Puiseux Expansions in Linear Chain Models

Nicolás Escobar Velásquez

Universidad de los Andes

Perturbation Theory in Finite Dimensional Spaces

Consider a finite dimensional, complex vector space V. Let g be a complex-valued parameter and $H(g)$ be an operator on V that depends on g as a polynomial:

$$H(g) = H_0 + gH_1 + g^2H_2 + \cdots + g^mH_m \tag{1}$$

for some operators H_i. There are several things we can say about the behavior of the eigenvalues of $H(g)$ as g varies [1]:

- The number of distinct eigenvalues is constant, except for a finite set of values of g, called *exceptional points*.
- Outside of those exceptional points, the multiplicity of each eigenvalue is constant.
- Each eigenvalue is a continuous function of g. They are analytic except at the exceptional points.

Puiseux Series

Around each exceptional point, analytic continuation along a region which completely surround the analytic point induces a permutation of the eigenvalues. Such permutation decomposes in disjoint cycles, called λ-groups:

$$\sigma = \left(\lambda_{i_1}\lambda_{i_1}\ldots\lambda_{i_{k_1}}\right)\ldots\left(\lambda_{j_1}\lambda_{j_1}\ldots\lambda_{j_{k_n}}\right) \tag{2}$$

If λ_i is in a cycle of length k, then it's a branch of a k-valued, analytic function defined on a punctured disk around the exceptional point. Let a be the exceptional point. Then it follows that the function

$$f_i(z) = \lambda_i(z^k + a) \tag{3}$$

is analytic in a punctured disk around zero. But, unlike λ, it's single-valued, so it must admit a Laurent decomposition. However, f must be continuous even at a, so it's actually a Taylor decomposition:

$$f_i(z) = \sum_{n=0}^{\infty} c_n z^n \tag{4}$$

Consequently, λ_i admits a decomposition [2]:

$$\lambda_i(g) = \sum_{n=0}^{\infty} c_n(\sqrt[k]{g-a})^n \tag{5}$$

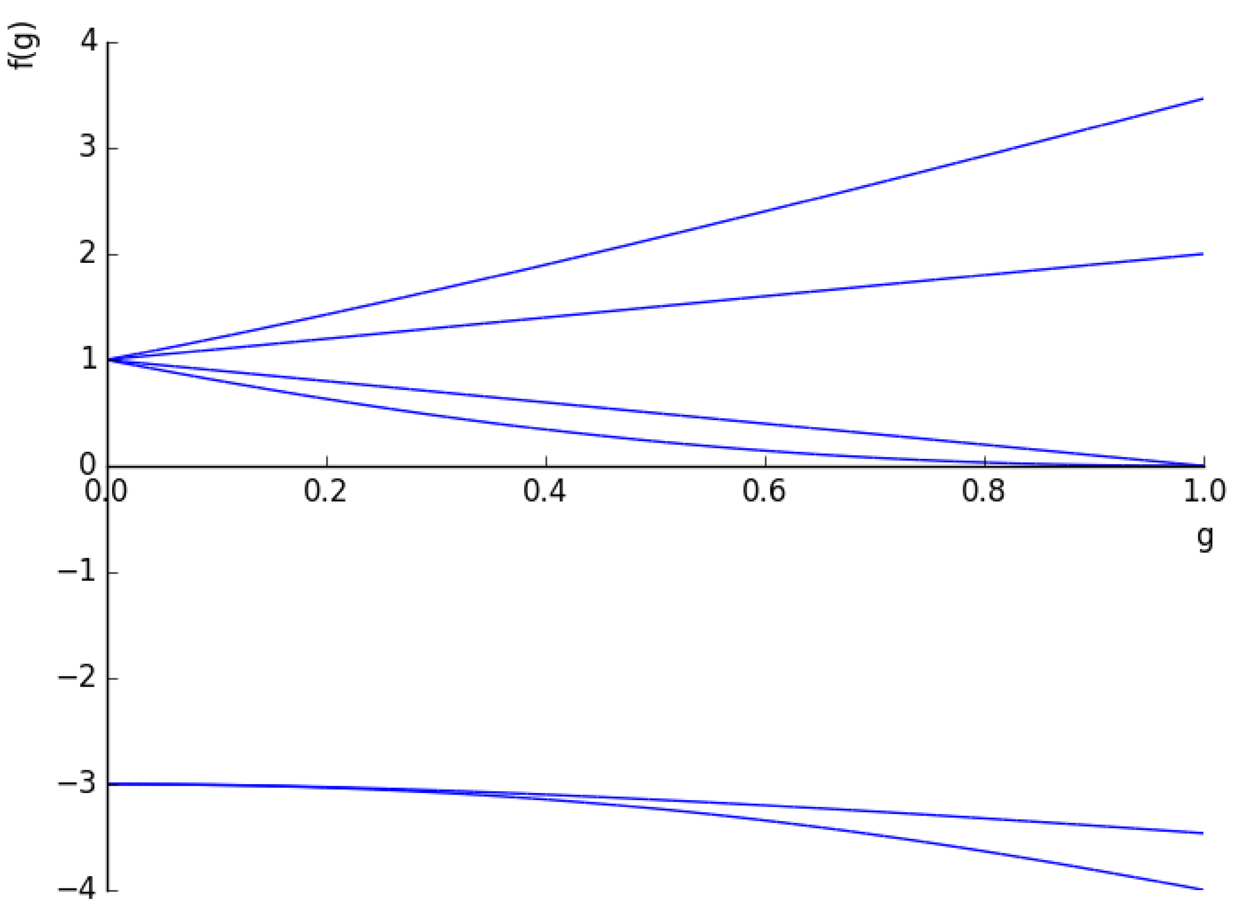

Figure 1: Spectrum of the Ising model for $N = 3$. From top to bottom, we have λ_2,λ_3, λ_1,λ_6, λ_5 and λ_4

.

Ising Model

The Ising model is an example of a parametrized operator on a finite-dimensional complex vector space. For N-sites, it's defined on $\mathbb{C}^{2^N}$, according to:

$$H(g) = -\sum_{i=1}^{N}\sigma_i^x\sigma_{i+1}^x - g\sum_{i=1}^{N}\sigma_i^z \tag{6}$$

The parameter g is proportional to the strength of the external magnetic field. We know by physical arguments [3] that zero is an exceptional point of H. We would like to characterize the associated permutation of the spectrum.

We consider first a chain with 3 sites, to illustrate the theory. The eigenvalues are given by:

$$\lambda_1 = -g + 1 \tag{7}$$
$$\lambda_2 = g + 2\sqrt{g^2 + g + 1} - 1 \tag{8}$$
$$\lambda_3 = g + 1 \tag{9}$$
$$\lambda_4 = -g - 2\sqrt{g^2 - g + 1} - 1 \tag{10}$$
$$\lambda_5 = g - 2\sqrt{g^2 + g + 1} - 1 \tag{11}$$
$$\lambda_6 = -g + 2\sqrt{g^2 - g + 1} - 1 \tag{12}$$

which are plotted in Figure 1. We can see that λ_2 and λ_5 are branches of the same function, the same as λ_4 and λ_6. Yet, none of them collide at $g = 0$, so the λ-groups are all trivial.

Other, more interesting, exceptional points can be found for complex values of g, for instance $g = -\frac{1}{2} - \frac{\sqrt{3}i}{2}$. There, λ_2 and λ_5 form a λ-group, creating meaningful Puiseux expansions. For λ_2, the first non-vanishing coefficients are:

$$c_0 = -\frac{3}{2} - \frac{\sqrt{3}i}{2}$$
$$c_2 = -0.938365889960524 - 0.824826797842661i$$
$$c_4 = -0.311879058800719 - 0.819010304101336i$$

In the general case, we can think of the spectrum as [4]

$$\epsilon = \pm 2\sqrt{g^2 + 1 - 2g\cos(ka)} \tag{13}$$

for appropriate values of k. Hence, the exceptional values are at $e^{\pm ika}$.The first non-trivial coefficient is

$$c_2 = -\frac{-2e^{ik} + 2\cos(k)}{2\sqrt{e^{2ik} - 2e^{ik}\cos(k) + 1}}. \tag{14}$$

References

[1] Konrad Knopp.
Elements of the Theory of Functions, volume 1.
Courier Corporation, 1952.

[2] Tosio Kato.
Perturbation theory for linear operators, volume 132.
Springer Science & Business Media, 2013.

[3] Subir Sachdev.
Quantum phase transitions.
Physics world, 12(4):33, 1999.

[4] Jacek Dziarmaga.
Dynamics of a quantum phase transition: Exact solution of the quantum ising model.
Physical review letters, 95(24):245701, 2005.