The Ayón-Beato–García Regular Black Hole and Topology Change

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1. INTRODUCTION

In 1998 Eloy Ayón-Beato and Alberto García proposed a regular black hole solution that has generated a lot of discussion until the present day. It was obtained in the scope of general relativity coupled to nonlinear electrodynamics. This work analyses such solution. In particular, it is shown that the metric has a de Sitter asymptotic when the radial coordinate r approaches zero, and a Reissner-Nordstrøm one when r tends to infinity; this is consistent with the behaviour of the electric field $E(r)$ which presents one maximum value. All this can be understood from a global perspective in terms of topology change.

2. BASIC EQUATIONS

Consider the action of general relativity coupled to nonlinear electrodynamics

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi}R - \frac{1}{4\pi}L(F)\right), \tag{1}$$

where R is the scalar curvature of spacetime and $L(F)$ is the Lagrangian for nonlinear electrodynamics, which depends on the Lorentz invariant

$$F = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\vec{E}^2. \tag{2}$$

Einstein's equations come from varying the action with respect to the inverse metric $g^{\mu\nu}$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{3}$$

where the general energy momentum tensor for nonlinear electrodynamics is

$$T_{\mu\nu} = -\frac{1}{4\pi}\left(\frac{dL}{dF}F_{\mu}{}^{\beta}F_{\nu\beta} + L(F)g_{\mu\nu}\right). \tag{4}$$

For a static, spherically symmetric geometry (given by the metric $ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega^2$), the corresponding electric field is

$$\begin{aligned} E(r) &= \frac{r^2}{4q}\sqrt{4R^{\mu\nu}R_{\mu\nu} - R^2} \\ &= \frac{r^2}{4q}\sqrt{\frac{\left(2 - 2f(r) + r^2f''(r)\right)^2}{r^4}}, \end{aligned} \tag{5}$$

where $R^{\mu\nu}$ is the Ricci tensor. This relation is very useful, since it allows one to directly calculate the electric field from a given metric.

An alternative formulation of nonlinear electrodynamics is the FP dual formalism, where an auxiliary field and its invariant are defined as

$$P_{\mu\nu} = \frac{dL}{dF}F_{\mu\nu}, \quad P = -\frac{1}{4}P_{\mu\nu}P^{\mu\nu} = \left(\frac{dL}{dF}\right)^2F; \tag{6}$$

then, a canonical description of the system can be obtained by means of a Legendre transformation

$$H = 2F\frac{dL}{dF} - L(F), \tag{7}$$

where $H = H(P)$ is called the structural function. In terms of this, the energy momentum tensor reads

$$T_{\mu\nu} = -\frac{1}{4\pi}\left[\frac{dH}{dP}P_{\mu}{}^{\beta}P_{\nu\beta} + \left(2P\frac{dH}{dP} - H(P)\right)g_{\mu\nu}\right]. \tag{8}$$

Also, one obtains a simpler relation analogous to (5), namely

$$M'(r) = -r^2H(P), \tag{9}$$

where $M(r)$ is a function defined by

$$f(r) = 1 - \frac{2M(r)}{r}, \tag{10}$$

and r and P are connected by $P = q^2/2r^4$

3. THE AYÓN-BEATO–GARCÍA REGULAR BLACK HOLE

A black hole is said to be regular if the invariants $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$, $R^{\mu\nu}R_{\mu\nu}$ and R are all finite everywhere in spacetime.

The regular black hole proposed by Ayón-Beato and García [1] is described by the metric

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2mr^2}{(r^2+q^2)^{3/2}} + \frac{q^2r^2}{(r^2+q^2)^2}\right)dt^2 \\ &+ \left(1 - \frac{2mr^2}{(r^2+q^2)^{3/2}} + \frac{q^2r^2}{(r^2+q^2)^2}\right)^{-1}dr^2 + r^2d\Omega^2, \end{aligned} \tag{11}$$

which is a suitable manipulation of the Reissner-Nordström metric, where q acts as a regularizing parameter. Asymptotic expansions of $f(r)$ yield

$$\begin{aligned} f(r) &\xrightarrow[r\to\infty]{} 1 - \frac{2m}{r} + \frac{q^2}{r^2} + O\left(\frac{1}{r^4}\right), \\ f(r) &\xrightarrow[r\to 0]{} 1 + \left(\frac{1}{q^2} - \frac{2m|q|}{q^4}\right)r^2 + O\left(r^4\right). \end{aligned} \tag{12}$$

At infinity one recovers the Reissner-Nordström metric, so m and q are the mass and the electric charge of the system, respectively. The asymptotic behavior around $r = 0$ is identified with a de Sitter spacetime, where the cosmological constant is given by

$$\Lambda = 3\left(\frac{2m|q|}{q^4} - \frac{1}{q^2}\right). \tag{13}$$

This spacetime describes a black hole if $g^{11} = -g_{00} = f(r)$ vanishes for some r; this happens for the relative value

$$|q| \leq 0.652m. \tag{14}$$

The electric field acting as the source of this geometry is

$$E(r) = qr^4\left(\frac{r^2 - 5q^2}{(r^2+q^2)^4} + \frac{15}{2}\frac{m}{(r^2+q^2)^{7/2}}\right), \tag{15}$$

which has the following qualitative behavior

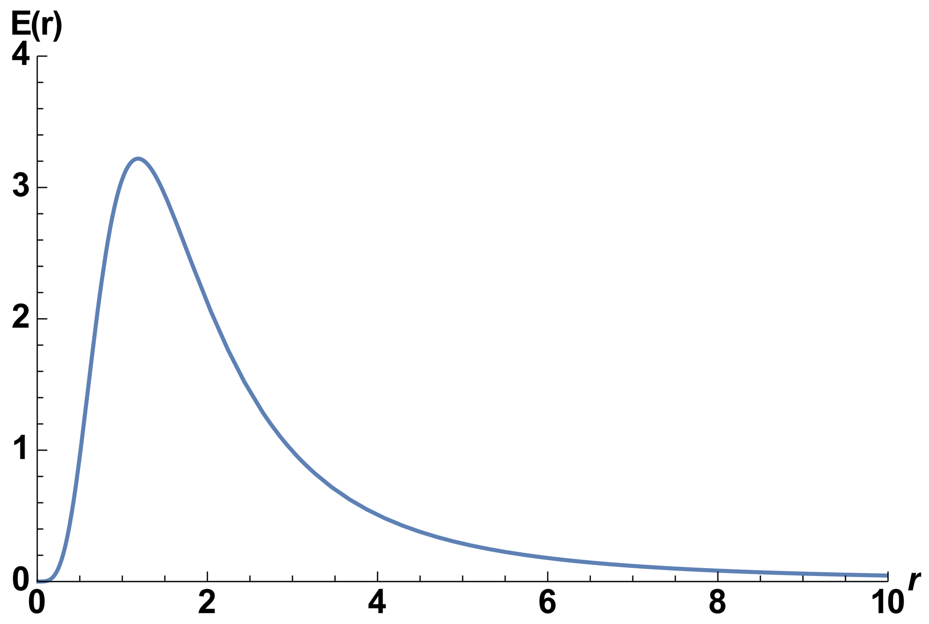

FIG I Ayón-Beato-García electric field for $q = 1$ and $m = 5$, these values satisfy the inequality of the black hole regime.

Finally, for weak fields the structural function of the system behaves as

$$\begin{aligned} H(P) = -P &- \frac{3\left(2q^2\right)^{1/4}P^{5/4}}{s} + 6\sqrt{2q^2}P^{3/2} \\ &+ \frac{15\left(2q^2\right)^{3/4}P^{7/4}}{2s} - 30q^2P^2 + O\left(P^{9/4}\right) \end{aligned} \tag{16}$$

where the first term corresponds to the Maxwell theory, but the following terms indicate that $H(P)$ cannot be regarded as some weak field limit of the Euler-Heisenberg Lagrangian, like the Born-Infeld theory does.

4. TOPOLOGY CHANGE IN SPHERICAL SYMMETRY

The above solution seems to contradict a no-go theorem proved by K. A. Bronnikov long before [2], it reads

Theorem 1 *The field system of equation (1) having a Maxwell asymptotic ($L \to 0$, $dL/dF \to -1$ as $F \to 0$), does not admit a static spherically symmetric solution with a regular center and a nonzero electric charge.*

As I. Dymnikova pointed out [3], if one considers the energy density of the system

$$\rho = T_0^0 = \frac{1}{4\pi}\left[\frac{dL}{dF}E(r)^2 - L(F)\right] \tag{17}$$

and the fact that $F \to 0$ near the center, then

$$\rho = T_0^0 = \frac{1}{4\pi}L(F); \tag{18}$$

therefore, it is not possible that $L \to 0$ as $F \to 0$, for the energy density is maximal at the origin of coordinates. To see this, calculate

$$-\frac{r}{2}\frac{dT_0^0}{dr} = T_0^0 - T_2^2 = \frac{1}{4\pi}\frac{dL}{dF}E(r)^2 \tag{19}$$

and express the WEC in terms of the components of the energy-momentum tensor

$$T_0^0 \geq 0, T_0^0 - T_k^k \geq 0,\ k = 1, 2, 3 \tag{20}$$

the above leads to $T_0^0 \geq 0$ and $dT_0^0/dr \leq 0$ for all r.

I. Dymnikova also showed [3] that for the system in (1) with a static, spherical symmetry the WEC always implies a de Sitter asymptotic at approaching a regular center. To prove this, consider (19) near the center

$$T_0^0 - T_2^2 = \frac{1}{4\pi}\frac{dL}{dF}E(r)^2 \longrightarrow 0 \Longrightarrow T_0^0 = T_2^2 \tag{21}$$

which means that $T_0^0 = T_1^1 = T_2^2 = T_3^3$ around $r = 0$. Such energy momentum tensor is a characteristic feature of a de Sitter spacetime.

Note that the topology of de Sitter spacetime is $\mathbb{R} \times \mathbb{S}^3$, whereas the topology for Reissner-Nordström is $\mathbb{R} \times \mathbb{R} \times \mathbb{S}^2$. Therefore, the topology of the spacelike slices changes from $\mathbb{R} \times \mathbb{S}^2$ (open) at infinity to $\mathbb{S}^3$ (closed) approaching the origin of coordinates.

5. TOPOLOGY CHANGE IN GENERAL

We see that the Ayón-Beato and García regular black hole gives a particular example of topology change, a way to avoid singularities in spacetime. This is explained by A. Borde [4] in the following

Theorem 2 *Suppose that there is a spacetime $\mathcal{M}$ that*
A. contains an eventually future-trapped surface τ
B. Obeys the null convergence condition
C. is null geodesically complete
D. Is future causally simple
Then there is a compact slice to the causal future of τ

Roughly speaking, the theorem states that for regular black holes satisfying the WEC and a causality condition there is always a topology change from open (at spatial infinity) to closed. Singularities are avoided becouse of this compact slice, since it violates one of the hypotheses of the Penrose singularity theorem [5].

The global structure of a portion of a regular black hole such as the Ayón-Beato–García one is pictured in the following diagram

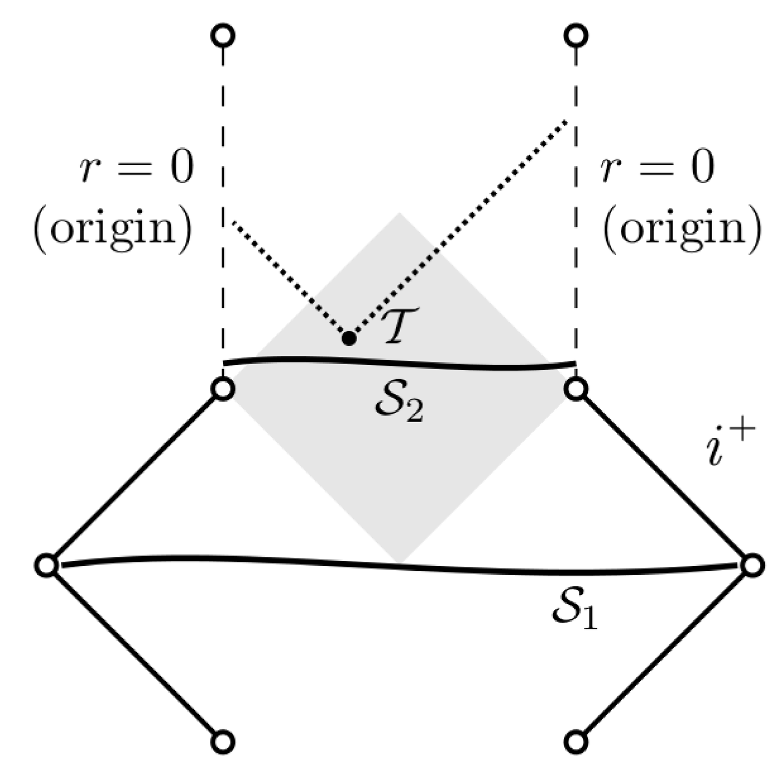

FIG II Part of the maximally extended spacetime of a regular black hole such as the Ayón-Beato–García one [4].

6. CONCLUSIONS

We derived the basic equations relating general relativity and nonlinear electrodynamics; those were applied to the Ayón-Beato–García regular black hole. The behavior of the electric field and the structural function shows that the electrodynamics is Born-Infeld-Like, and no more physical interpretation can be given to it.

We also showed that the avoidance of singularities in this black hole can be understood as the spherically symmetric case of topology change. The fulfillment of the WEC condition is crucial and is what makes this solution special among the others presented by the authors.

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