Problem Set 2 (Tuesday)

Fixing the Matrix Measure

Given a random matrix model, as the action is given as the invariant scalar trace of potential function of the matrix, it is typically invariant under a large set of symmetry transformations that are symmetries of the particular ensemble chosen. For example in the unitary ensemble of $N \times N$ hermitian matrices M, the action is invariant under unitary transformations $M \to U^\dagger M U$ for any unitary matrix $U \in \mathsf{U}(N)$. Additionally, the flat Lebesgue measure $\mathrm{d}M$ is also invariant under both a change of basis and a unitary transformation. This results in a large amount of degeneracy in computing the path integral. In order to be able to perform any real computations we need to be factor out (in some sense) the volume of this symmetry ("gauge") group.

(1) What are the number $\mathcal{N}$ of real "gauge fixing" conditions that are required to bring a Hermitian matrix M of size N to its diagonal form (i.e. $M = U^\dagger \Lambda U$ for some unitary U and $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$)?

(2) For the unitary case, as $\mathsf{U}(N)$ is a connected Lie group, without loss of generality, we can study its action on any matrix (by conjugation) around the identity ($U = \mathrm{Id}$). As any hermitian matrix M can be brought to diagonal form by some (non-unique) unitary transformation, an infintesimal change of a diagonal matrix Λ under an infinitesimal unitary transformation is given by

$$\Lambda \to \delta_\epsilon M = U^\dagger \Lambda U, \quad U = 1 + \epsilon \tag{1}$$

As $U^\dagger U = 1$, ϵ is an antihermitian ($\epsilon^\dagger = -\epsilon$) generator of the group (i.e. an element of the Lie algebra $\mathfrak{u}(N)$). How many (real) independent components does ϵ have? Compare this with $\mathcal{N}$.

(3) In order to change variables from M_{ij} to $\{\lambda_i, U_{ij}\}$, one needs to compute the jacobian determinant of this transformation. Using the previous calculation, show that $\delta M = \delta\Lambda + [\delta U, \Lambda]$ (δU is the same as ϵ from above). Hence compute the big $\mathcal{N} \times \mathcal{N}$ Jacobian derivative matrix $\mathcal{J}$ given by

$$\mathcal{J} = \frac{\delta M}{\delta U} \tag{2}$$

and the change in the measure is $\mathrm{d}M = |\det(\mathcal{J})| \prod_i^N \mathrm{d}\lambda_i \mathrm{d}U_{\mathrm{Haar}}$, the final part being the invariant *Haar measure* on the unitary group.

(4) Show that the determinant of $\mathcal{J}$ is the square of the *Vandermonde determinant* corresponding to the diagonal matrix Λ.

$$\det(\mathcal{J}) = \Delta(\Lambda)^2, \quad \Delta(\Lambda) = \prod_{i<j} (\lambda_i - \lambda_j) \tag{3}$$

(5) For the real symmetric ensemble, the corresponding symmetry group is the orthogonal group $\mathsf{O}(N)$. Correspondingly repeat the above calculation and show that the jacobian determinant involved in the change of variables is now $|\Delta(\Lambda)|$.

The Plateau Problem: Minimal Surfaces

Consider a $2d$ liquid membrane (soap film) embedded into three dimensional Euclidean space ($\mathbb{R}^3$). Representing the abstract $2d$ surface as Σ, $\boldsymbol{X} : \Sigma \to \mathbb{R}^d$ is the embedding. For a soap film drawn over a fixed rigid frame, the primary energetic contribution is from the surface tension. Choosing local coordinates $\{\xi\}$ in a chart, for a given Riemannian metric g, the free energy functional is given by

$$F = \sigma_0 A = \sigma_0 \int_\Sigma \mathrm{d}^2\xi \sqrt{|g|} \tag{4}$$

where σ_0 is the surface tension and A is the total area of the surface. The metric g is induced via the embedding and is hence a dynamical variable that must itself be obtained as a solution of the minimization of F. Writing $\boldsymbol{t}_\alpha = \partial_\alpha \boldsymbol{X}$ as a basis of the tangent space, the metric tensor componenets are given by $g_{\alpha\beta} = \boldsymbol{t}_\alpha \cdot \boldsymbol{t}_\beta$. Hence minimizing the energy is equivalent to minimizing the area of the surface given boundary conditions. This is the old and famous *Plateau problem* and its solutions are said to be local *minimal surfaces* as they have vanishing mean curvature.

(1) Consider a variation of the embedding $\boldsymbol{X} \to \boldsymbol{X} + \delta\boldsymbol{X}$ with Dirichlet boundary conditions (so the variation on the boundary vanishes: $\delta\boldsymbol{X}|_{\partial\Sigma} = 0$). Compute the corresponding variation of $|g|$ and show that

$$\delta\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}g^{\alpha\beta}\delta g_{\alpha\beta} \tag{5}$$

(Hint: Use the identity $\ln\det M = \operatorname{Tr}\ln M$ for any finite matrix M)

(2) Relate the variation of the metric tensor $\delta g_{\alpha\beta}$ to the variation in the embedding $\delta\boldsymbol{X}$ to get

$$\delta F = \sigma_0 \delta A = \int_\Sigma \mathrm{d}^2\xi \sqrt{|g|}g^{\alpha\beta}(\partial_\alpha \boldsymbol{X} \cdot \delta\partial_\beta \boldsymbol{X}) \tag{6}$$

(3) Integrate by parts and use the boundary condition to set the boundary term to zero. Demanding the free energy variation vanish for arbitrary $\delta\boldsymbol{X}$, derive the corresponding Euler-Lagrange equation to be

$$\partial_\alpha(\sqrt{|g|}g^{\alpha\beta}\partial_\beta\boldsymbol{X}) = 0 \tag{7}$$

(4) Using the definition of the covariant derivative $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\eta}V^\eta$, show that th eabove equation is nothing but $g^{\alpha\beta}\nabla_{alpha}\partial_\beta\boldsymbol{X} = \Delta_g\boldsymbol{X} = 0$, where Δ_g is the scalar Laplace-Beltrami operator with metric g.

(5) As the extrinsic curvature tensor (also called the second fundamental form) is given by $K_{\alpha\beta} = \boldsymbol{n}\cdot\nabla_\alpha \boldsymbol{t}_\beta$ ($\boldsymbol{n}$ being the normal to the surface), show that

$$\Delta_g\boldsymbol{X} = 0 \implies \operatorname{Tr}(K_{\alpha\beta}) = H = 0 \tag{8}$$

H being the mean curvature of the surface.

Enumerative Combinatorics of some Maps

Let $k \geq 2$. Define a k-ary tree as a planar map with a unique face, such that every vertex has either degree $k+1$ (these vertices shall be called nodes) or degree 1 (these vertices being called leaves). Let there also be a marked leaf (which would play the role of a root). Let a_n be the number of k-ary trees with n nodes. We write the associated generating function as

$$A(z) = \sum_{n\geq 1} a_n z^n \tag{9}$$

(1) Using recursive decomposition, show that $A(z)$ satisfies the following functional relation

$$A(z) = z(1 + A(z))^k \tag{10}$$

(2) The classical Lagrange inversion theorem ensures that if a generating function $A(z)$ is known to satisfy a functional equation of the form $A(z) = z\phi(A(z))$ for some power series $\phi(y)$ then n times the n^{th} coefficient of $A(z)$ (as a formal power series) is equal to the $(n-1)^{\text{th}}$ coefficient in the power series of $\phi(y)^n$. Using this formula, find an explicit expression for a_n (now involving some binomial coefficients).

(3) Verify that this formula for $k = 2$ matches with the number of rooted planar trees with n edges (which is given by the Catalan number). Find a bijective explanation for this.

(4) For a connected graph $G = (V, E)$ with a marked vertex v_0, consider the geodesic labelling with respect to v_0, that is, give every vertex $v \in V$ a label $d(v_0, v)$ which is the length of the shortest path connecting v_0 to v. Show that this labelling is the unique labelling $\ell(v)$ of the vertices of G (with labels in $\mathbb{Z}$) satisfying the following properties:

- For every edge $\{u, v\} \in E$ of G we have $|\ell(u) - \ell(v)| \leq 1$,
- Every vertex $v \neq v_0$ has a neighbour with a smaller label and
- The label of v_0 is 0.

Moreover show that the graph is bipartite if and only if there is no edge $\{u, v\}$ with $\ell(u) = \ell(v)$.

(5) For $n, k \geq 0$ let $a_{n,k}$ now be the number of rooted bipartite maps (so all the faces have an even degree) with n edges and an outer face of degree $2k$. The associated generating function is given as

$$A(z,u) = \sum_{n,k \geq 0} a_{n,k} z^n u^k \tag{11}$$

Show that $A(z,u)$ satisfies the following functional equation:

$$A(z,u) = 1 + zuA(z,u)^2 + zu\frac{A(z,u) - A(z,1)}{u-1} \tag{12}$$