limit ($N \to \infty$), only the planar diagrams contribute to the free energy. Let $F_{\text{planar}}(g)$ be the generating functional for planar quadrangulations (using the quartic potential)

$$F_{\text{planar}}(g) = \lim_{N\to\infty} N^{-2} \log\left[\int dM \; \exp(-N \; tr(M^2/2 - g/4M^4))\right] \tag{1}$$

Writing its derivative as a function of the resolvent,

(1) Deduce that -

$$F'_{\text{planar}} = \frac{d}{dg} F_{\text{planar}}(g) = 1/4 \int dx \; x^4 \; \rho(x) \tag{2}$$

with $\rho(x)$ as the density of eigenvalues.

(2) The explicit form for ρ has been found to be

$$\rho(x) = \frac{1}{2\pi} g(b^2 - a^2)(a^2 - x^2)^{1/2}$$

with

$$a^2 = \frac{2\,(1 - \sqrt{1-12g})}{3g} \;, \quad b^2 = \frac{\sqrt{1-12g} + 2}{3g}$$

(3) Compute F'_{planar} and show that it has a singularity at $g = g_c$ of the form

$$F'_{\text{planar}}(g) = \text{regular part} \; + \; cst \; (g_c - g)^{3/2}(1 + \mathcal{O}(g_c - g))$$

You may use a formal calculational software to compute this (only if you spent too much time with pen calculations).

(4) Alternatively, show that the third derivative of F_{planar} diverges when $g \to g_c = 1/12$ as

$$F'''_{\text{planar}}(g) \propto (g_c - g)^{-1/2}(1 + \mathcal{O}(g_c - g))$$

by looking at the behaviour of the corresponding x integral near the endpoint $x = a$ of the eigenvalue distribution.

(5) Deduce that $F_{\text{planar}}(g)$ has a singularity as

$$(g_c - g)^{5/2}$$

and that the number of (unmarked) quadrangulations with K squares, N_K, scales as

$$N_K \propto 12^K K^{-7/2}$$

(Hint: Use the fact that you can write

$$N_K = \frac{1}{2i\pi} \oint dx \; F_{\text{planar}}(g) \; g^{-K-1}$$

with a small counter-clockwise contour around the origin, and deform the contour around the singularity at g_c.

Playing with Planar Folding

Starting with a Heisenberg like discrete lattice model for bending ($F = \tilde{\kappa} \sum_{\langle i,j \rangle} \boldsymbol{n}_i \cdot \boldsymbol{n}_j$), the continuum limit of the free energy (using the Monge gauge) in terms of the height field (h) becomes

$$F_{\text{cont}} = \int \mathrm{d}^2x \frac{\tilde{\kappa}}{2} |\nabla \boldsymbol{n}|^2 \to \frac{\tilde{\kappa}}{2} \int \mathrm{d}^2x \left[H^2 - 2K_G\right] \tag{3}$$

where $H = \nabla^2 h$ is the mean curvature and $K_G = \det(\partial_i \partial_j h)$ is the gaussian curvature. The Gauss-Bonnet theorem states that the integrated gaussian curvature over a closed surface (with no boundary) is a constant, given by the *Euler characteristic* of the surface (χ).

$$\int \mathrm{d}^2 x \sqrt{g} K_G = 2\pi\chi \tag{4}$$

allowing us to disregard its contribution to the energy as long as the topology of the manifold is fixed.

(1) In a Monge patch, show that K_G can be written as a total derivative.

$$K_G = -\frac{1}{2}\epsilon_{im}\epsilon_{jn}\partial_m\partial_n(\partial_i h \partial_j h) \tag{5}$$

This directly implies that the gaussian curvature integrates to a boundary term and does not affect the Euler-Lagrange equations obtained upon minimization of the free energy. So the only important contribution is the Willmore energy.

Going back to the discrete case, considering an extreme simplification of the problem, take the plaquette normals to be Ising spins. As discussed in the lecture, aligned spins would correspond to the sheet being flat, anti-aligned to the flat folded phase.

(1) Take a regular square lattice, with the normal "spins" being defined on the square plaqettes. For an $L\times W$ lattice, show that the number of all possible folding configurations is 2^{L+W-2}. Generalize this to higher dimensional embeddings, i.e. for a two dimensional square lattice embedded in $\mathbb{R}^d$, show that the number of configurations is $(2(d-1))^{L+W-2}$.

(2) Compute the thermodynamic entropy of folding (per face) as

$$q = \lim_{L,W\to\infty} \frac{1}{LW} \ln Z \tag{6}$$

where Z is the partition function (in this case just the number of folding configurations). If $Z \sim z^{LW}$ asymptotically, then $q = \ln z$. For the square lattice, show that $q = 0$ (unlike the nontrivial result for the triangular lattice).

(3) Now take a sheet of paper with a triangular lattice printed on it and verify that the local folding rules lead to only 22 folded configurations for each hexagonal plaquette. This provides the equivalence of the planar folding problem and the 11-vertex model. By colouring the edges, also verify that the total number of folded configurations is the total number of 3-edge colourings of the lattice.

On Counting by Bijections

The Schaeffer bijection admits a natural extension (due to Bouttier, Di Francesco and Guitter) to vertex-pointed bipartite maps on the sphere. Given such a map M (whose vertices are considered white), one endows M with its geodesic labelling with respect to the marked vertex v_0 (of label 0), that is, each vertex v receives a label $\ell(v)$ giving the length of a shortest path in M connecting v_0 to v; each edge has labels differing by 1. Then one inserts a black vertex in each face of M, and for each edge $e = \{u, v\}$ of M (considered as a black edge), one draws a new (blue) edge from the extremity of larger label to the black vertex in the face on the right of e (seeing e as traversed from the extremity of smaller label to the extremity of larger label). Finally one deletes v_0 and the original (black) edges. Let G be the resulting embedded graph, see Figure 1 for an example.

(1) Show that G is a tree covering all the black and white vertices except for v_0 (Hint: show that G covers all these vertices, has one more vertex than the number of edges, and is acyclic).

(2) In addition show that -

- The degree of each black vertex is half the degree of the corresponding face of M,
- The minimum label over the white vertices of G is 1 and
- For two white vertices u and v that are consecutive in clockwise order around a black vertex, the label of v is at most the label of u plus 1.

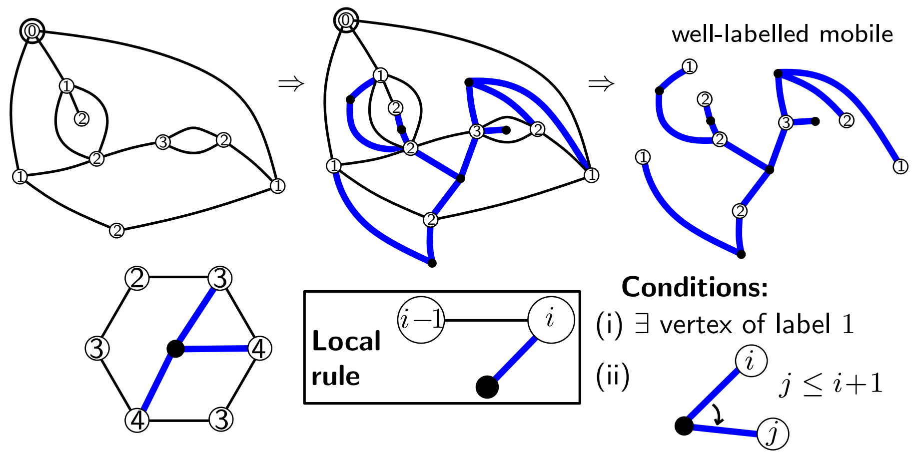

Figure 1: An example of the Bouttier - Di Francesco - Guitter bijection.

A tree (with labelled white vertices, unlabelled black vertices) satisfying these conditions is called a *well-labelled mobile*.

(3) Using the bouncing path argument (as was used for the Schaeffer bijection), show that M is recovered from G as follows: for each edge $e = \{b, w\}$ of G (with b the black extremity and w the white extremity), insert a "leg" at w just after e in clockwise order around w. Then each time $\ell(w) \geq 2$, complete the leg into an edge reaching the next corner of the label $\ell(w) - 1$ in a counterclockwise walk around G starting at e. Finally, create a new vertex of label 0 outside of G, connect all the legs of label 1 to this vertex, and delete all the original edges of G and all black vertices.

(4) Define a *rooted well-labelled mobile* as a well-labelled mobile with a marked edge. Let a_n be the number of rooted well-labelled mobiles with n black vertices all of degree 3 (via the bijection, these correspond to hexangulations with n faces, with a marked vertex and a marked edge). Let $A(z) = \sum_{n\geq 1} a_n z^n$ be the corresponding generating function. Show that $A(z)$ satisfies the following functional equation

$$A(z) = 10z(1 + A(z))^3 \tag{7}$$

(Hint: It could help to consider erasing the labels, and instead storing the δ-label $\ell(v) - \ell(u)$, for each pair of successive white neighbours u, v in clockwise order around a black vertex. The sum of the three δ-labels around each black vertex is clearly 0, and each δ-label is at most 1 according to the condition stated in second question).

(5) Find an explicit formula for a_n (using the Lagrange inversion formula given in problem set 2). Deduce from it that the number of rooted hexangulations with n faces is equal to

$$\frac{(3n)!}{n!(2n+2)!}10^n \tag{8}$$

(6) Show more generally that, for any $p \geq 2$, the number of rooted $2p$-angulations with n faces is equal to

$$\frac{(pn)!}{n!((p-1)n+2)!}\binom{2p-1}{p-1}^n \tag{9}$$