Approximation of Eigenvalues

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Abstract

In this poster, we explain the main theorem of [1] which describes the spectrum of pencils in terms of the zeros of certain functions. After that, we will explain a general example where the classical variational principles may fail and this one works.

Introduction

The idea of variational principles is to show that isolated eigenvalues are solutions to certain maximization or minimization problems. Roughly speaking, if T is a self-adjoint linear operator which is bounded from below, then the classical variational principle gives the following formula for the eigenvalues of T below its essential spectrum:

$$\lambda_n = \min_{\substack{L\subset\mathcal{D}(T)\\ \dim L=n}} \max_{\substack{x\in L\\ \|x\|=1}} \langle Tx, x\rangle .$$

Even if this procedure does not immediately lead to an expression that can be evaluated easily numerically, it can be used to obtain at least bounds for the eigenvalues.

Pencils and the Classical Variational Principle

We will show a generalized variational principle that applies to certain operator valued functions. A *pencil* is a function $\mathcal{T} : \Omega \subset \mathbb{R} \to \mathfrak{C}(H)$; here, Ω is a subset of $\mathbb{C}$ and $\mathfrak{C}(H)$ is the set of closed operators on a Hilbert space H. Probably the easiest non-trivial example is the following. Fix a closed linear operator $T \in \mathfrak{C}(H)$ and define

$$\mathcal{T}(\lambda) := T - \lambda. \tag{1}$$

Note that in this case

$$\lambda \in \sigma_p(T) \quad\Longleftrightarrow\quad 0 \in \sigma_p(\mathcal{T}(\lambda)).$$

So a somewhat natural definition of the spectrum of a pencil is to say that $\lambda \in \Omega$ belongs to the spectrum of a pencil if and only if 0 is in the spectrum of $\mathcal{T}(\lambda)$. Moreover, for $x \in \mathcal{D}(T)$ with $x \neq 0$, let us define $p(x)$ as the number λ where $\langle \mathcal{T}(\lambda)x, x\rangle = 0$. Then, if T is selfadjoint, it is easy to see that the classical variational principle of min-max (see [3]) in the case of the pencil (1) can be rewritten as

$$\lambda_n = \min_{\substack{L\subset\mathcal{D}\\ \dim L=n}} \max_{\substack{x\in L\\ x\neq 0}} p(x). \tag{2}$$

So it seems natural to look for a variational principle for a general pencil in terms of zeros of certain functionals.

Generalized Variational Principle

Before addressing the variational principle, we will need the following definitions and notation.

Notation

- $\Delta \subseteq \mathbb{R}$ will be an open, half-open or closed interval with end points α and β, such that $-\infty \leq \alpha < \beta \leq \infty$.
- From now on $\mathcal{T}$ is a pencil with domain $\Delta \subset \mathbb{R}$ and such that $\mathcal{T}(\lambda)$ is self-adjoint for all λ in Δ.
- Let $A(H \to H)$ be a self-adjoint operator with spectral resolution E_λ and projection valued measure $E(\cdot)$. If B is a Borel set of $\mathbb{R}$, we set $\mathcal{L}_B(A) := \mathrm{Rg}(E(B))$.
- For all λ in Δ, we denote the dimension of $\mathcal{L}_{(-\infty,0)}(\mathcal{T}(\lambda))$ by $k_-(\lambda)$.

Definition

- Given a pencil $\mathcal{T}$, we define $t_0(\lambda)[x,y] := \langle \mathcal{T}(\lambda)x, y\rangle$ and $t_0(\lambda)[x] := \langle \mathcal{T}(\lambda)x, x\rangle$, where x, y belong to $\mathcal{D}(\mathcal{T}(\lambda))$ and λ belong to Ω. If $t_0(\lambda)[\cdot,\cdot]$ is closable we denote its closure by $t(\lambda)[\cdot,\cdot]$.
- The spectrum, essential spectrum and discrete spectrum of a pencil $\mathcal{T}$ is defined respectively as

$$\sigma(\mathcal{T}) := \{\lambda \in \Omega : 0 \in \sigma(T(\lambda))\},$$
$$\sigma_{ess}(\mathcal{T}) := \{\lambda \in \Omega : 0 \in \sigma_{ess}(T(\lambda))\},$$
$$\sigma_d(\mathcal{T}) := \{\lambda \in \Omega : 0 \in \sigma_d(T(\lambda))\}.$$

Hypotheses

Let $\mathcal{T}$ be a pencil. Now we will introduce the main hypotheses that will be needed for the generalized variational principle.

(A1) Either $\mathcal{D}(\mathcal{T}(\lambda)) \equiv \mathcal{D}$ is independent of λ or the form $t_0(\lambda)[\cdot]$ is closable and there exists a dense subspace of H such that

$$\mathcal{D}(\mathcal{T}(\lambda)) \subset \mathcal{D} \subseteq \mathcal{D}(t(\lambda)), \qquad \forall \lambda \in \Delta.$$

(A2) The pencil $\mathcal{T}$ is continuous in the norm resolvent topology. Moreover, if the first case of (A1) happens, then the function $\lambda \mapsto t_0(\lambda)[x]$, is continuous in Δ, for every $x \in \mathcal{D}$ fixed. If the second one occurs, the same would be true for the function $\lambda \mapsto t(\lambda)[x]$.

(A3) For all $x \in \mathcal{D}$ with $x \neq 0$, the function $t_0(\cdot)[x]$, or $t(\cdot)[x]$ if the second case of (A1) occurs, is decreasing at value zero in Δ, i.e., if $t(\lambda_0)[x] = 0$ for some $\lambda_0 \in \Delta$, this implies

$$t_0(\lambda)[x] > 0 \quad\text{for}\quad \lambda < \lambda_0,$$
$$t_0(\lambda)[x] < 0 \quad\text{for}\quad \lambda > \lambda_0.$$

(A4) There exists $\gamma \in \Delta$ such that $\dim \mathcal{L}_{(-\infty,0)}(\mathcal{T}(\gamma)) < \infty$.

Main Theorem

Theorem. *Assume that $\mathcal{T}$ satisfies the assumptions* (A1)-(A4) *and that Δ' is not empty. If Δ is closed at the left end point, set $k := k_-(\alpha)$. Otherwise there exists an $\alpha' \in \Delta'$ such that $(\alpha, \alpha') \subset \rho(\mathcal{T})$ and we set $k := k_-(\alpha')$. In both cases, k is a finite number.*
Then $\sigma(\mathcal{T}) \cap \Delta'$ consists only of a finite or infinite sequence of isolated eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N$, $N \in \mathbb{N}_0 \cup \{\infty\}$, counted with their multiplicity, and for $n = 1, \ldots, N$:

$$\lambda_n = \mu_{k+n} = \min_{\substack{L\subset\mathcal{D}\\ \dim L=k+n}} \max_{\substack{x\in L\\ x\neq 0}} p(x), \tag{3}$$

$$\lambda_n = \mu'_{k+n} = \max_{\substack{L\subset H\\ \dim L=k+n-1}} \inf_{\substack{x\in\mathcal{D}, x\neq 0\\ x\perp L}} p(x). \tag{4}$$

Moreover

$$N = \begin{cases} k_-(\beta) - k + \dim\ker\mathcal{T}(\beta) & \text{if } \beta \in \Delta \text{ and } \sigma_{ess}(\mathcal{T}) = \emptyset,\\ k_-(\lambda_e) - k & \text{otherwise.}\end{cases} \tag{5}$$

(i) *If $N = \infty$, then $\lim_{n\to\infty} \lambda_n = \lambda_e$.*

(ii) *If $N < \infty$ and $\sigma_{ess}(\mathcal{T}) = \emptyset$, then $\mu_n = \infty$ for $n > k + N$.*

Schur Decomposition

Consider the block operator

$$M = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}.$$

in $H \oplus H$, with H a Hilbert space. We denote $\mathcal{D}(A)$ and $\mathcal{D}(T)$ as the domains of A and T respectively. Assume that both A and T are densely defined. Also assume that

(D1) Both A and D are self-adjoint.

(D2) The operator B is bounded.

now notice that M is self-adjoint, since both A and D are so. Moreover, according to [1] we have the following,

$$M - \lambda = \begin{pmatrix} Id & B(D-\lambda)^{-1} \\ 0 & Id \end{pmatrix} \begin{pmatrix} S(\lambda) & 0 \\ 0 & D-\lambda \end{pmatrix} \begin{pmatrix} Id & 0 \\ (D-\lambda)^{-1}B^* & Id \end{pmatrix},$$

where $\lambda \in \rho(D)$ and $S(\lambda) = A - \lambda - (T-\lambda)^{-1}$, one of the Schur complements of the matrix M. Due to the previous decomposition, for all $\lambda \in \rho(D)$ we have that $\lambda \in \sigma(M)$ if and only if $\lambda \in \sigma(S)$. Therefore, to claculate the eigenvalues of M in $\rho(D)$ we just need to find the eigenvalues of the pencil $S(\cdot)$ using the generalized variational principle.

References

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