Problem Set 1 (Monday)

Random Matrices and Perturbative Feynman Diagrams

Like functional integrals, matrix integrals can also be computed in a perturbative series, whose terms are products of correlation functions of a free gaussian theory. Interpreting the resulting Feynman graphs as geometrical objects gives rise to the connection between random matrix models and problems involving random geometries like in quantum gravity, string theory and statistical physics on random lattices. To illustrate this, consider the Hermitian matrix model with $M_{ji} = \overline{M}_{ij}$ and a potential $V(M)$. The partition function is then simply given by

$$Z = \int \mathrm{d}M \; e^{-N\mathrm{Tr}[V(M)]} \tag{1}$$

where $\mathrm{d}M = \prod_i \mathrm{d}M_{ii} \prod_{i<j} \mathrm{d}\Re(M_{ij}) \; \mathrm{d}\Im(M_{ij})$ is the flat Lebesgue measure on the unitary ensemble. The corresponding free energy is then given by the logarithm of the partition function ($\ln Z$).

(1) For a cubic potential of the form

$$V(M) = \frac{1}{2}M^2 - \frac{g}{3}M^3, \tag{2}$$

write down the various diagrams contributing to the free energy (i.e, only the connected vacuum diagrams) at order g^2. Collect them by powers of N (using Wick's theorem to compute the diagrams) and draw them on a surface with the appropriate genus.

(2) Repeat the same for the quartic model with the potential

$$V(M) = \frac{1}{2}M^2 - \frac{g}{4}M^4 \tag{3}$$

Now instead of complex matrices, consider symmetric real matrices. In particular, let us look at the Gaussian Orthogonal Ensemble (GOE) with $N \times N$ real symmetric matrices (so $M_{ij} = M_{ji}$). The appropriate Lebesgue measure now is given by $\mathrm{d}M = \prod_i \mathrm{d}M_{ii} \prod_{i<j} \mathrm{d}M_{ij}$. As the theory is gaussian, the potential to be used is just

$$V_0(M) = \frac{1}{2}M^2 \tag{4}$$

with the partition function defined just as before.

(3) Compute the gaussian propagator and show it to be

$$\langle M_{ij} M_{kl} \rangle_0 = 1/(2N)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{5}$$

(3) Consider just as before the cubic and quartic potentials (equations (2) and (3)) but now for the case of symmetric matrices. Write down the diagrams for the free energy upto order g^2 (Bonus: go to order g^4 as well). Show that unlike in the hermitian case, one additionally generates unoriented diagrams (i.e. fat graphs that can be drawn only on unoriented surfaces like the Klein bottle or the Projective Plane).

(3) Once again use Wick's theorem to compute the N factors associated with the unoriented diagrams.

Random Flight Models and Polymers

Consider a simple unbiased random walk in d-dimensions. As we shall be working in continuous space ($\mathbb{R}^d$) but discrete time, this is more appropriately called a random *flight*. Starting from the origin, at each time instant n, take a step $\boldsymbol{r}_n$ of fixed length ℓ but oriented randomly (uniformly on the sphere). This corresponds to the simplest model of a polymer, the so-called ideal gaussian or freely jointed chain model, where the "time" corresponds to the length along the backbone of the polymer (when discrete, it would correspond to the number of monomers). As each step is taken independent of the previous one (making this a *Markov chain*), this model neglects crucial properties of real polymers like self-avoidance and semi-flexibility. Nevertheless, it is still a useful simple model that illustrates some of the most basic properties of random walks and polymers.

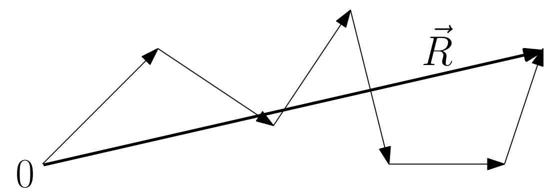

Figure 1: A cartoon of a configuration of the random flight model

(1) As each step is of constant length, the bond probability density $p(\boldsymbol{r})$ is given by

$$p(\boldsymbol{r}) = \frac{1}{c}\delta(|\boldsymbol{r}| - \ell); \quad \int \mathrm{d}^d r \; p(\boldsymbol{r}) = 1 \tag{6}$$

where c is a normalization constant. Show that $c = S_{d-1}\ell^{d-1}$ where $S_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ is the volume of a unit $d-1$-sphere.

(2) Let $\boldsymbol{R}$ be the end to end distance for an n-step flight. Show that $\langle \boldsymbol{R} \rangle = 0$ and $\langle |\boldsymbol{R}|^2 \rangle = nd\ell^2$ (ℓ is said to be the Kuhn length of the polymer chain). Hence we immediately see that the average size of the polymer (or random flight) $R = \sqrt{\langle |\boldsymbol{R}|^2 \rangle} \sim \sqrt{n}$. This scaling is characteristic of diffusion, one needs to wait four times longer to go twice as far.

(3) Using the fact that each step is chosen independently, write down the probability density $P_n(\boldsymbol{R})$ of a flight of n-steps having an end-to-end vector $\boldsymbol{R}$.

(4) Fourier transform this probability density to obtain the characteristic function:

$$\tilde{P}_n(\boldsymbol{k}) = \int \mathrm{d}^d R \; e^{-i\boldsymbol{k}\cdot\boldsymbol{R}} P_n(\boldsymbol{R}) \tag{7}$$

and show that $\tilde{P}_n(\boldsymbol{k}) = [\tilde{p}(\boldsymbol{k})]^n$. This is now in principle the exact solution for all n and dimensionality d.

(5) Restricting now to $d = 3$, compute $\tilde{p}(\boldsymbol{k})$ and show it to be equal to $\sin(k\ell)/k\ell$.

(6) For long polymer chain (i.e in the long time limit), one can take a continuum limit to obtain the universal properties of this model. Writing $n = t/\tau$, keeping t fixed, we send both $\ell \to 0$ and $\tau \to 0$ simultaneously keeping $\ell^2/6\tau = D$ constant (the numerical factors are chosen to set D as the standard diffusion constant in three dimensions). In this nontrivial scaling limit, show that

$$P_t(\boldsymbol{R}) = \frac{1}{(4\pi D t)^{3/2}} e^{-|\boldsymbol{R}|^2/(4Dt)} \tag{8}$$

This is nothing but the fundamental solution to the diffusion equation in three dimensions. The fact that the end-to-end vector has a gaussian distribution is a direct demonstration of the *Central Limit Theorem*. For a long polymer with a large number of monomer units $n \gg 1$, one would equivalently write (in terms of n now instead of t),

$$P_n(\boldsymbol{R}) \propto e^{-3|\boldsymbol{R}|^2/2n\ell^2} \tag{9}$$

(7) In thermodynamic equilibrium, given the postulate of apriori equal probabilities, for a macrostate of the polymer chain (for fixed but large n) labelled by $\boldsymbol{R}$, the number of microstate configurations corresponding to it is $\Omega(\boldsymbol{R}) \propto P_n(\boldsymbol{R})$. Using the Boltzmann formula for the entropy, compute $S(\boldsymbol{R})$. As there is no internal energy in this model, only the entropy contributes to the Helmholtz free energy $F(\boldsymbol{R})$. Show that

$$F(\boldsymbol{R}) = \text{const.} + \frac{1}{2}KR^2; \quad K = \frac{3k_B T}{n\ell^2} \tag{10}$$

This has exactly the same form as that of a Hookean spring, though now the "spring constant" K depends on temperature. This is precisely what is meant by an *entropic spring*. Note that as temperature increases, the polymer effectively becomes *stiffer* instead of softer, which we would näively expect for regular crystalline solids. This counterintutive behaviour is essentially because all work done on the system goes to decrease its entropy as there is no potential energy to store the work as internal energy.

Exercises on Planar Graphs and Maps

Given a surface S and a graph $G = (V, E)$, a drawing of G on S has vertices drawn as points on S and edges drawn as closed curves on S (whose ends correspond to the positions of the two extremities). A graph is called planar if it admits a proper drawing (i.e. without edge crossings) in the plane (which we shall actually consider to be the sphere having included the point at infinity).

(1) Using the Euler relation ($|E| = |V| + |F| - 2$), show that any triangulation with n vertices has $3n - 6$ edges and any quadrangulation with n vertices has $2n - 4$ edges.

(2) The girth of a graph is defined as the length of the shortest cycle within the graph. Using the Euler relation show that a planar graph with n vertices, m edges and girth $g \geq 3$ satisfies

$$m \leq \frac{g}{g-2}(n-2) \tag{11}$$

(3) Using the previous result, deduce that in particularK_5 (the complete graph on 5 vertices) and $K_{3,3}$ (the complete bipartite graph on $3+3$ vertices) are non-planar.

(4) Show that any loop-less triangulation is 2-connected (i.e. it has no separating vertex). Similarly show that any simple triangulation is 3-connected (i.e. it has no separating vertex nor separating vertex-pair).

(5) A bipartite graph is a graph whose vertices can be partitioned into black and white vertices such that every edge connects a black to a white vertex. It is easy to see that a graph is bipartite if and only if every cycle of edges has even length. Show that a planar map is bipartite if and only if all its faces have an even degree.

(6) Let $\mathcal{M}_n$ be the set of rooted maps with n edges, and let M be a random map under the uniform distribution on $\mathcal{M}_n$. Show that the expected number of vertices (and similarly the expected number of faces) of M is given by

$$\mathbb{E}[|F|] = \mathbb{E}[|V|] = \frac{n}{2} + 1 \tag{12}$$