Geometric representations of planar graphs and maps

Éric Fusy (CNRS/LIX)

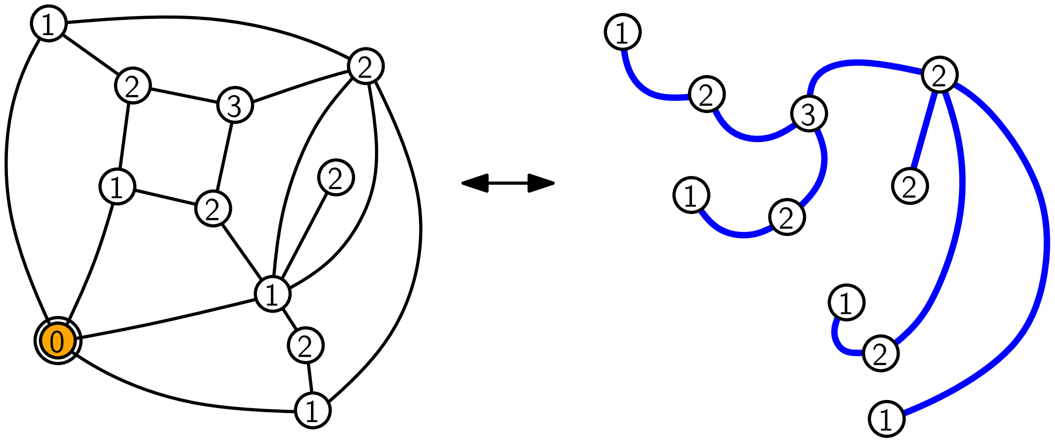
Summer school on random geometry, Bogota, may 2016

Overview of the course

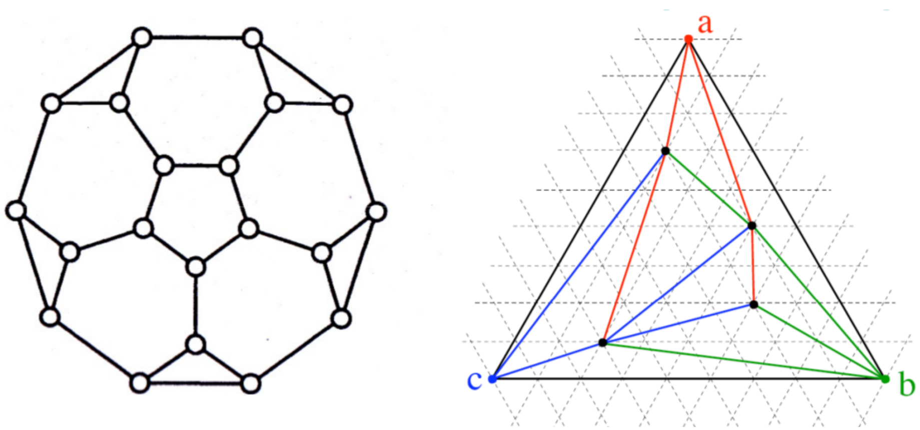
- Planar graphs and planar maps

  - structural aspects

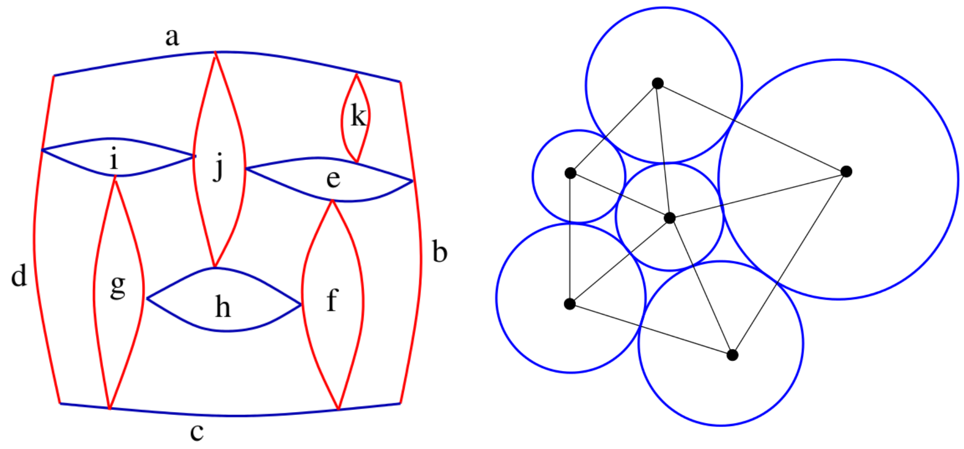
  - combinatorial aspects



- Geometric representations



straight-line drawings



contact representations

\+ applications & links to physical models

Structural aspects of planar graphs and maps

Graphs

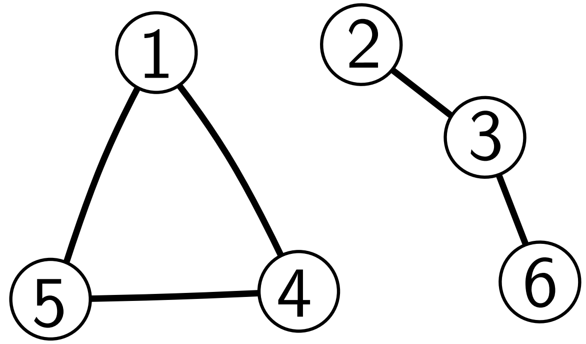
A graph $G = (V, E)$ is given by two sets V, E such that each $e \in E$ is an (unordered) pair of elements from V

V is the set of vertices, E is the set of edges (links, relations)

Example:

$V = \{1, 2, 3, 4, 5, 6\}$

$E = \{\{1, 5\}, \{3, 6\}, \{1, 5\}, \{4, 5\}, \{2, 3\}, \{1, 4\}\}$



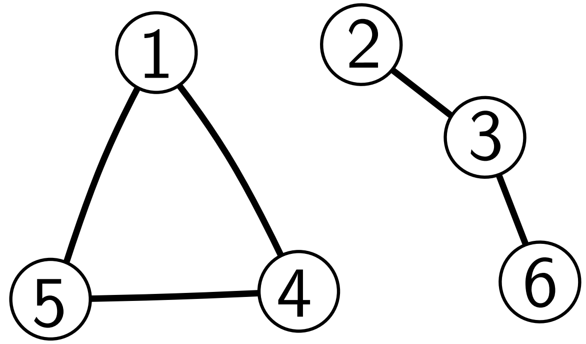
Graphs

A graph $G = (V, E)$ is given by two sets V, E such that each $e \in E$ is an (unordered) pair of elements from V

V is the set of vertices, E is the set of edges (links, relations)

Example:

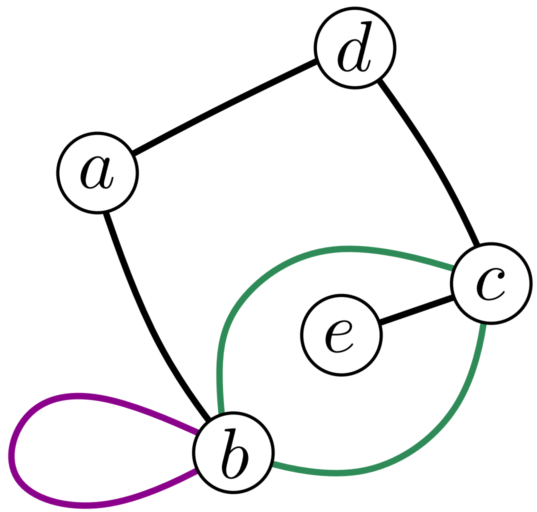
$V = \{1, 2, 3, 4, 5, 6\}$
$E = \{\{1, 5\}, \{3, 6\}, \{1, 5\}, \{4, 5\}, \{2, 3\}, \{1, 4\}\}$



Can also allow for loops and multiple edges

Example:

$V = \{a, b, c, d, e\}$
$E = \{\{a, b\}, \{b, b\}, \{b, c\}, \{c, e\}, \{b, c\}, \{a, d\}, \{d, c\}\}$



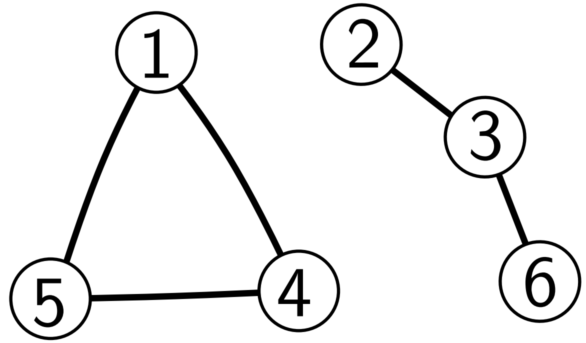
Graphs

A graph $G = (V, E)$ is given by two sets V, E such that each $e \in E$ is an (unordered) pair of elements from V

V is the set of vertices, E is the set of edges (links, relations)

Example:

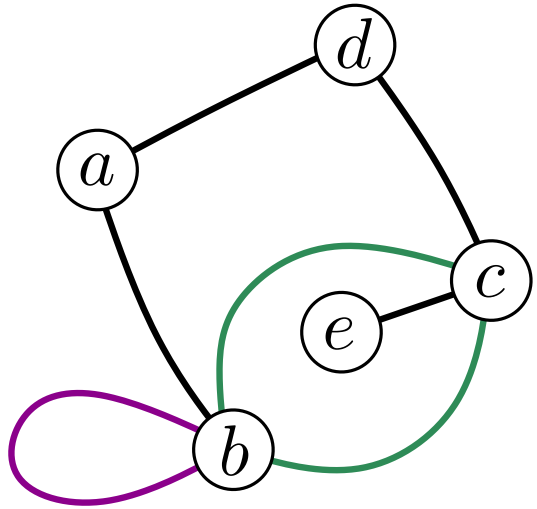
$V = \{1, 2, 3, 4, 5, 6\}$
$E = \{\{1, 5\}, \{3, 6\}, \{1, 5\}, \{4, 5\}, \{2, 3\}, \{1, 4\}\}$



Can also allow for loops and multiple edges

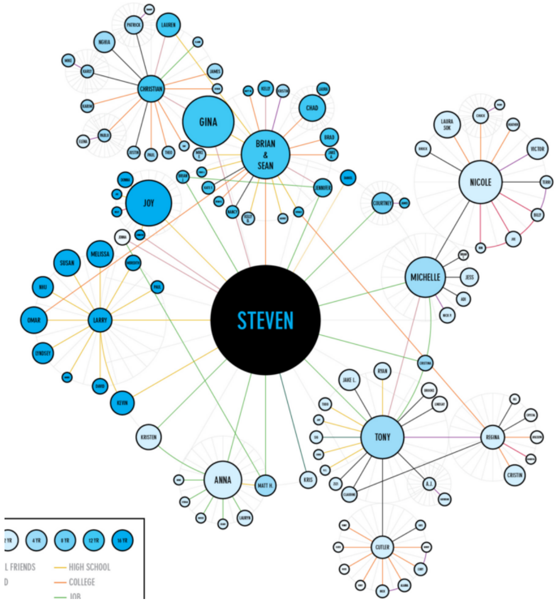
Example:

$V = \{a, b, c, d, e\}$
$E = \{\{a, b\}, \{b, b\}, \{b, c\}, \{c, e\}, \{b, c\}, \{a, d\}, \{d, c\}\}$

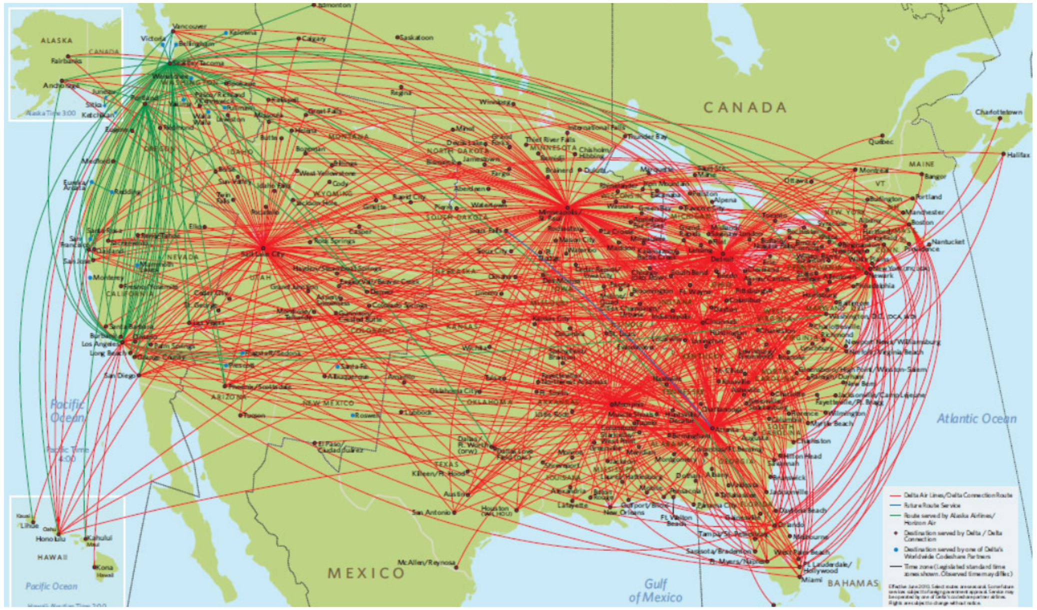


Def: A graph is called simple if it has no loop nor multiple edges
a graph is called connected if it is "in one piece"

The natural abstraction for networks

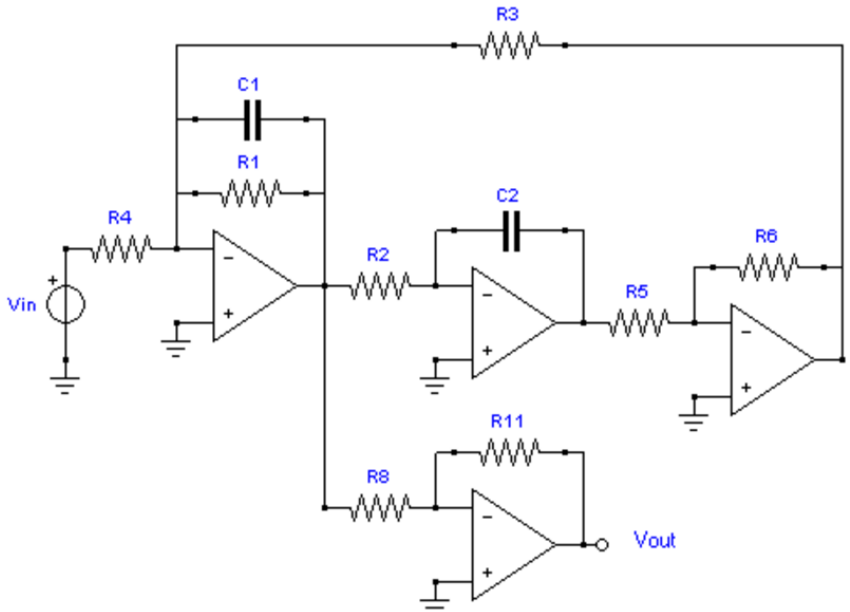


social network



airline connections network

road network

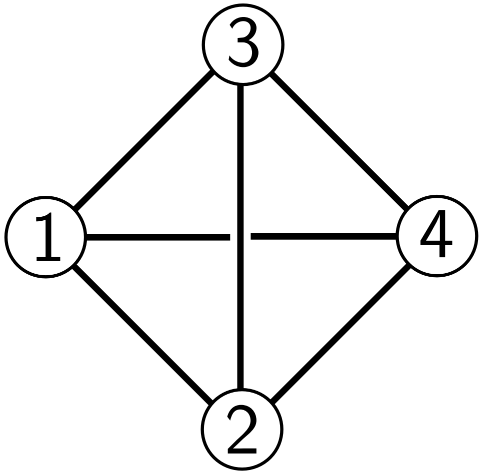


electronic network

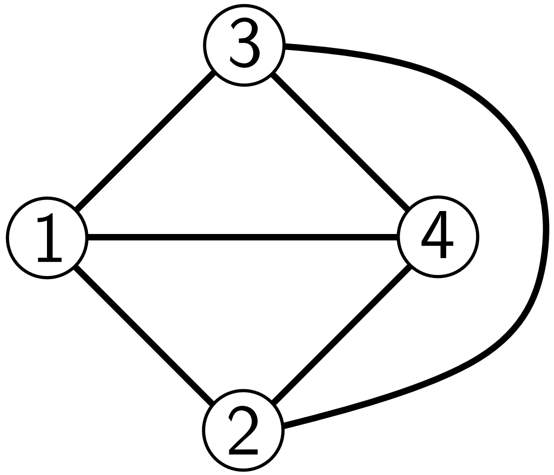
Planar graphs

A graph is called **planar** if it can be drawn crossing-free in the plane

K_4 is planar

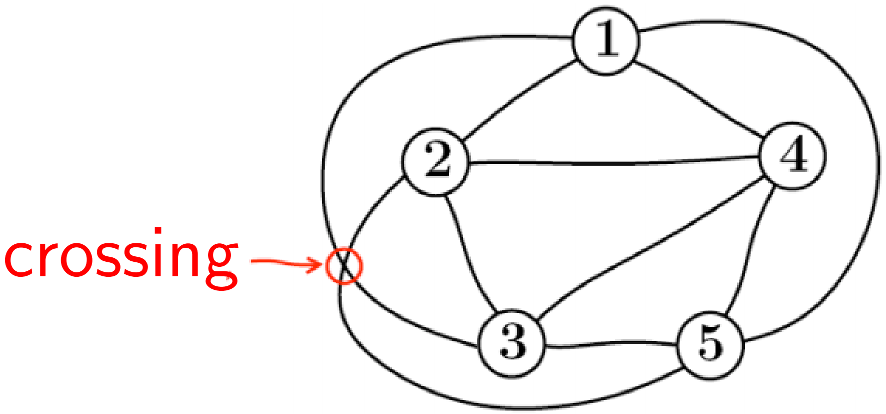


non-planar drawing



planar drawing

K_5 is not planar

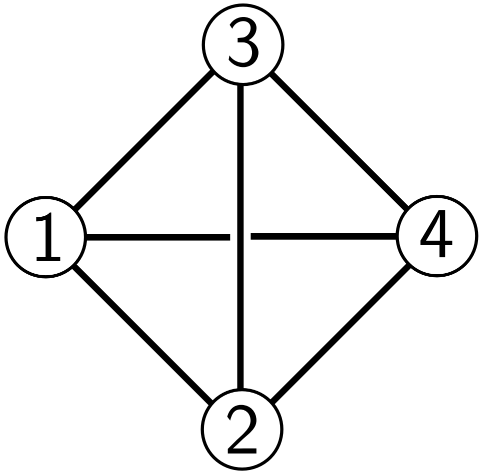


(whatever drawing, there is always a crossing)

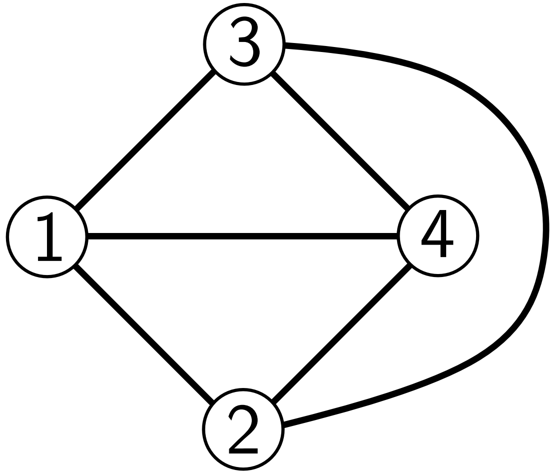
Planar graphs

A graph is called **planar** if it can be drawn crossing-free in the plane

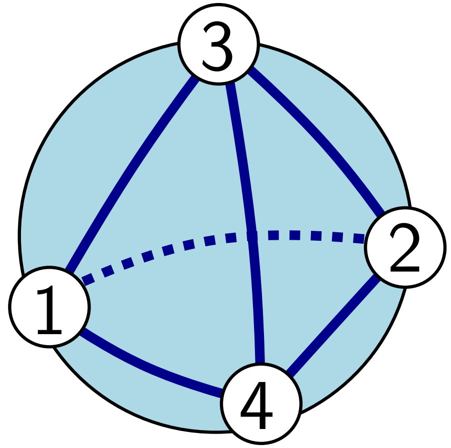
K_4 is planar



non-planar drawing

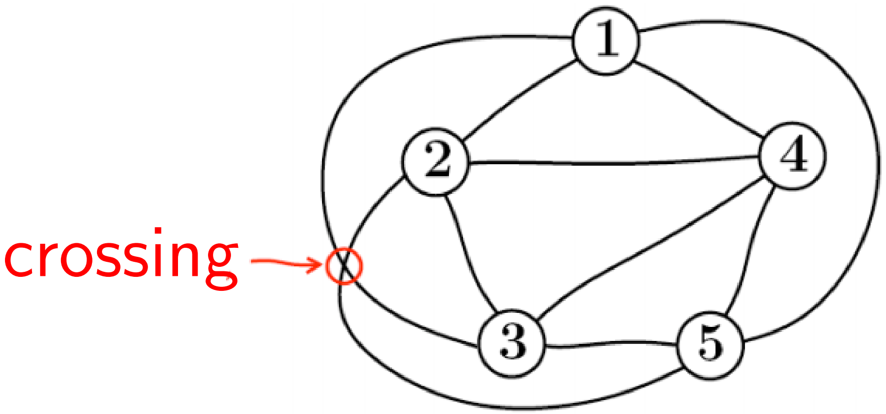


planar drawing



on the sphere

K_5 is not planar

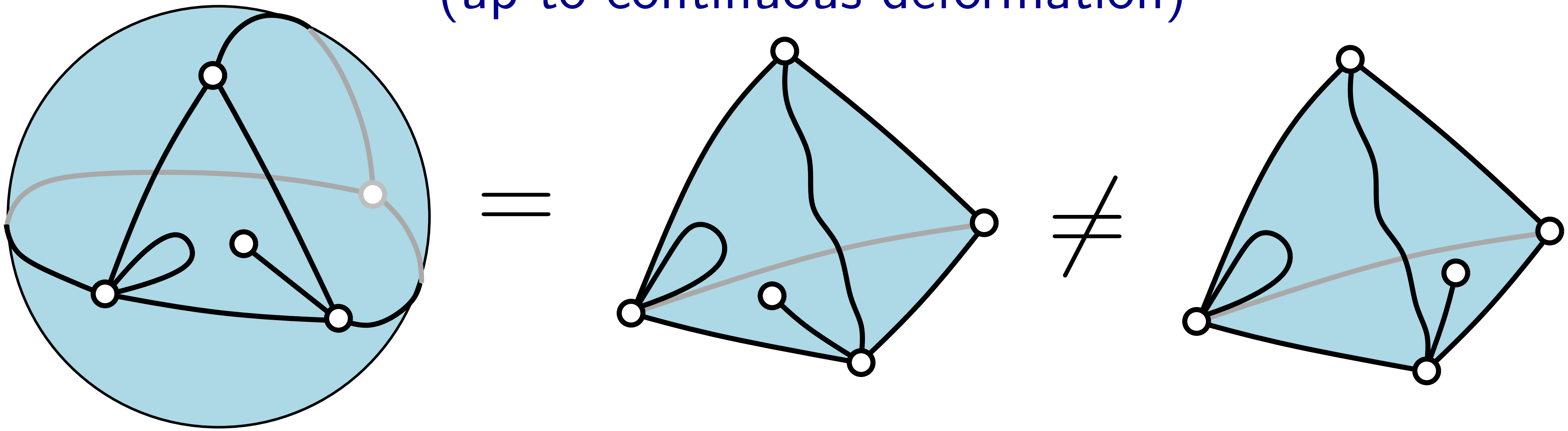


(whatever drawing, there is always a crossing)

Rk: planar $\leftrightarrow$ can be drawn crossing-free on the sphere

Planar maps

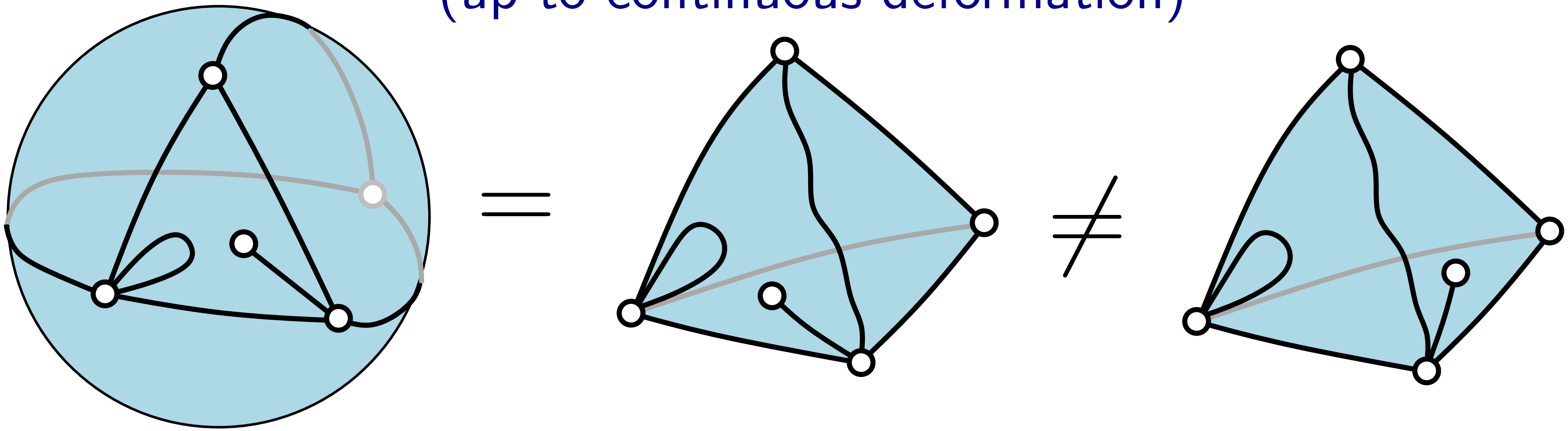
Def. Planar map = connected graph embedded on the sphere
(up to continuous deformation)



Rk: a planar graph can have several embeddings on the sphere

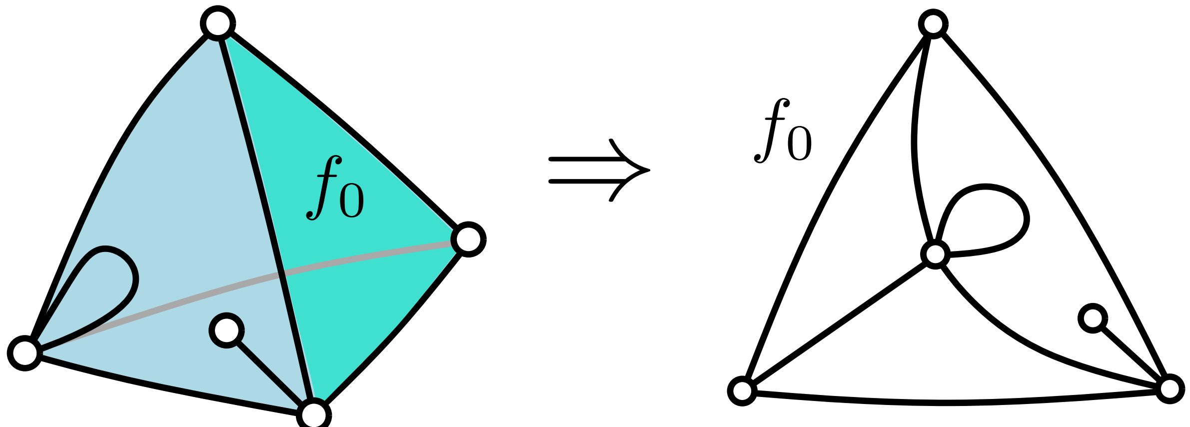
Planar maps

Def. Planar map = connected graph embedded on the sphere
(up to continuous deformation)



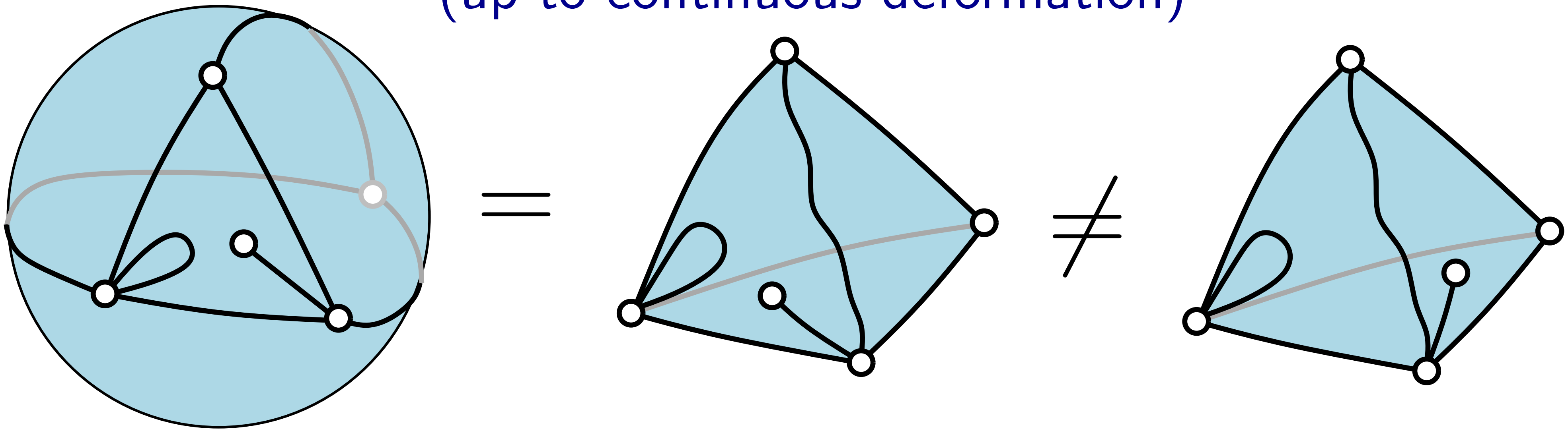
Rk: a planar graph can have several embeddings on the sphere

A map is easier to draw in the plane (implicit choice of an outer face f_0)



Planar maps

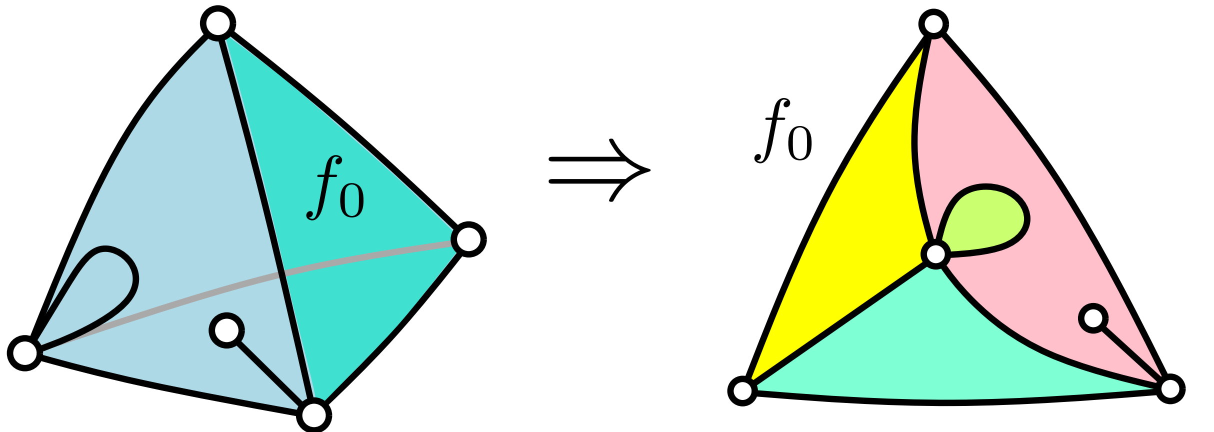
Def. Planar map = connected graph embedded on the sphere
(up to continuous deformation)



Rk: a planar graph can have several embeddings on the sphere
a map has vertices, edges, and faces

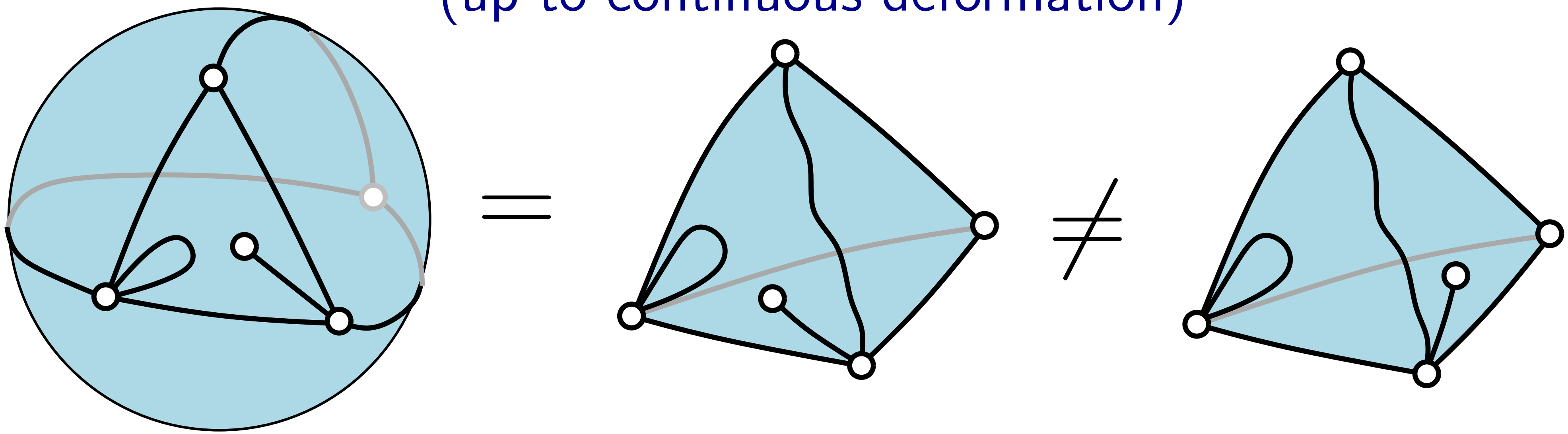
A map is easier to draw in the plane (implicit choice of an outer face f_0)

5 faces (including outer one)



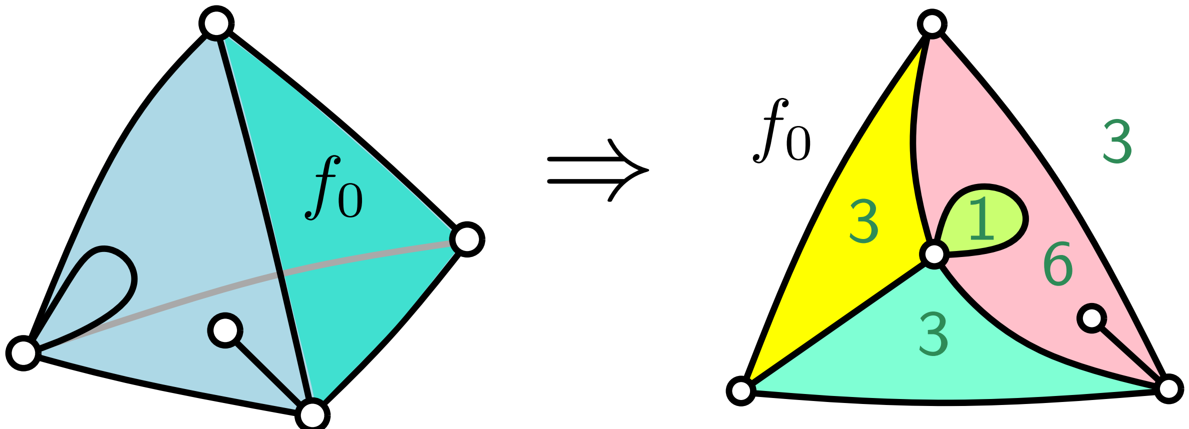
Planar maps

Def. Planar map = connected graph embedded on the sphere
(up to continuous deformation)



Rk: a planar graph can have several embeddings on the sphere
a map has vertices, edges, and faces

A map is easier to draw in the plane (implicit choice of an outer face f_0)

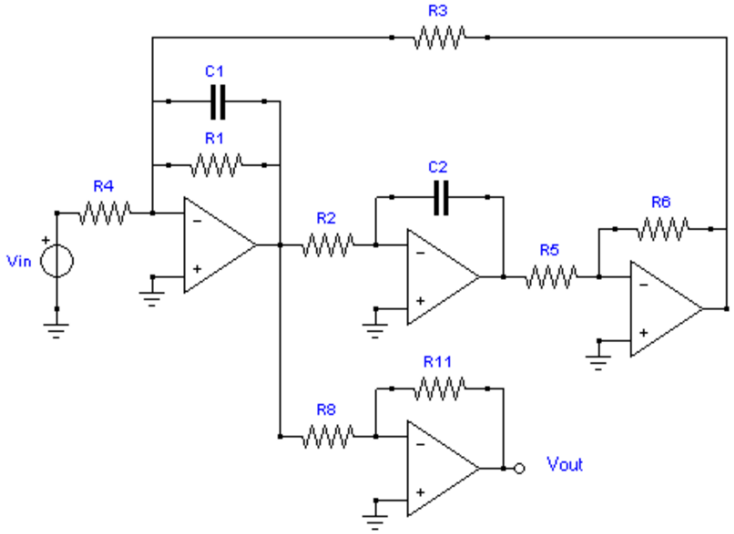


5 faces (including outer one)

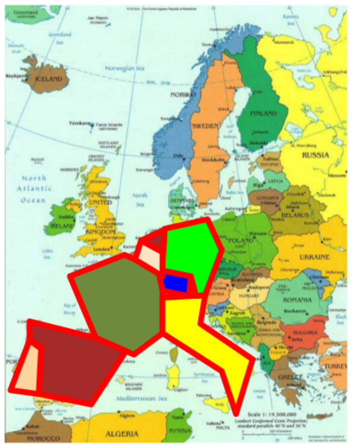
degree of a face
= length of walk around f

Motivations for studying planar maps

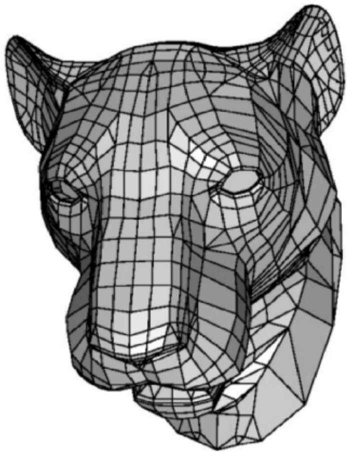
- Planar networks usually **come with an explicit planar embedding**



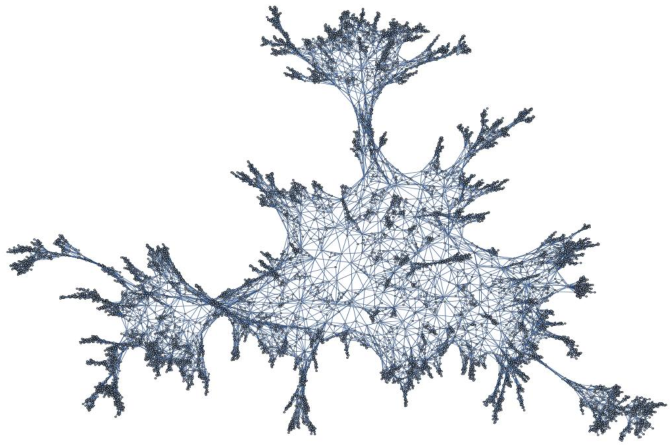
- A natural model of discrete surface (formed from glued polygons)



abstraction of geographic maps



meshes

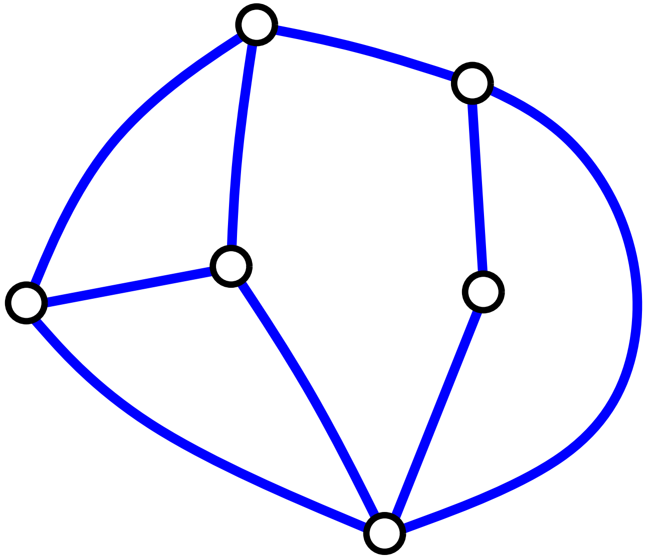


random discrete surfaces (2D quantum gravity)

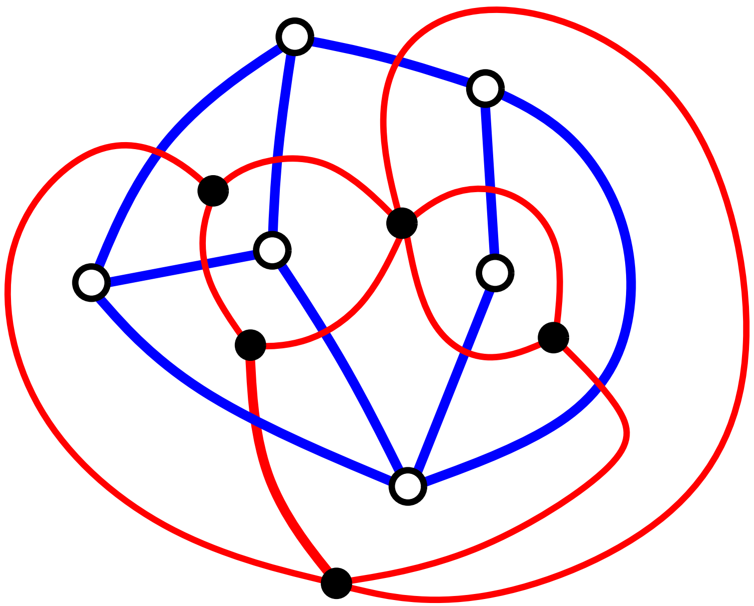
- Nice combinatorial properties!

Duality for planar maps

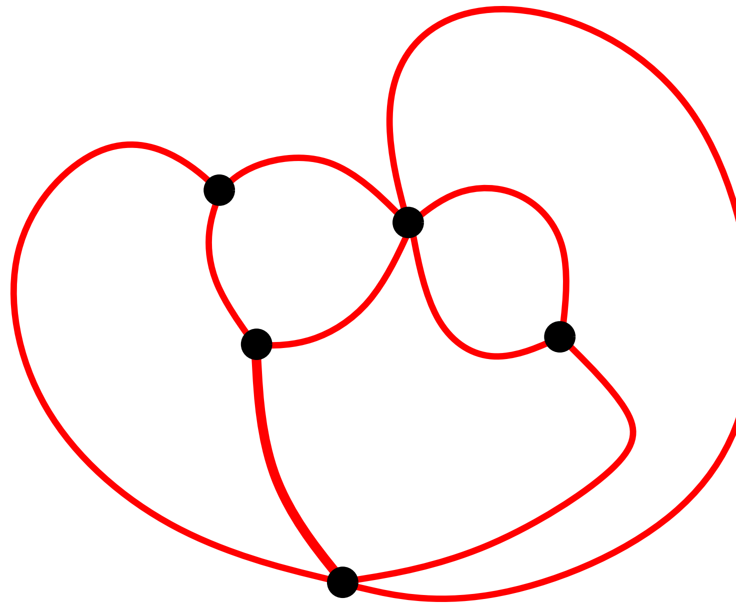
6 vertices, 9 edges, 5 faces



a planar map



the dual map



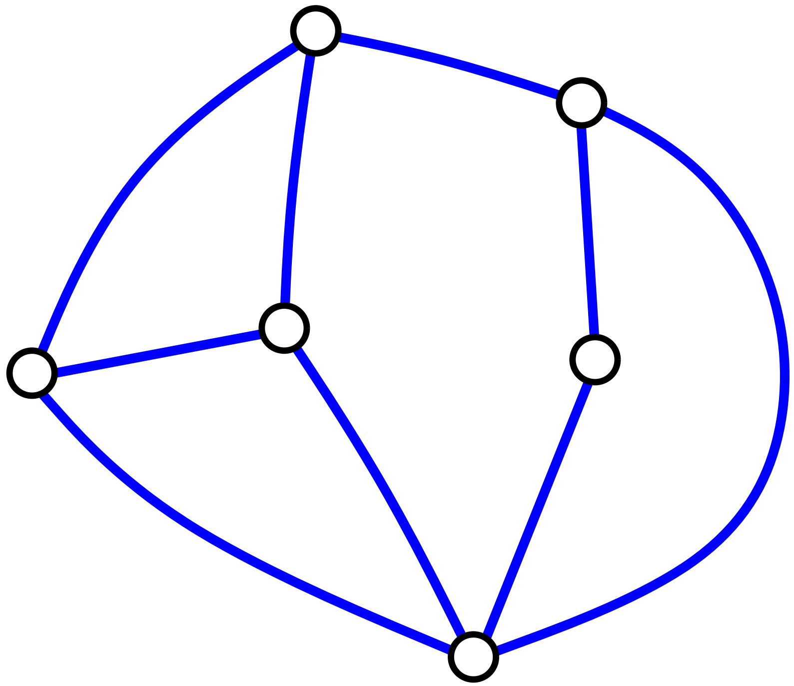
5 vertices, 9 edges, 6 faces

preserves #(edges), **exchanges #(vertices) and #(faces)**

The Euler relation

Let $M = (V, E, F)$ be a planar map. Then

$$|E| = |V| + |F| - 2$$

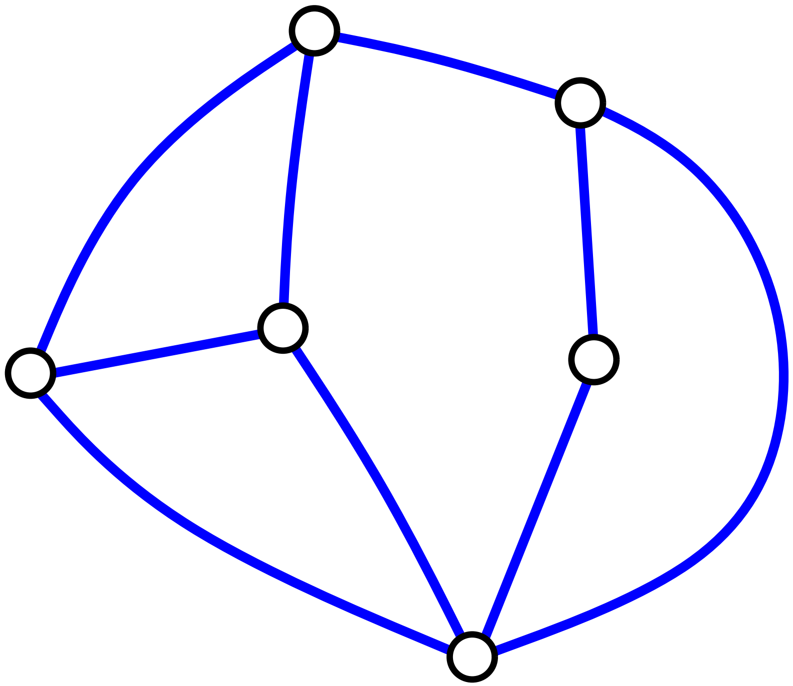


$|V| = 6, |E| = 9, |F| = 5$

The Euler relation

Let $M = (V, E, F)$ be a planar map. Then

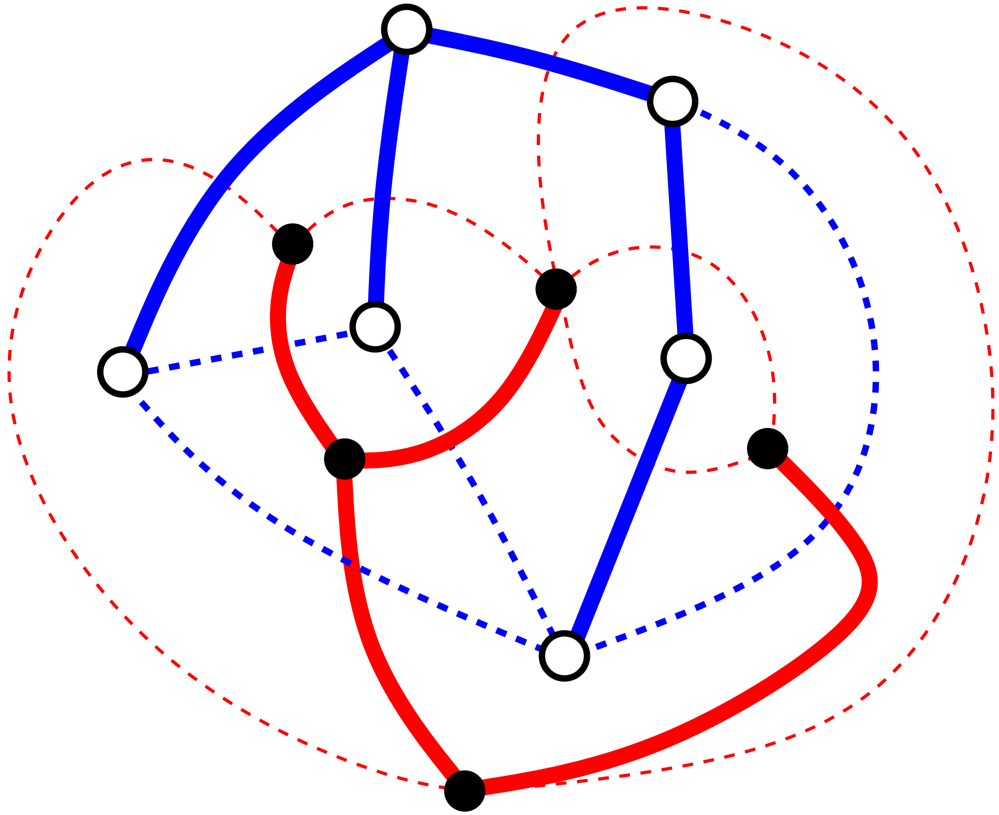
$$|E| = |V| + |F| - 2$$



$|V| = 6, |E| = 9, |F| = 5$

Proof using spanning trees

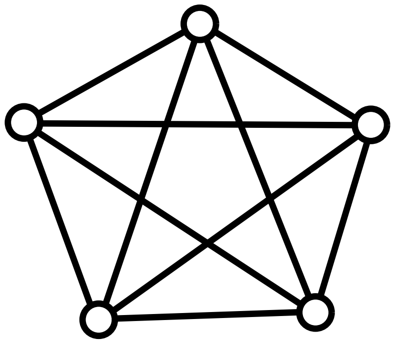
$$|E| = (|V| - 1) + (|F| - 1)$$



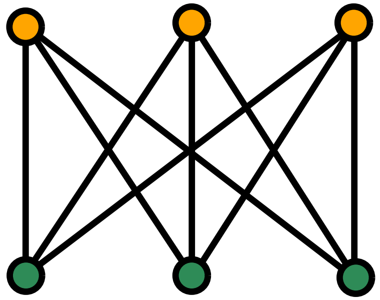
Kuratowski's theorem for planar graphs

The Euler relation implies (exercise!) that K_5 and $K_{3,3}$ are not planar

K_5



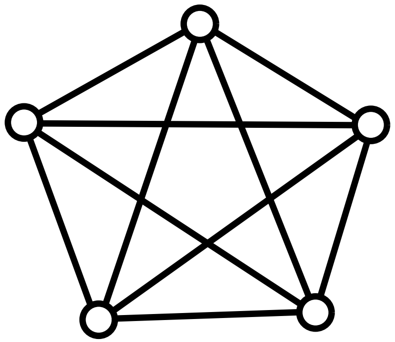
$K_{3,3}$



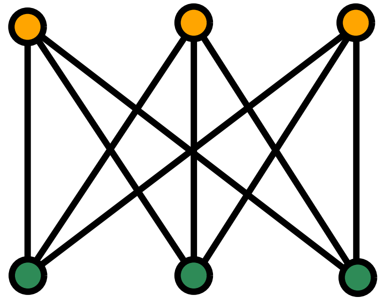
Kuratowski's theorem for planar graphs

The Euler relation implies (exercise!) that K_5 and $K_{3,3}$ are not planar

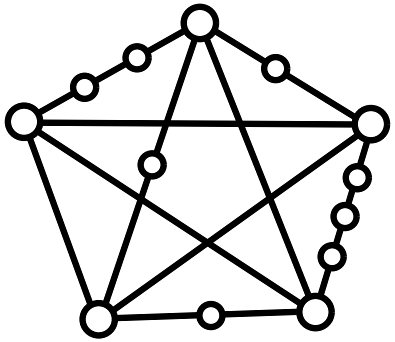
K_5



$K_{3,3}$



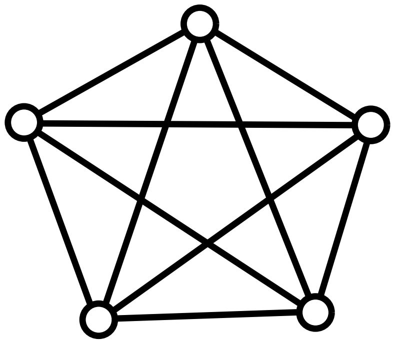
Hence any subdivision of K_5 or $K_{3,3}$ is not planar either

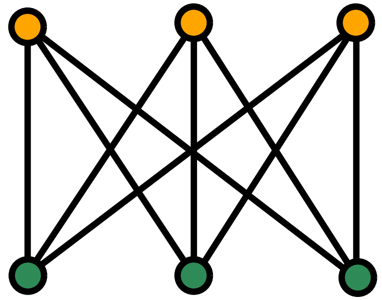


a subdivision of K_5

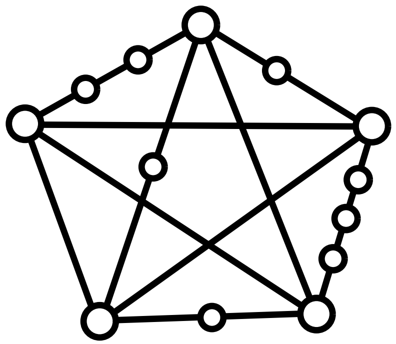
Kuratowski's theorem for planar graphs

The Euler relation implies (exercise!) that K_5 and $K_{3,3}$ are not planar

K_5 

$K_{3,3}$ 

Hence any subdivision of K_5 or $K_{3,3}$ is not planar either



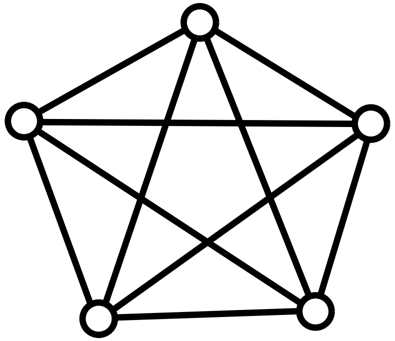
a subdivision of K_5

Kuratowski: any non-planar graph contains a subdivision of K_5 or $K_{3,3}$

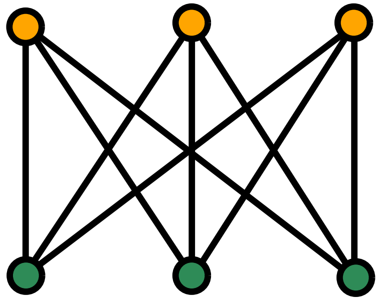
Kuratowski's theorem for planar graphs

The Euler relation implies (exercise!) that K_5 and $K_{3,3}$ are not planar

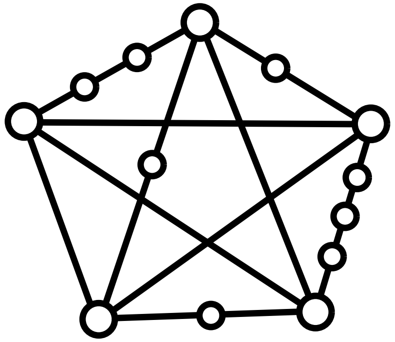
K_5



$K_{3,3}$

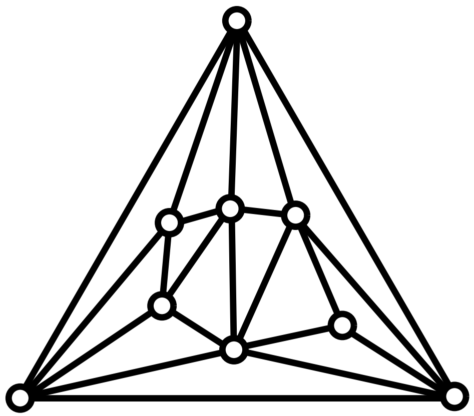


Hence any subdivision of K_5 or $K_{3,3}$ is not planar either



a subdivision of K_5

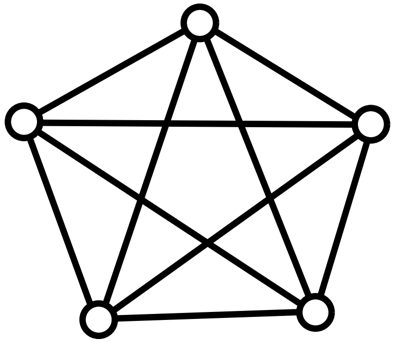
Kuratowski: any non-planar graph contains a subdivision of K_5 or $K_{3,3}$



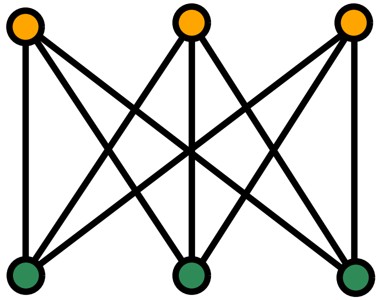
Kuratowski's theorem for planar graphs

The Euler relation implies (exercise!) that K_5 and $K_{3,3}$ are not planar

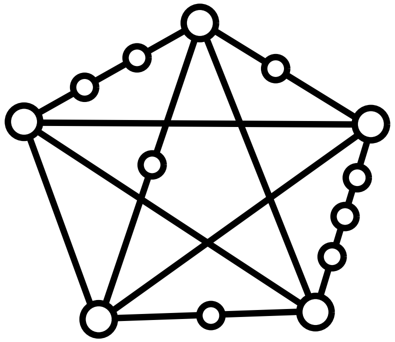
K_5



$K_{3,3}$

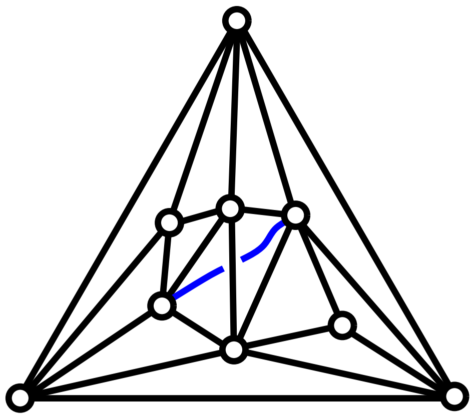


Hence any subdivision of K_5 or $K_{3,3}$ is not planar either



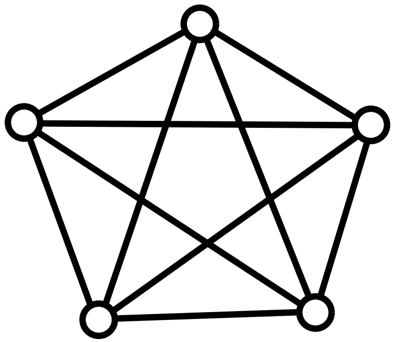
a subdivision of K_5

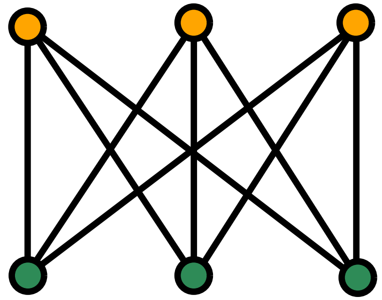
Kuratowski: any non-planar graph contains a subdivision of K_5 or $K_{3,3}$



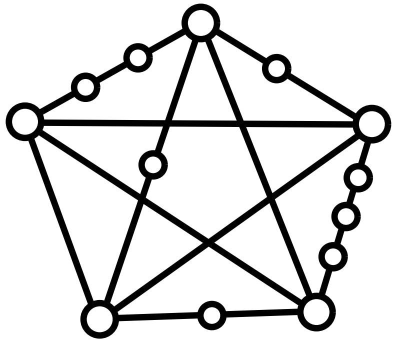
Kuratowski's theorem for planar graphs

The Euler relation implies (exercise!) that K_5 and $K_{3,3}$ are not planar

K_5 

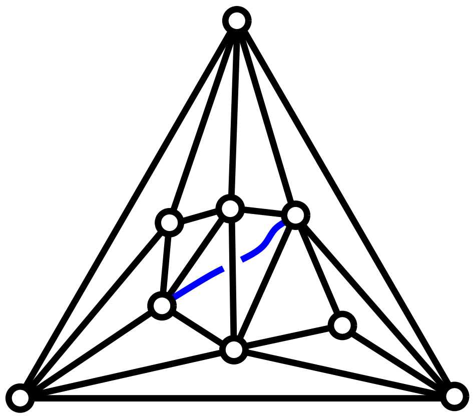
$K_{3,3}$ 

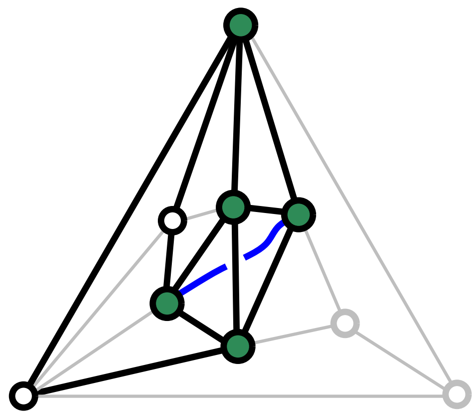
Hence any subdivision of K_5 or $K_{3,3}$ is not planar either



a subdivision of K_5

Kuratowski: any non-planar graph contains a subdivision of K_5 or $K_{3,3}$





contains subdivision of K_5

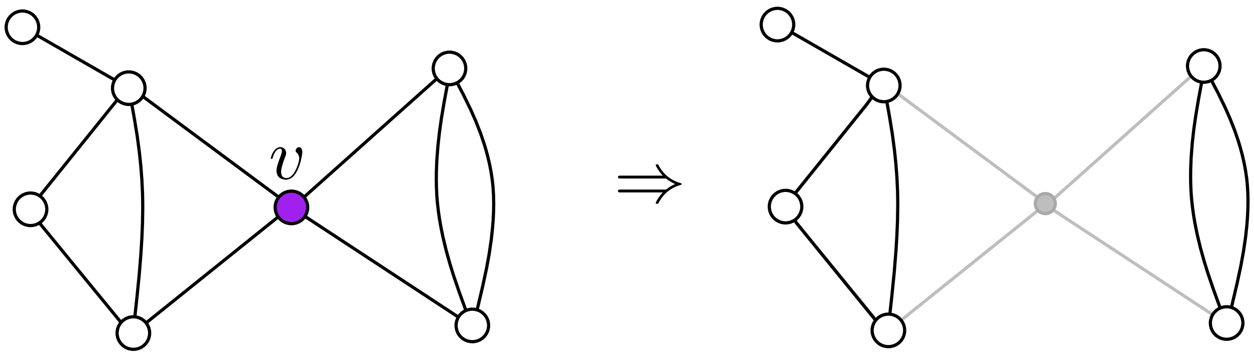
k-connectivity in graphs

For $k \geq 2$ a graph G is called k-connected if G is connected and remains connected when deleting any $(k-1)$-subset of vertices

k-connectivity in graphs

For $k \geq 2$ a graph G is called k-connected if G is connected and remains connected when deleting any $(k-1)$-subset of vertices

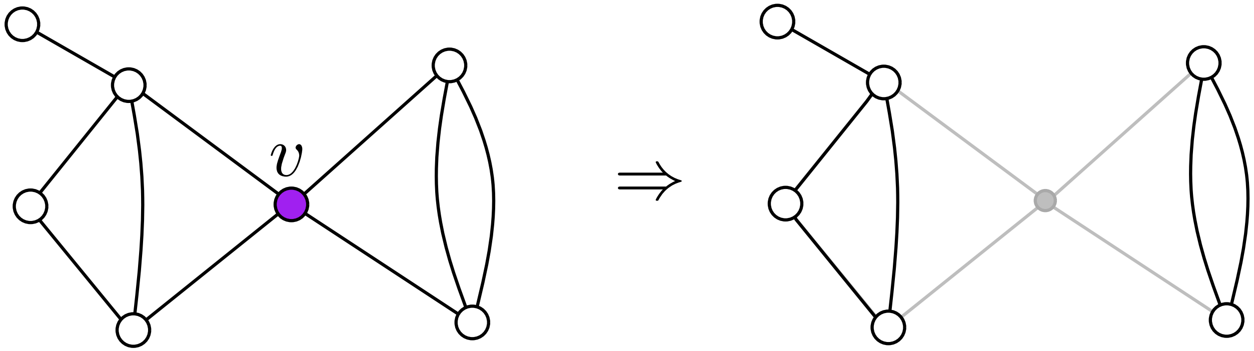
- not 2-connected $\Leftrightarrow$ $\exists$ separating vertex



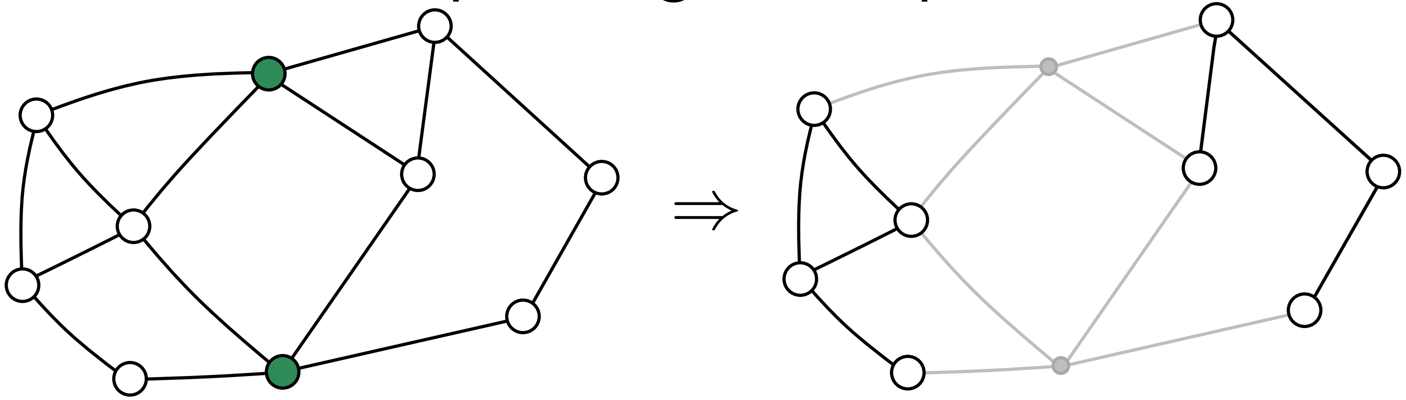
k-connectivity in graphs

For $k \geq 2$ a graph G is called k-connected if G is connected and remains connected when deleting any $(k-1)$-subset of vertices

- not 2-connected $\Leftrightarrow$ $\exists$ separating vertex



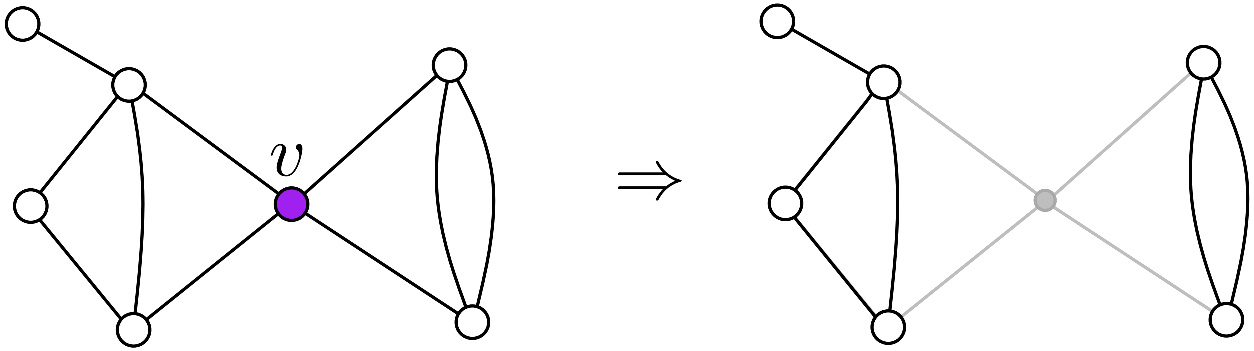
- not 3-connected $\Leftrightarrow$ $\exists$ separating vertex-pair



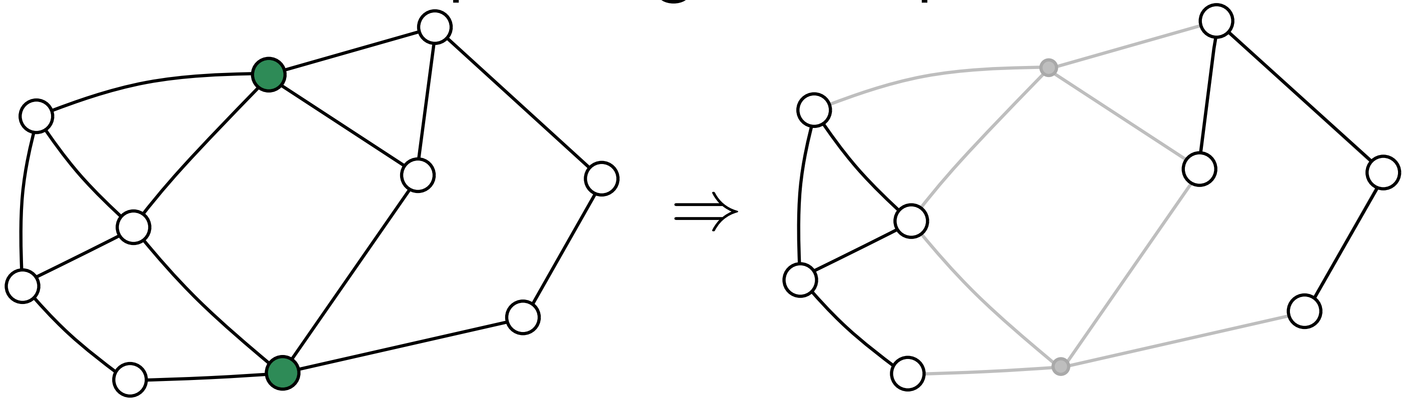
k-connectivity in graphs

For $k \geq 2$ a graph G is called k-connected if G is connected and remains connected when deleting any $(k-1)$-subset of vertices

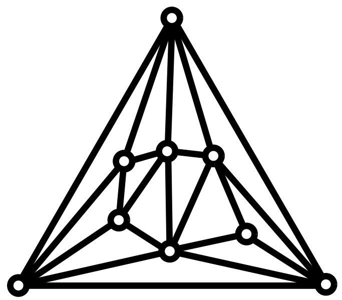
- not 2-connected $\Leftrightarrow$ $\exists$ separating vertex



- not 3-connected $\Leftrightarrow$ $\exists$ separating vertex-pair



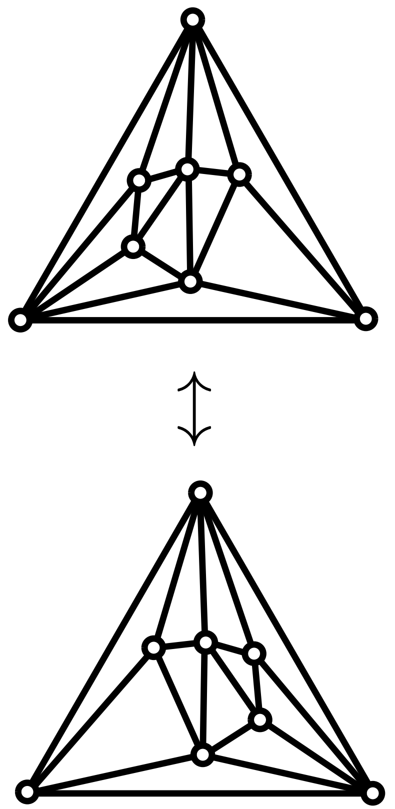
Exercise: for triangulations (faces have degree 3)
2-connected $\Leftrightarrow$ loopless
3-connected $\Leftrightarrow$ simple



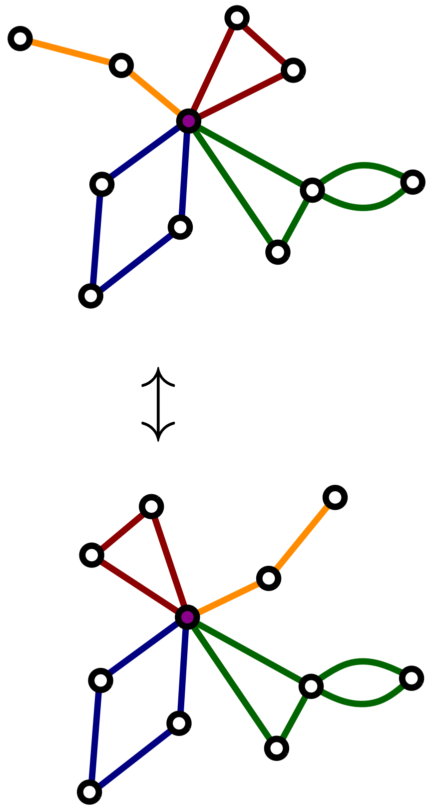
The structure of the set of embeddings

For G a connected planar graph, operations to change the embedding are:

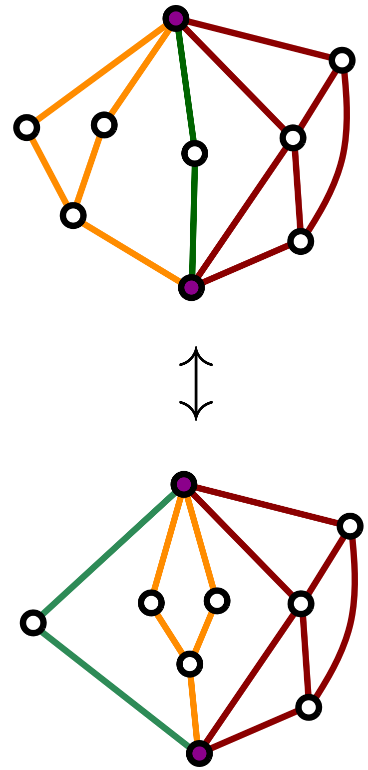
mirror



flip at separating vertex

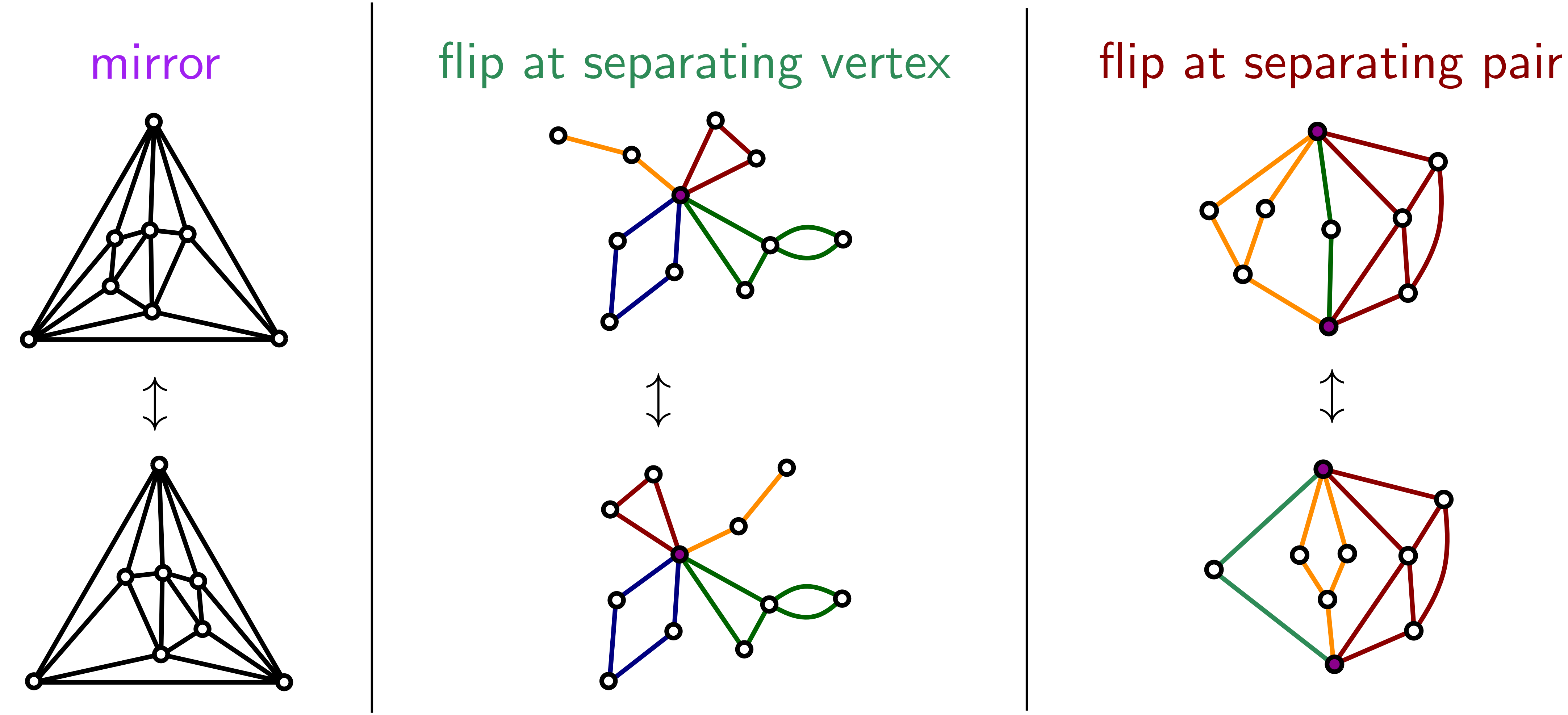


flip at separating pair



The structure of the set of embeddings

For G a connected planar graph, operations to change the embedding are:

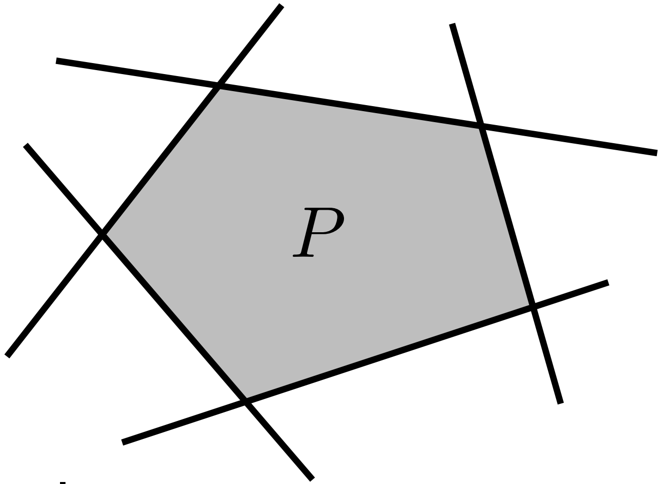


Theorem (Tutte, Whitney): any two embeddings of G are related by a sequence of such operations

Hence 3-connected planar graphs have a unique embedding (up to mirror)

Relation with polytopes

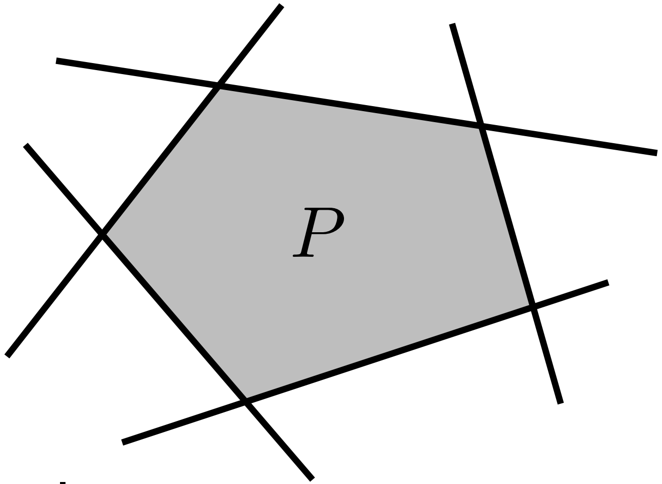
A d-dimensional polytope is a bounded region $P \subset \mathbb{R}^d$ that can be obtained as $P = H_1 \cap H_2 \cap \cdots \cap H_k$ for some half-spaces $H_1, \ldots, H_k$



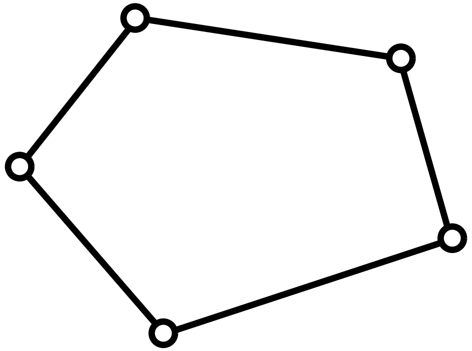
a 2D-polytope

Relation with polytopes

A d-dimensional polytope is a bounded region $P \subset \mathbb{R}^d$ that can be obtained as $P = H_1 \cap H_2 \cap \cdots \cap H_k$ for some half-spaces $H_1, \ldots, H_k$



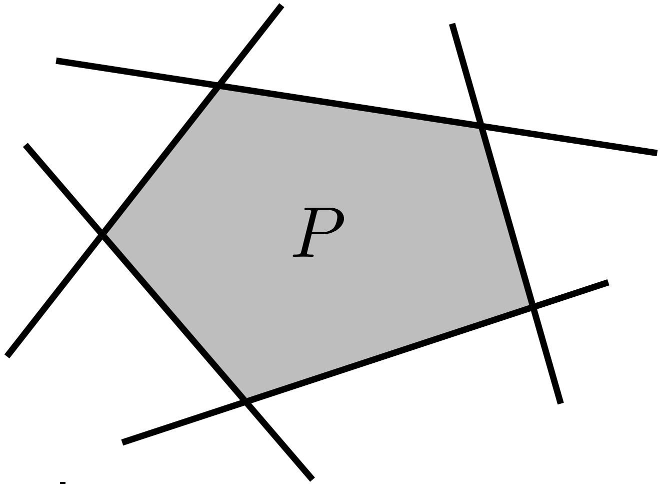
a 2D-polytope



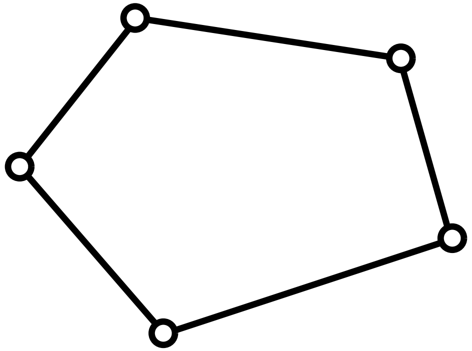
Rk: a polytope P induces a graph G_P (vertices & edges)

Relation with polytopes

A d-dimensional polytope is a bounded region $P \subset \mathbb{R}^d$ that can be obtained as $P = H_1 \cap H_2 \cap \cdots \cap H_k$ for some half-spaces $H_1, \ldots, H_k$



a 2D-polytope

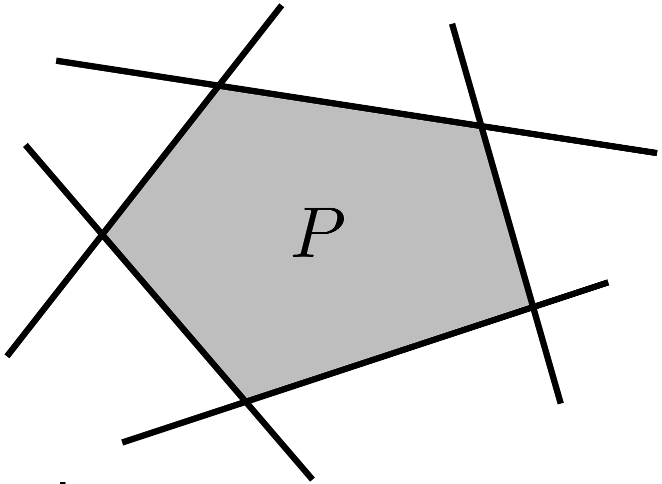


Rk: a polytope P induces a graph G_P (vertices & edges)

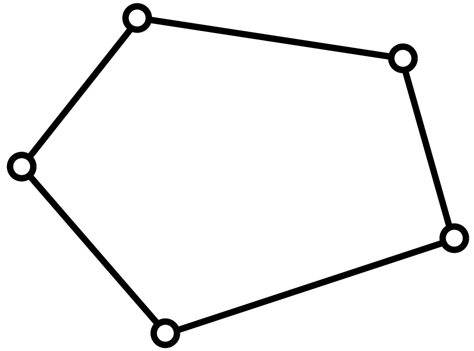
Balinsky'61: if P has dimension d, then G_P is d-connected

Relation with polytopes

A d-dimensional polytope is a bounded region $P \subset \mathbb{R}^d$ that can be obtained as $P = H_1 \cap H_2 \cap \cdots \cap H_k$ for some half-spaces $H_1, \ldots, H_k$

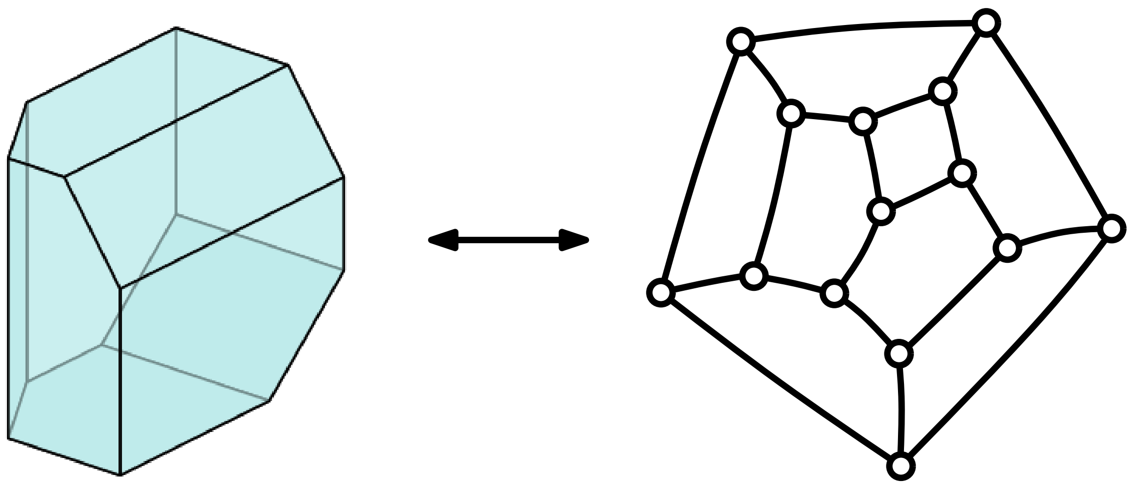


a 2D-polytope



Rk: a polytope P induces a graph G_P (vertices & edges)
Balinsky'61: if P has dimension d, then G_P is d-connected

Steinitz'16: a planar graph is 3-connected iff it can be obtained as the graph of a 3D polytope

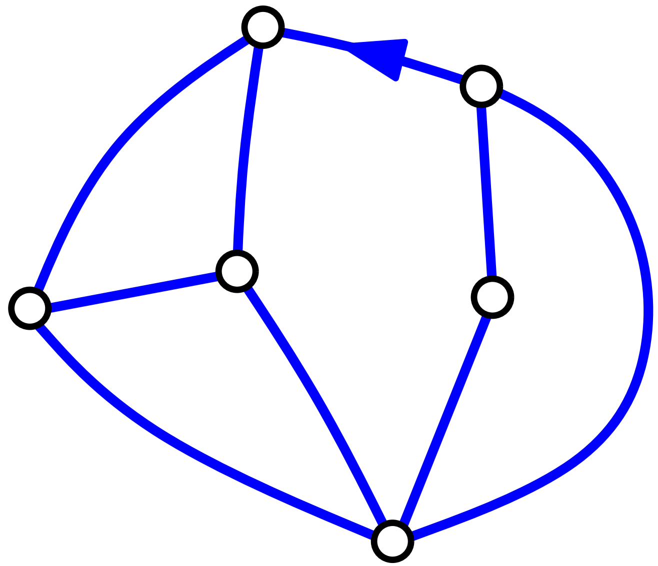


Combinatorial aspects of planar maps

Rooted maps

A map is **rooted** by marking and orienting an edge

a rooted map



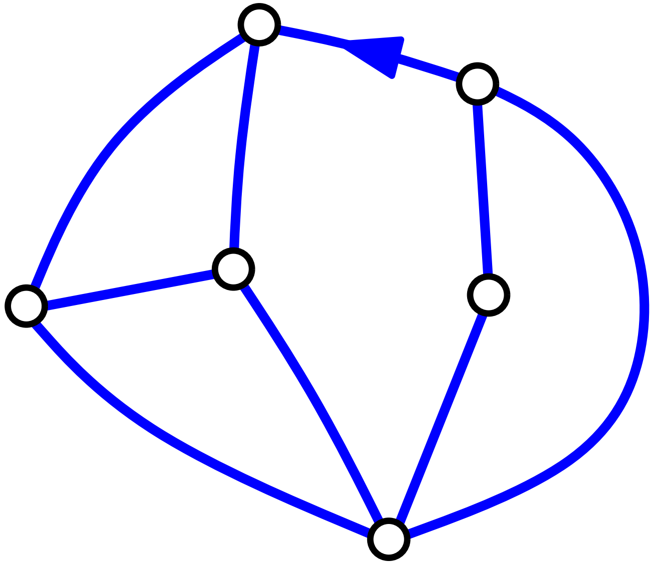
the face on the right
of the root is taken
as the outer face

Rooted maps are combinatorially easier than maps
(no symmetry issue, root gives starting point for recursive decomposition)

Rooted maps

A map is **rooted** by marking and orienting an edge

a rooted map

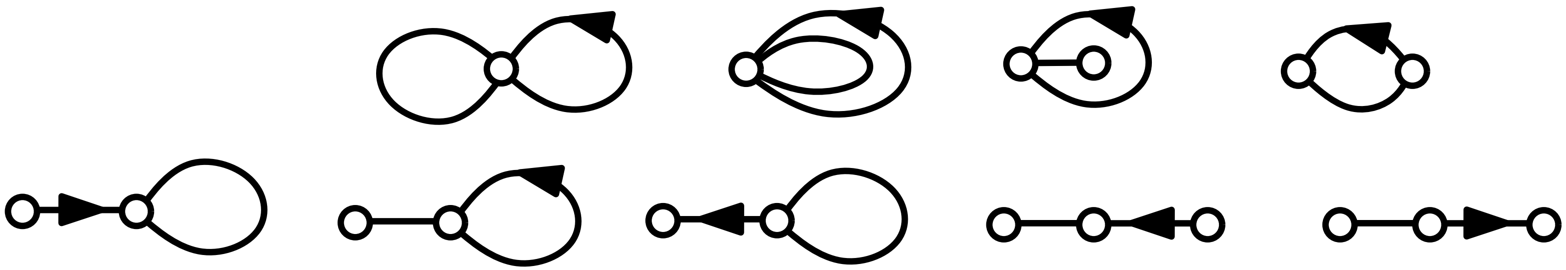


the face on the right of the root is taken as the outer face

Rooted maps are combinatorially easier than maps
(no symmetry issue, root gives starting point for recursive decomposition)

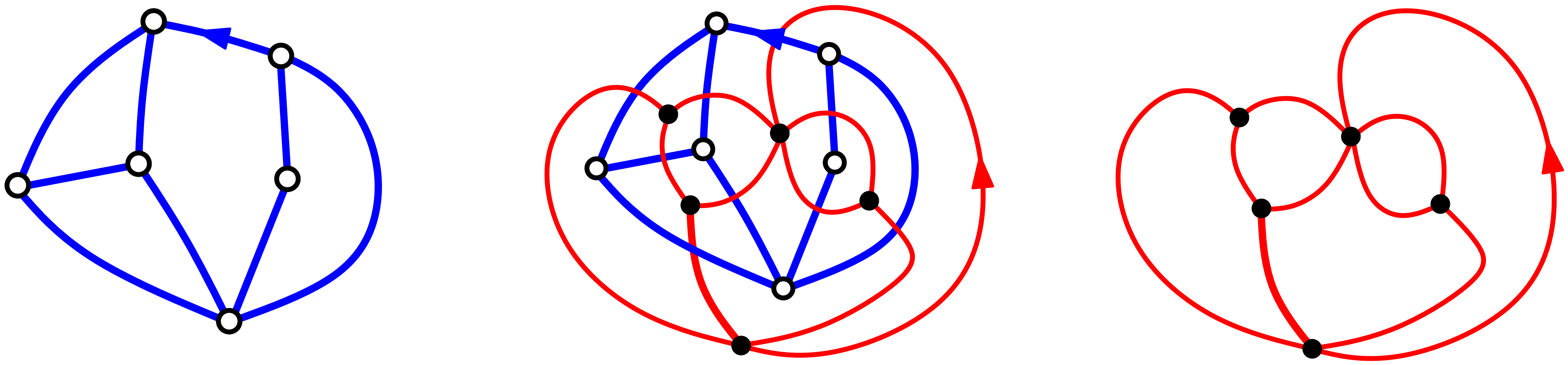
The 2 rooted maps with one edge

The 9 rooted maps with two edges



Duality for rooted maps

same as for maps (root the dual at the dual of the root-edge)



vertices and faces play a symmetric role in rooted maps

Counting rooted maps

Let a_n be the number of rooted maps with n edges

n	1	2	3	4	5	6	7	$\cdots$
a_n	2	9	54	378	2916	24057	208494	$\cdots$

Counting rooted maps

Let a_n be the number of rooted maps with n edges

n	1	2	3	4	5	6	7	$\cdots$
a_n	2	9	54	378	2916	24057	208494	$\cdots$

Theorem: (Tutte'63) $$\frac{2 \cdot 3^n}{(n+1)(n+2)}\binom{2n}{n}$$

Counting rooted maps

Let a_n be the number of rooted maps with n edges

n	1	2	3	4	5	6	7	$\cdots$
a_n	2	9	54	378	2916	24057	208494	$\cdots$

Theorem: (Tutte'63) $$\frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}$$

Not an isolated case:

- Triangulations ($2n$ faces)

Loopless: $\frac{2^n}{(n+1)(2n+1)} \binom{3n}{n}$ Simple: $\frac{1}{n(2n-1)} \binom{4n-2}{n-1}$

- Quadrangulations (n faces)

General: $\frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}$ Simple: $\frac{2}{n(n+1)} \binom{3n}{n-1}$

Counting rooted maps

Let a_n be the number of rooted maps with n edges

n	1	2	3	4	5	6	7	$\cdots$
a_n	2	9	54	378	2916	24057	208494	$\cdots$

Theorem: (Tutte'63) $$\frac{2 \cdot 3^n}{(n+1)(n+2)}\binom{2n}{n}$$

Not an isolated case:

- Triangulations ($2n$ faces)

Loopless: $\frac{2^n}{(n+1)(2n+1)}\binom{3n}{n}$

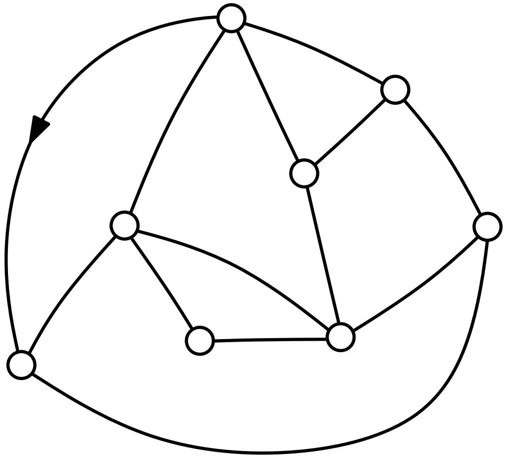
Simple: $\frac{1}{n(2n-1)}\binom{4n-2}{n-1}$

- Quadrangulations (n faces)

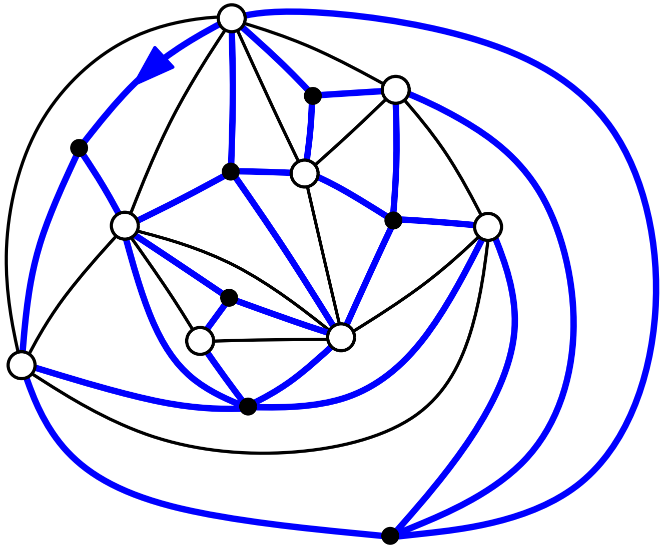
General: $\frac{2 \cdot 3^n}{(n+1)(n+2)}\binom{2n}{n}$

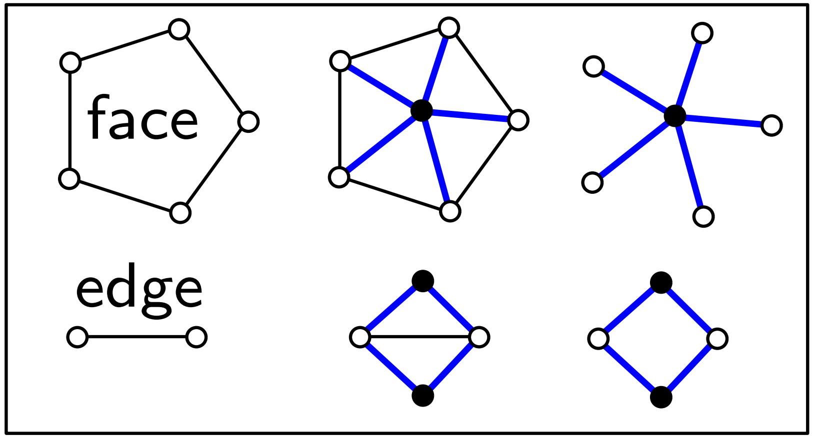
Simple: $\frac{2}{n(n+1)}\binom{3n}{n-1}$

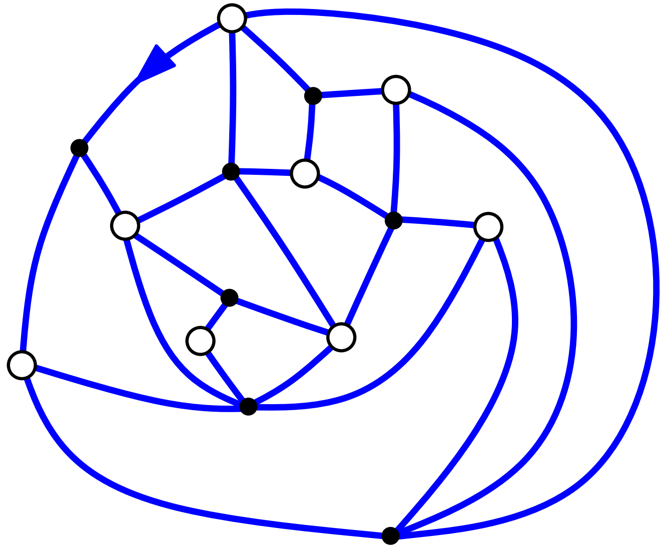
Bijection maps $\leftrightarrow$ quadrangulations



n edges
i vertices
j faces

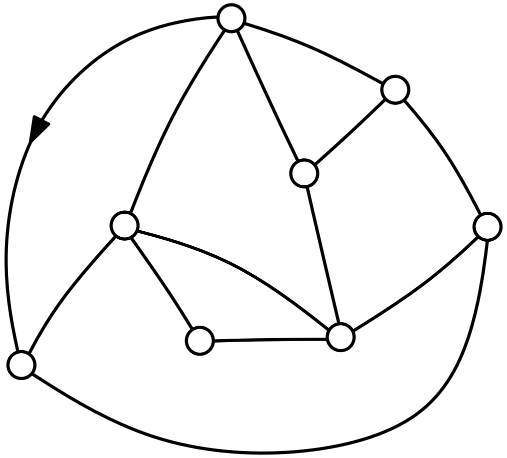


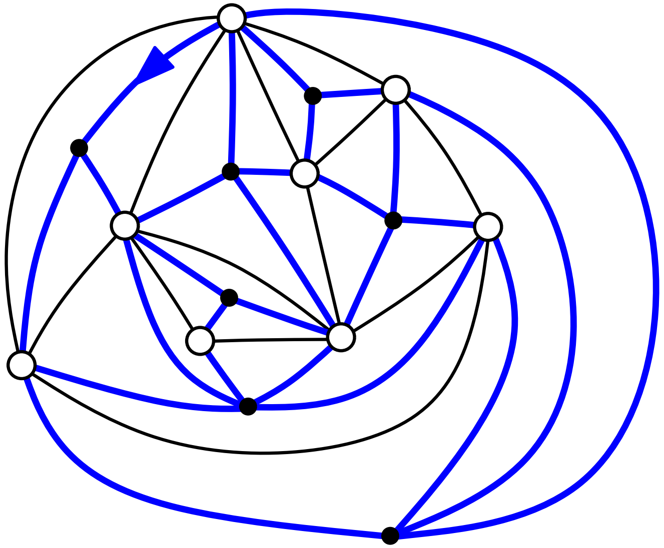


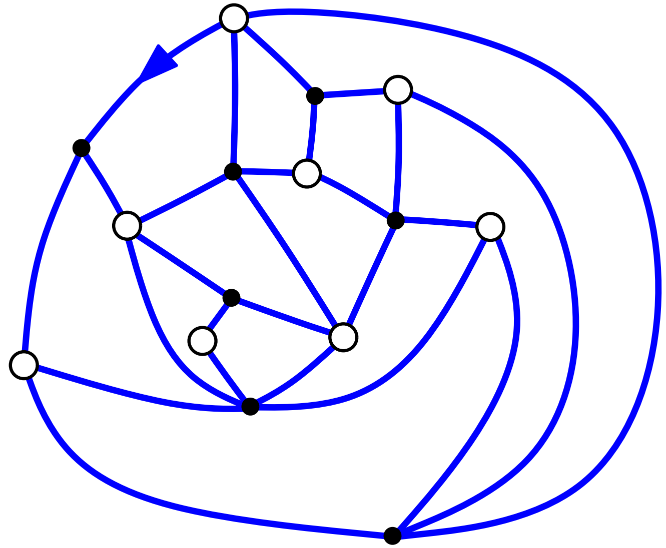


n faces
i white vertices
j black vertices

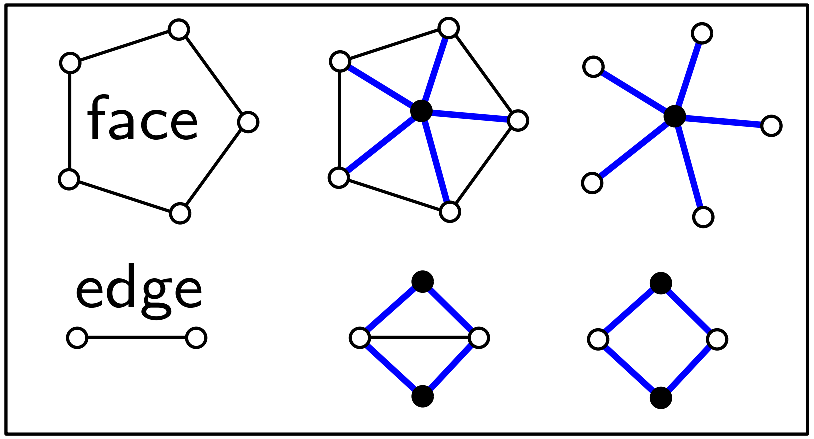
Bijection maps $\leftrightarrow$ quadrangulations







n edges
i vertices
j faces

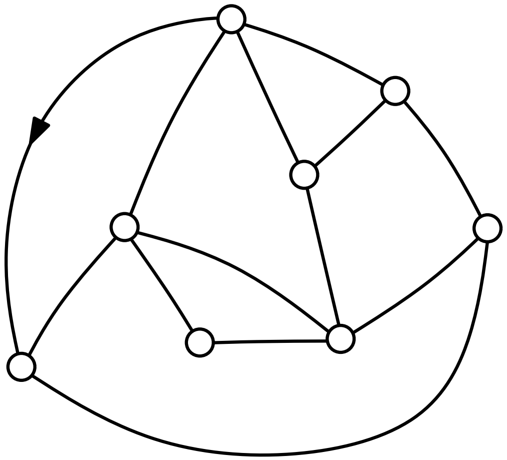


n faces
i white vertices
j black vertices

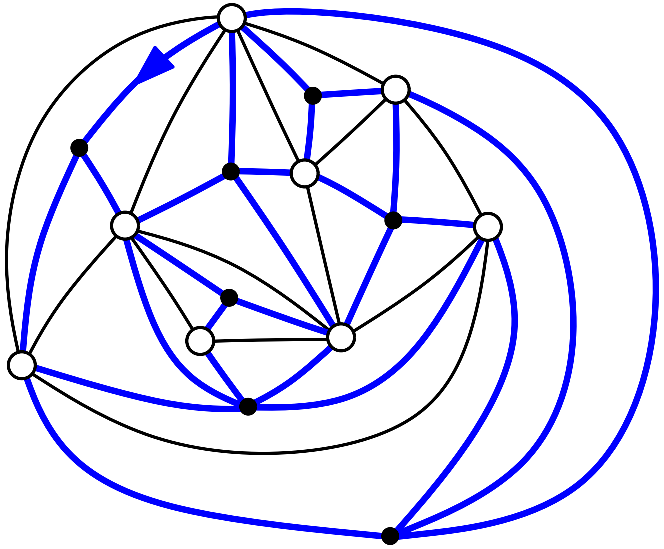
Consequence:

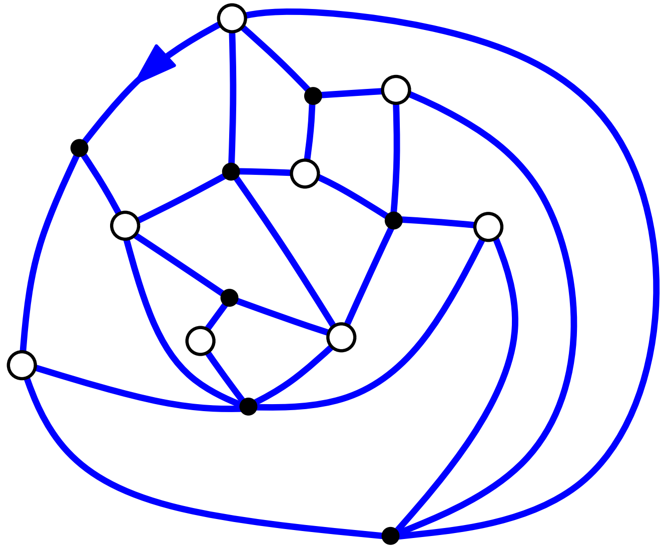
#(rooted maps with n edges) $=$ #(rooted quadrangulations with n faces)

Bijection maps $\leftrightarrow$ quadrangulations

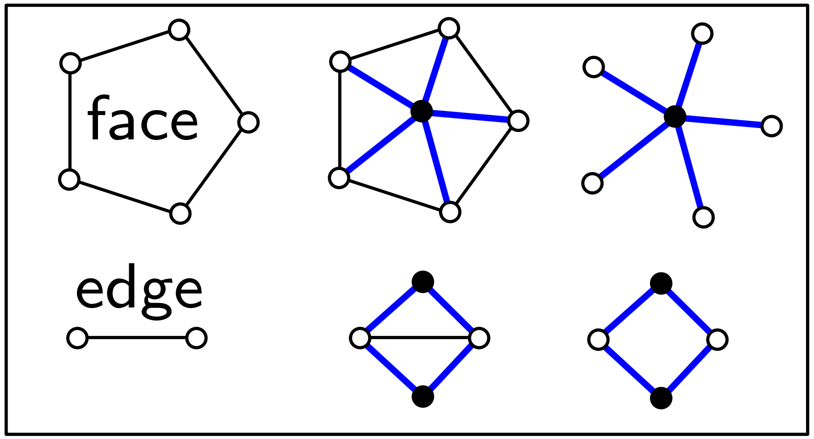


n edges
i vertices
j faces





n faces
i white vertices
j black vertices



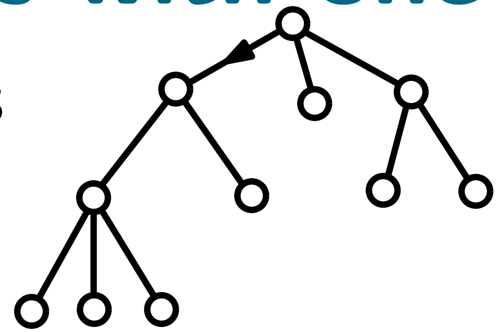
Consequence:

#(rooted maps with n edges) $=$ #(rooted quadrangulations with n faces)

It remains to see why this common number is $\dfrac{2 \cdot 3^n}{(n+1)(n+2)}\dbinom{2n}{n}$

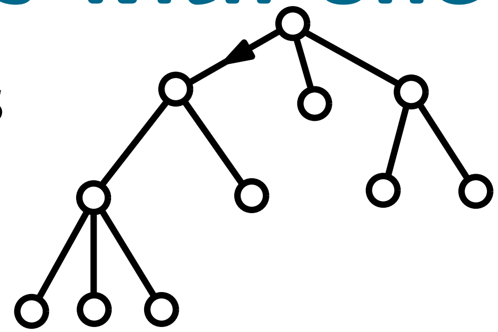
Counting rooted maps with one face

A rooted map with one face is called a **rooted plane tree**



Counting rooted maps with one face

A rooted map with one face is called a **rooted plane tree**



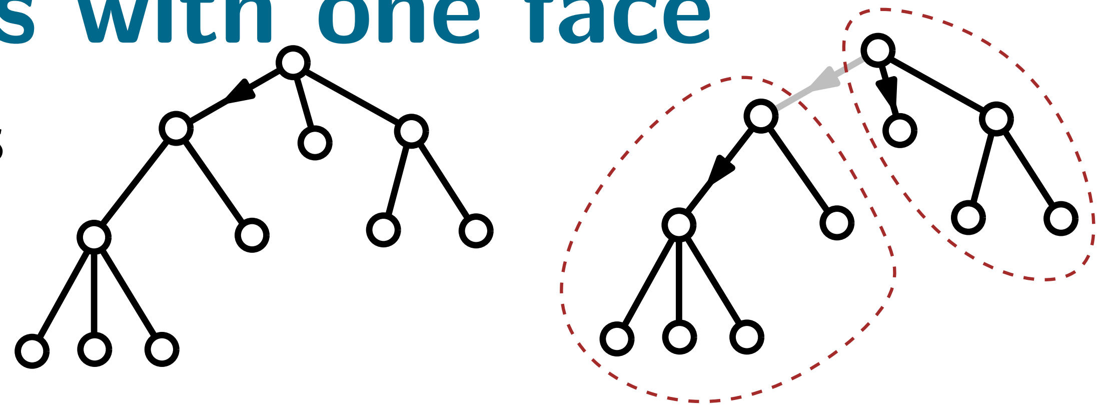
Let c_n be the number of rooted plane trees with n edges

Let $C(z) = \sum_{n\geq 0} c_n z^n$ be the associated generating function

$$C(z) = 1 + z + 2z^2 + 5z^3 + 14z^4 + \cdots$$

Counting rooted maps with one face

A rooted map with one face is called a **rooted plane tree**

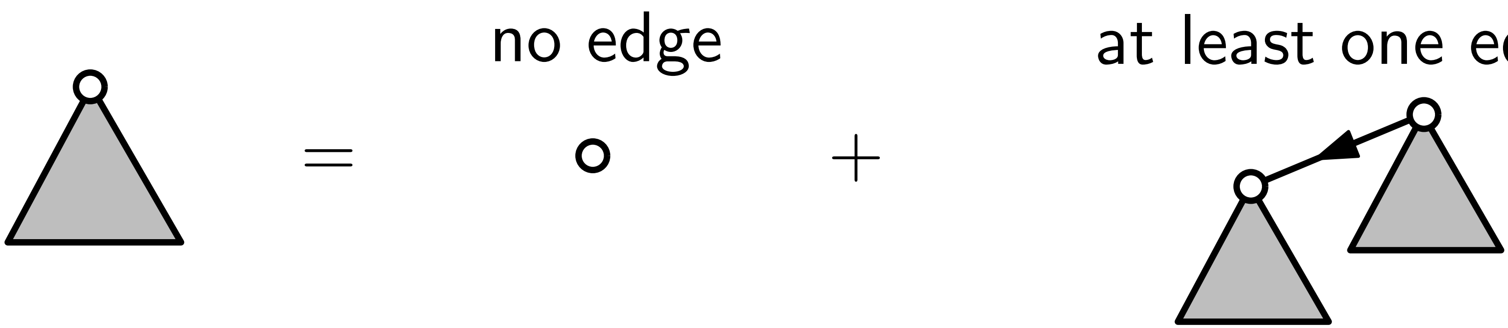


Let c_n be the number of rooted plane trees with n edges

Let $C(z) = \sum_{n \geq 0} c_n z^n$ be the associated generating function

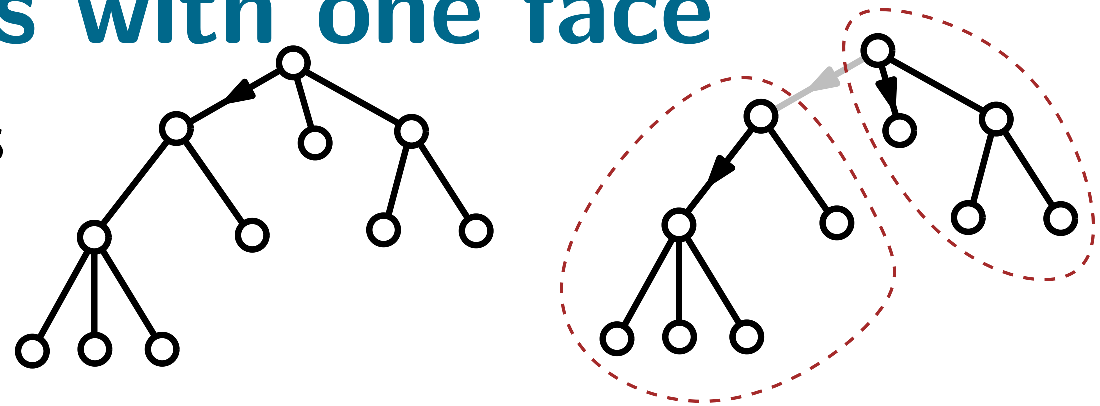
$$C(z) = 1 + z + 2z^2 + 5z^3 + 14z^4 + \cdots$$

Decomposition at the root:



Counting rooted maps with one face

A rooted map with one face is called a **rooted plane tree**

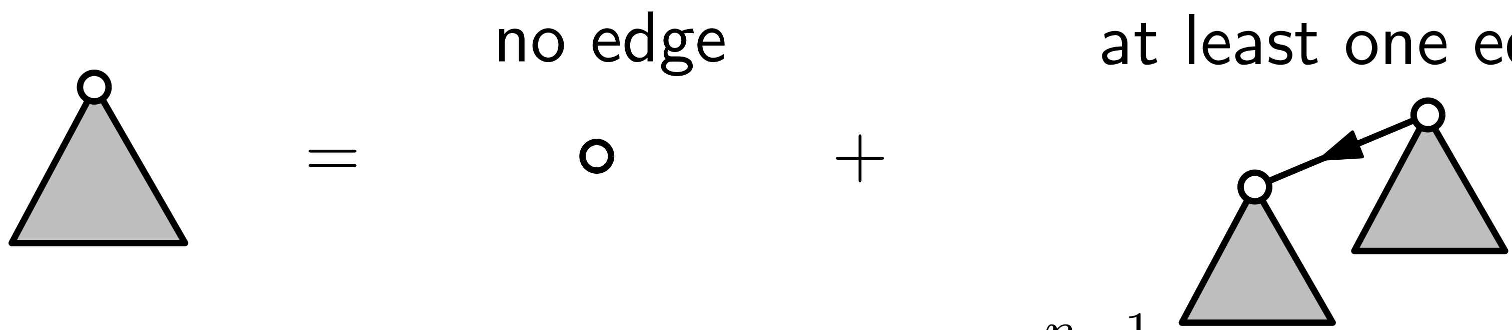


Let c_n be the number of rooted plane trees with n edges

Let $C(z) = \sum_{n \geq 0} c_n z^n$ be the associated generating function

$$C(z) = 1 + z + 2z^2 + 5z^3 + 14z^4 + \cdots$$

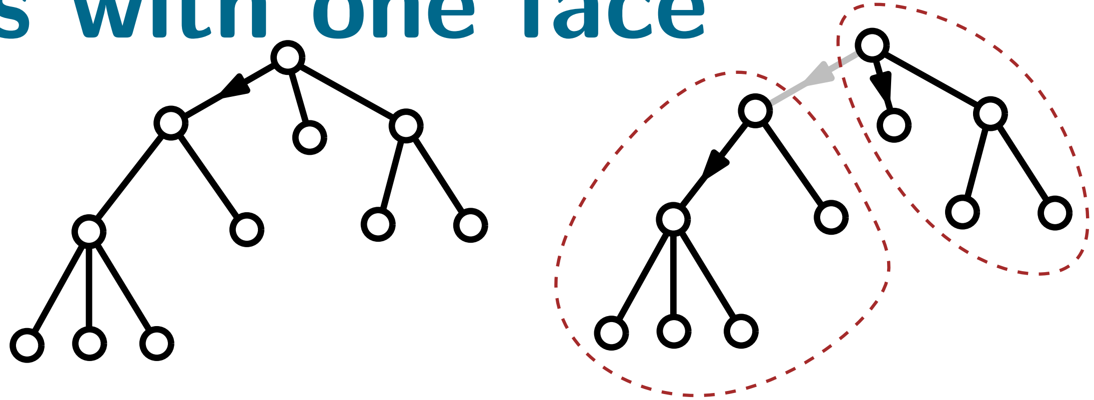
Decomposition at the root:



recurrence: $c_0 = 1$ and $c_n = \sum_{k=0}^{n-1} c_k c_{n-1-k}$ for $n \geq 1$

Counting rooted maps with one face

A rooted map with one face is called a **rooted plane tree**

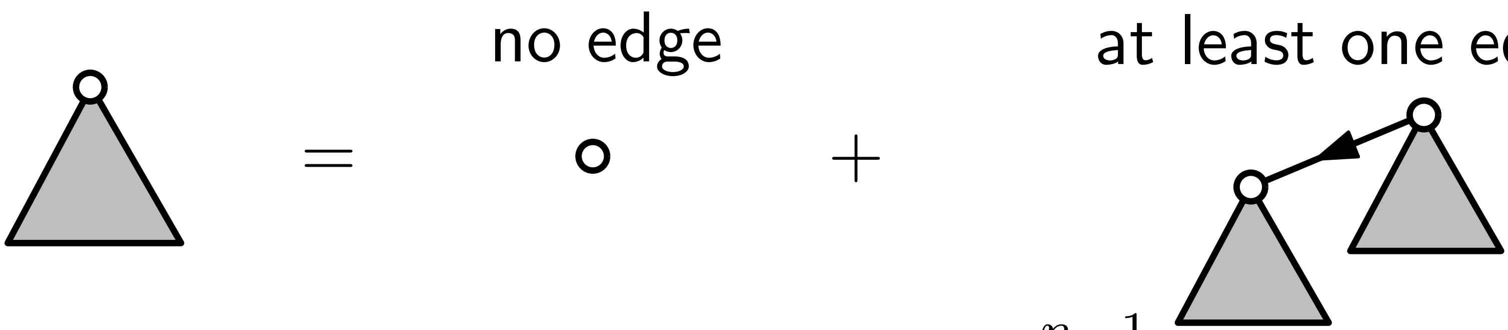


Let c_n be the number of rooted plane trees with n edges

Let $C(z) = \sum_{n \geq 0} c_n z^n$ be the associated generating function

$$C(z) = 1 + z + 2z^2 + 5z^3 + 14z^4 + \cdots$$

Decomposition at the root:

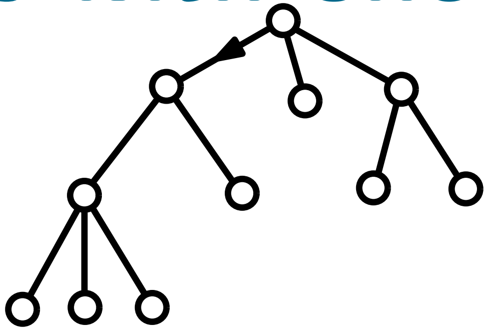


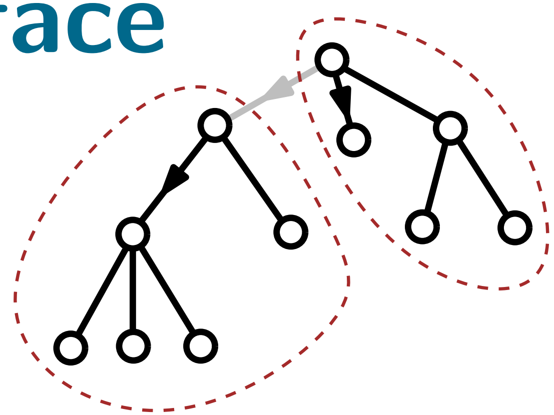
recurrence: $c_0 = 1$ and $c_n = \sum_{k=0}^{n-1} c_k c_{n-1-k}$ for $n \geq 1$

GF equation: $C(z) = 1 + z \cdot C(z)^2$

Counting rooted maps with one face

A rooted map with one face is called a **rooted plane tree**



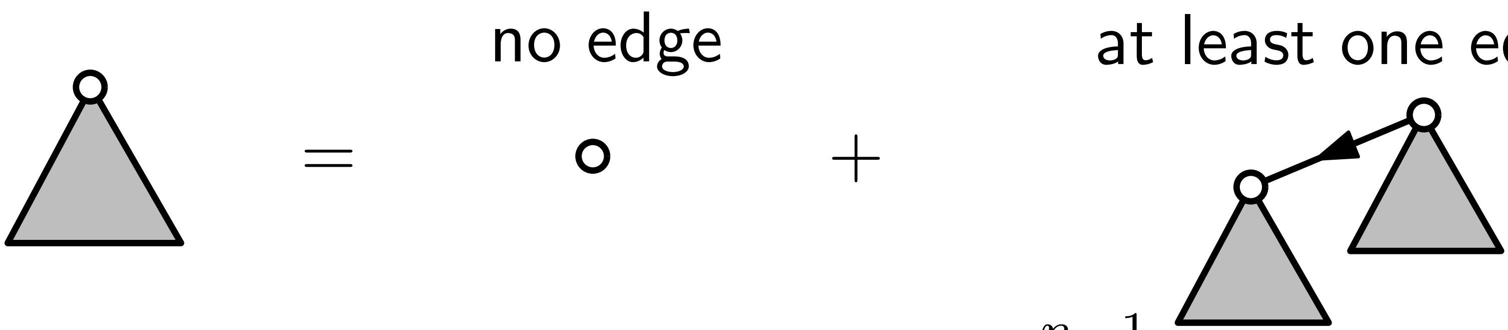


Let c_n be the number of rooted plane trees with n edges

Let $C(z) = \sum_{n \geq 0} c_n z^n$ be the associated generating function

$$C(z) = 1 + z + 2z^2 + 5z^3 + 14z^4 + \cdots$$

Decomposition at the root:

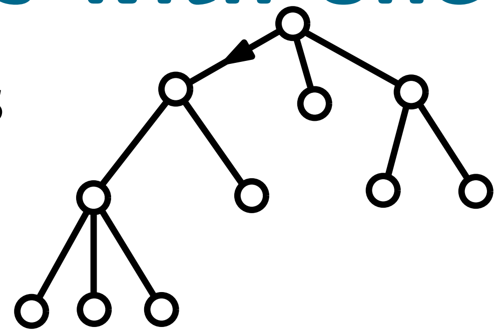


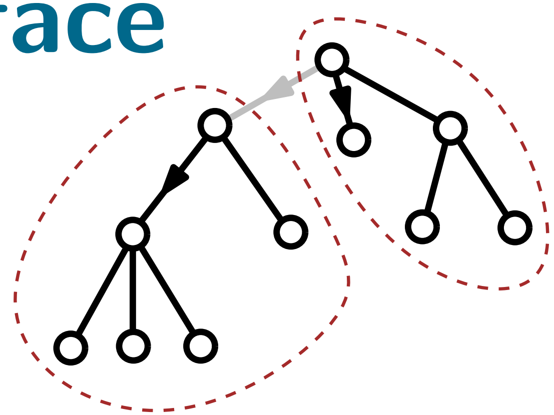
recurrence: $c_0 = 1$ and $c_n = \sum_{k=0}^{n-1} c_k c_{n-1-k}$ for $n \geq 1$

GF equation: $C(z) = 1 + z \cdot C(z)^2$ solved as $C(z) = \frac{1-\sqrt{1-4z}}{2z}$

Counting rooted maps with one face

A rooted map with one face is called a **rooted plane tree**





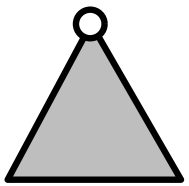
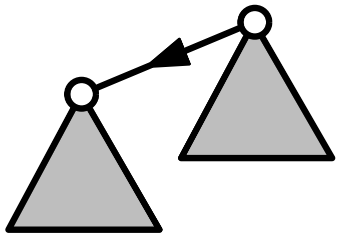
Let c_n be the number of rooted plane trees with n edges

Let $C(z) = \sum_{n \geq 0} c_n z^n$ be the associated generating function

$$C(z) = 1 + z + 2z^2 + 5z^3 + 14z^4 + \cdots$$

Decomposition at the root:

no edge   at least one edge

 $=$  $+$ 

recurrence: $c_0 = 1$ and $c_n = \sum_{k=0}^{n-1} c_k c_{n-1-k}$ for $n \geq 1$

GF equation: $C(z) = 1 + z \cdot C(z)^2$ solved as $C(z) = \frac{1-\sqrt{1-4z}}{2z}$

Taylor expansion: $C(z) = \sum_{n \geq 0} \frac{(2n)!}{n!(n+1)!}$ $\Rightarrow$ $\boxed{c_n = \frac{(2n)!}{n!(n+1)!}}$ Catalan numbers

Adaptation to rooted maps

Let m_n be the number of rooted maps with n edges

Let $M(z) = \sum_{n\geq 0} m_n z^n$ be the associated generating function

$$= 1 + 2z + 9z^2 + 54z^3 + 378z^4 + 2916z^5 + \cdots$$

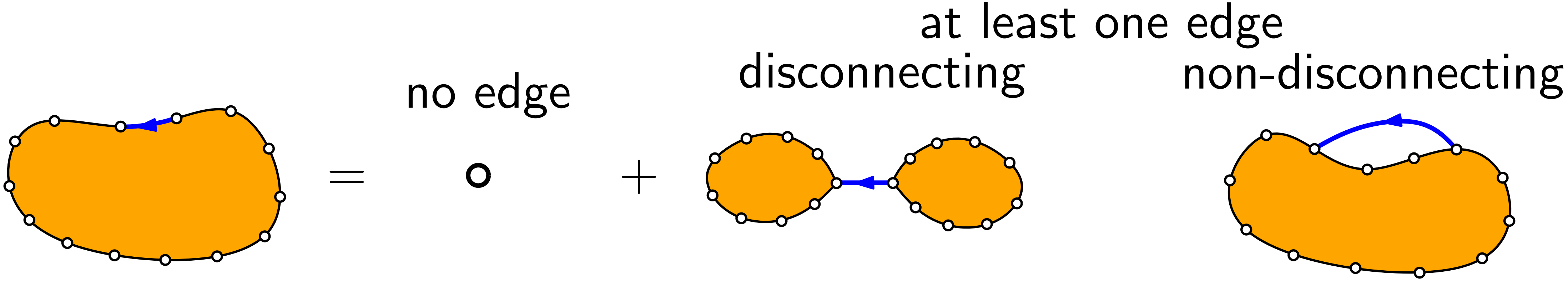
Adaptation to rooted maps

Let m_n be the number of rooted maps with n edges

Let $M(z) = \sum_{n \geq 0} m_n z^n$ be the associated generating function

$$= 1 + 2z + 9z^2 + 54z^3 + 378z^4 + 2916z^5 + \cdots$$

Decomposition by deleting the root:



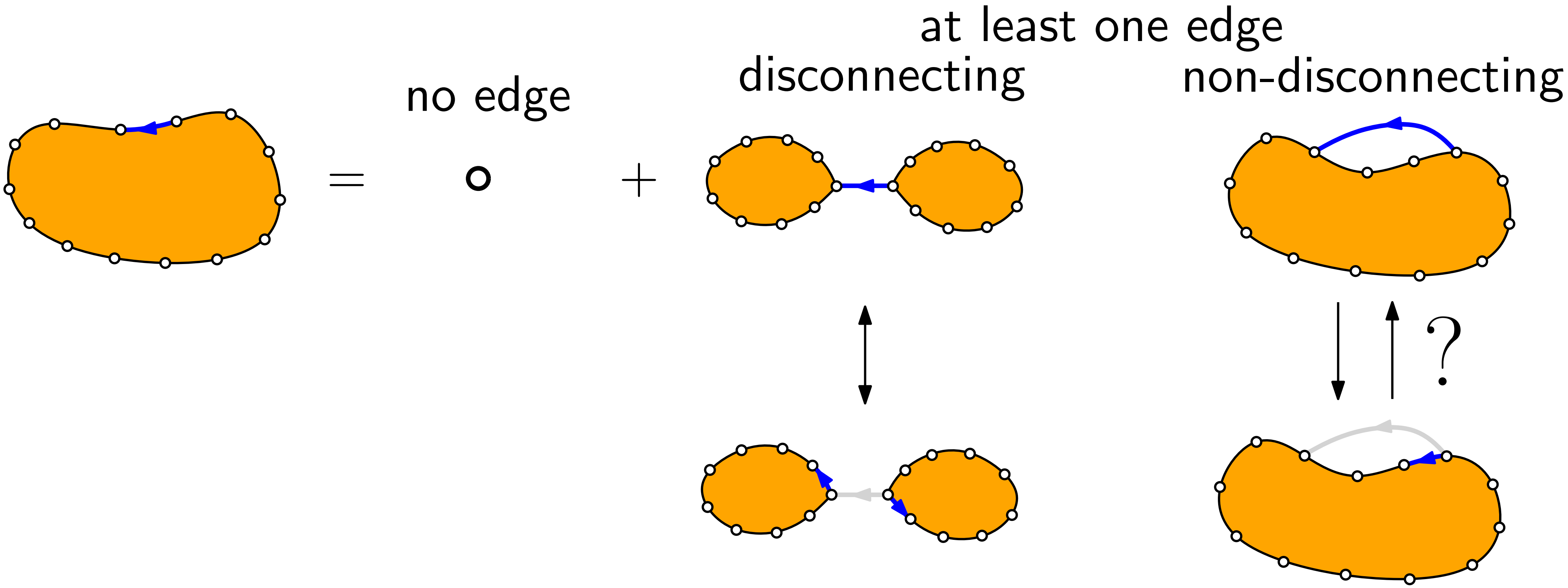
Adaptation to rooted maps

Let m_n be the number of rooted maps with n edges

Let $M(z) = \sum_{n \geq 0} m_n z^n$ be the associated generating function

$$= 1 + 2z + 9z^2 + 54z^3 + 378z^4 + 2916z^5 + \cdots$$

Decomposition by deleting the root:



$$M(z) \quad = \quad 1 \quad + \quad M(z)^2 \quad + \quad ?$$

Adding a secondary variable

Let $m_{n,k}$ be the number of rooted maps with n edges and outer degree k

Let $M(z,u) = \sum_{n,k\geq 0} m_{n,k} z^n u^k$ be the associated generating function

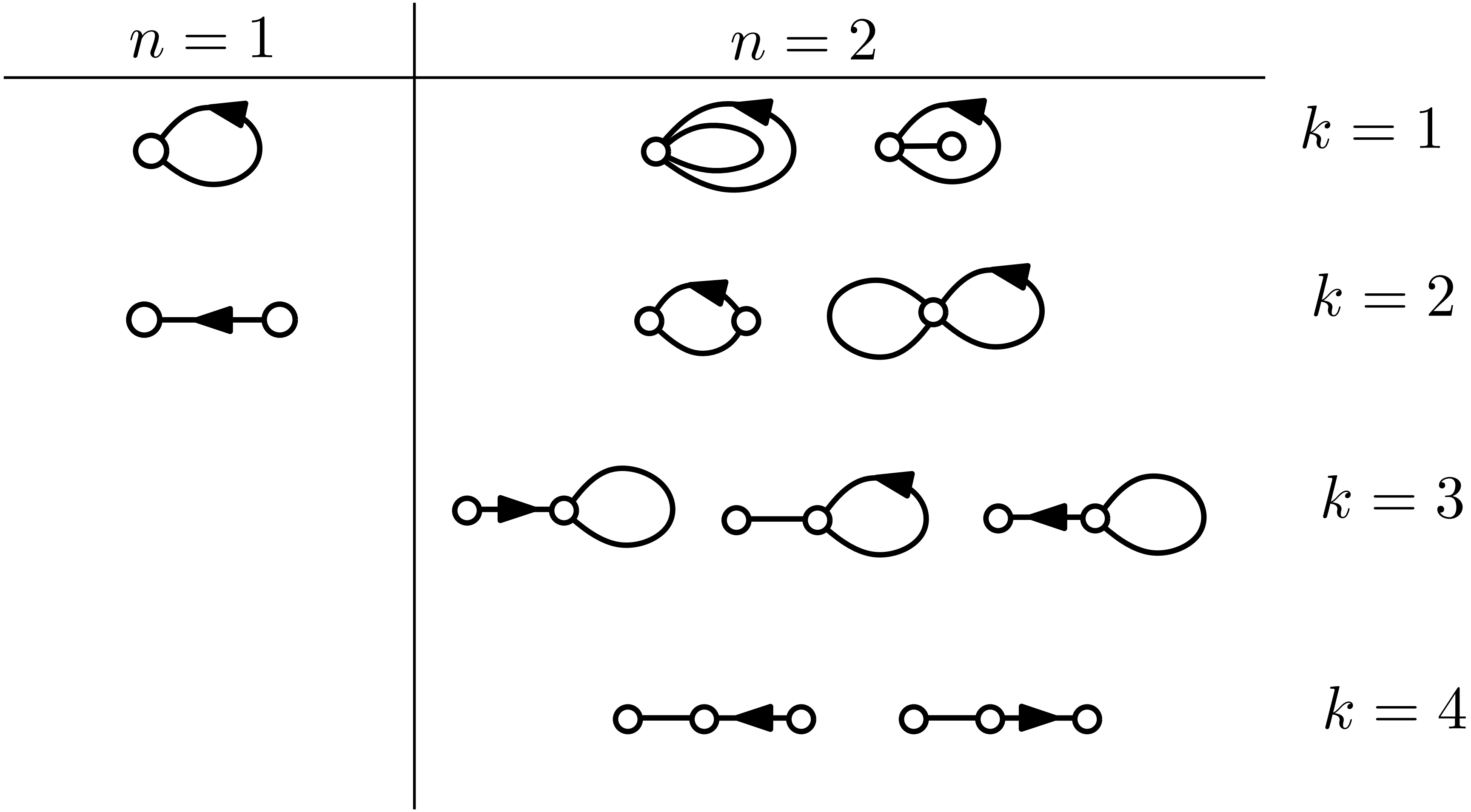
$$= 1 + z(u + u^2) + z^2(2u + 2u^2 + 3u^3 + 2u^4) + \cdots$$

Adding a secondary variable

Let $m_{n,k}$ be the number of rooted maps with n edges and outer degree k

Let $M(z,u) = \sum_{n,k\geq 0} m_{n,k} z^n u^k$ be the associated generating function

$$= 1 + z(u + u^2) + z^2(2u + 2u^2 + 3u^3 + 2u^4) + \cdots$$



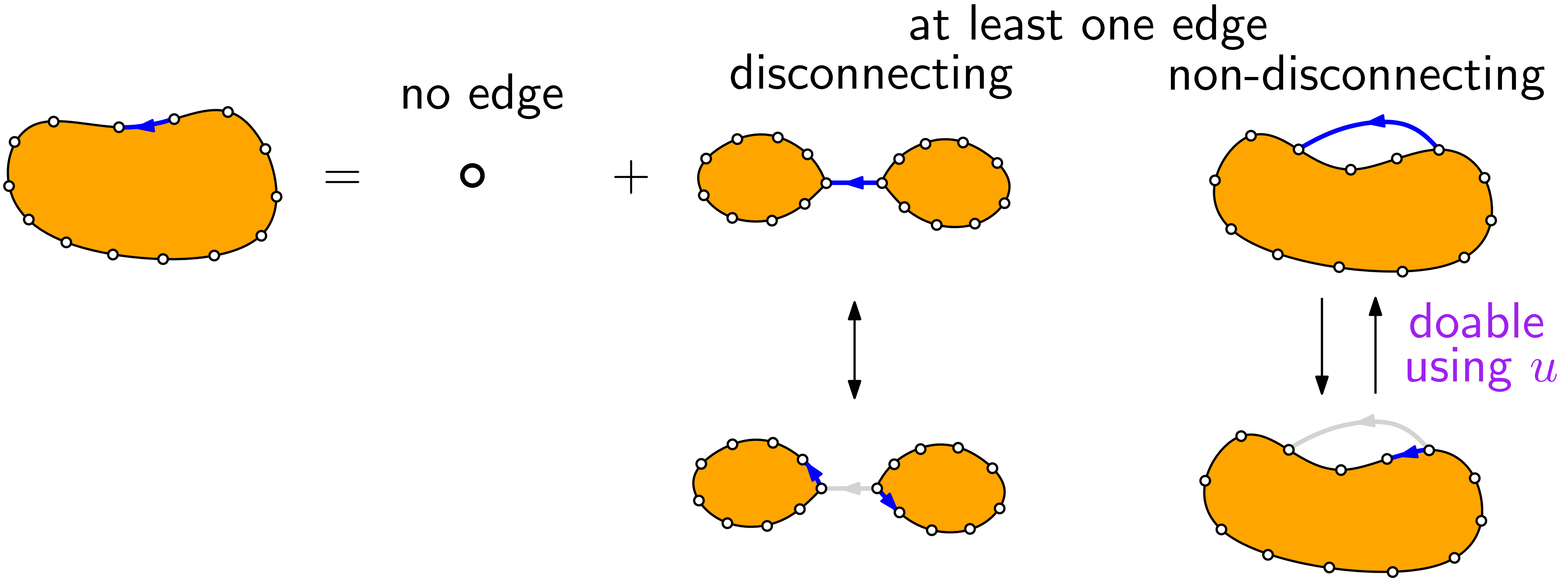
Adding a secondary variable

Let $m_{n,k}$ be the number of rooted maps with n edges and outer degree k

Let $M(z,u) = \sum_{n,k\geq 0} m_{n,k} z^n u^k$ be the associated generating function

$$= 1 + z(u+u^2) + z^2(2u + 2u^2 + 3u^3 + 2u^4) + \cdots$$

Decomposition by deleting the root:



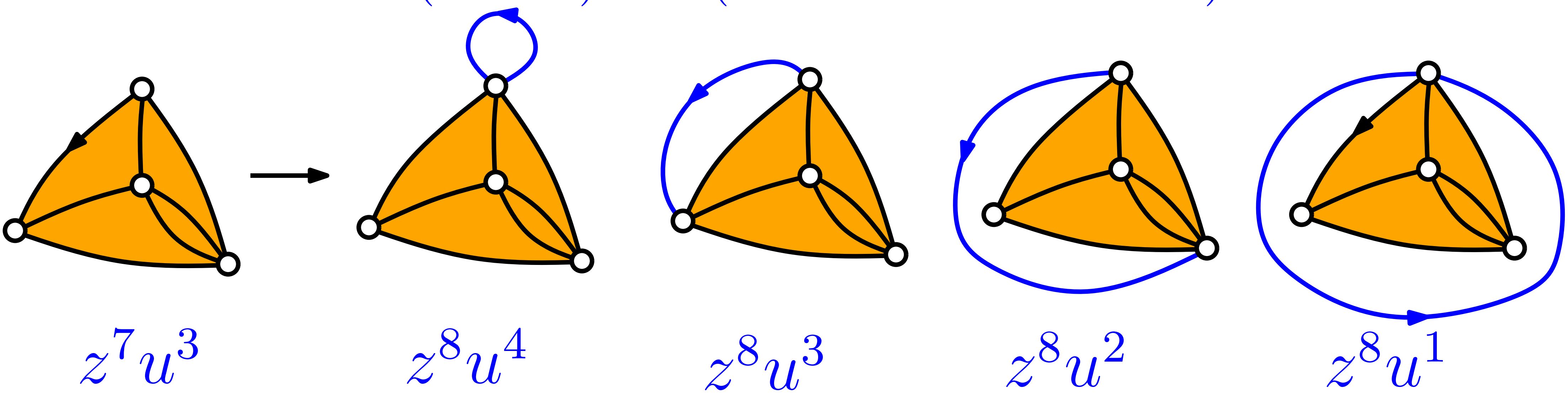
$$M(z,u) \quad = \quad 1 \quad + \quad zu^2 \cdot M(z,u)^2 \quad + \quad A(z,u)$$

Adding a secondary variable

Let $m_{n,k}$ be the number of rooted maps with n edges and outer degree k

Let $M(z,u) = \sum_{n,k\geq 0} m_{n,k} z^n u^k$ be the associated generating function

$$= 1 + z(u + u^2) + z^2(2u + 2u^2 + 3u^3 + 2u^4) + \cdots$$



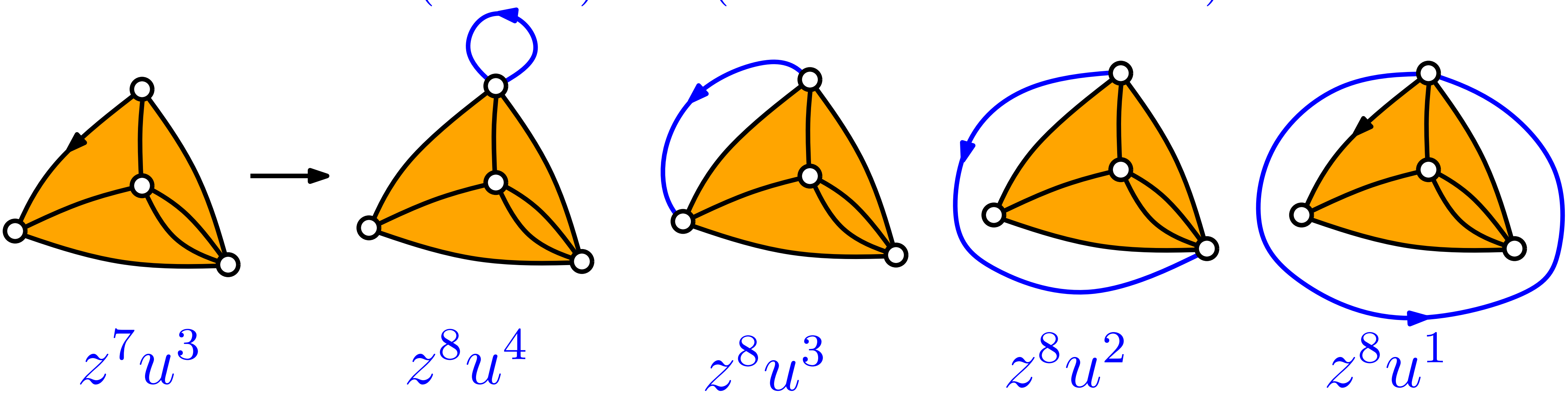
More generally $\quad z^n u^k \to z^{n+1} \cdot (u + u^2 + \cdots + u^{k+1})$

Adding a secondary variable

Let $m_{n,k}$ be the number of rooted maps with n edges and outer degree k

Let $M(z,u) = \sum_{n,k\geq 0} m_{n,k} z^n u^k$ be the associated generating function

$$= 1 + z(u+u^2) + z^2(2u+2u^2+3u^3+2u^4) + \cdots$$



More generally $\quad z^n u^k \to z^{n+1} \cdot (u + u^2 + \cdots + u^{k+1})$

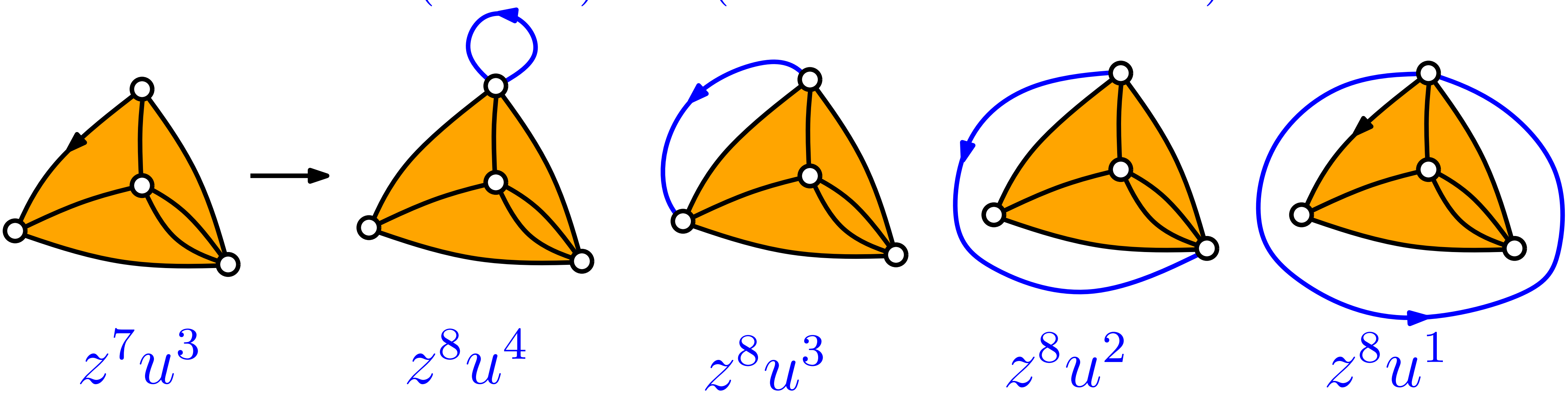
$$\Rightarrow A(z,u) = \sum_{n,k} m_{n,k}\, z^{n+1} \cdot \underbrace{(u + \cdots + u^{k+1})}_{u \cdot \frac{u^{k+1}-1}{u-1}}$$

Adding a secondary variable

Let $m_{n,k}$ be the number of rooted maps with n edges and outer degree k

Let $M(z,u) = \sum_{n,k\geq 0} m_{n,k} z^n u^k$ be the associated generating function

$$= 1 + z(u + u^2) + z^2(2u + 2u^2 + 3u^3 + 2u^4) + \cdots$$



More generally $\quad z^n u^k \to z^{n+1} \cdot (u + u^2 + \cdots + u^{k+1})$

$$\Rightarrow A(z,u) = \sum_{n,k} m_{n,k}\, z^{n+1} \cdot \underbrace{(u + \cdots + u^{k+1})}_{u \cdot \frac{u^{k+1}-1}{u-1}} = zu\frac{uM(z,u) - M(z,1)}{u-1}$$

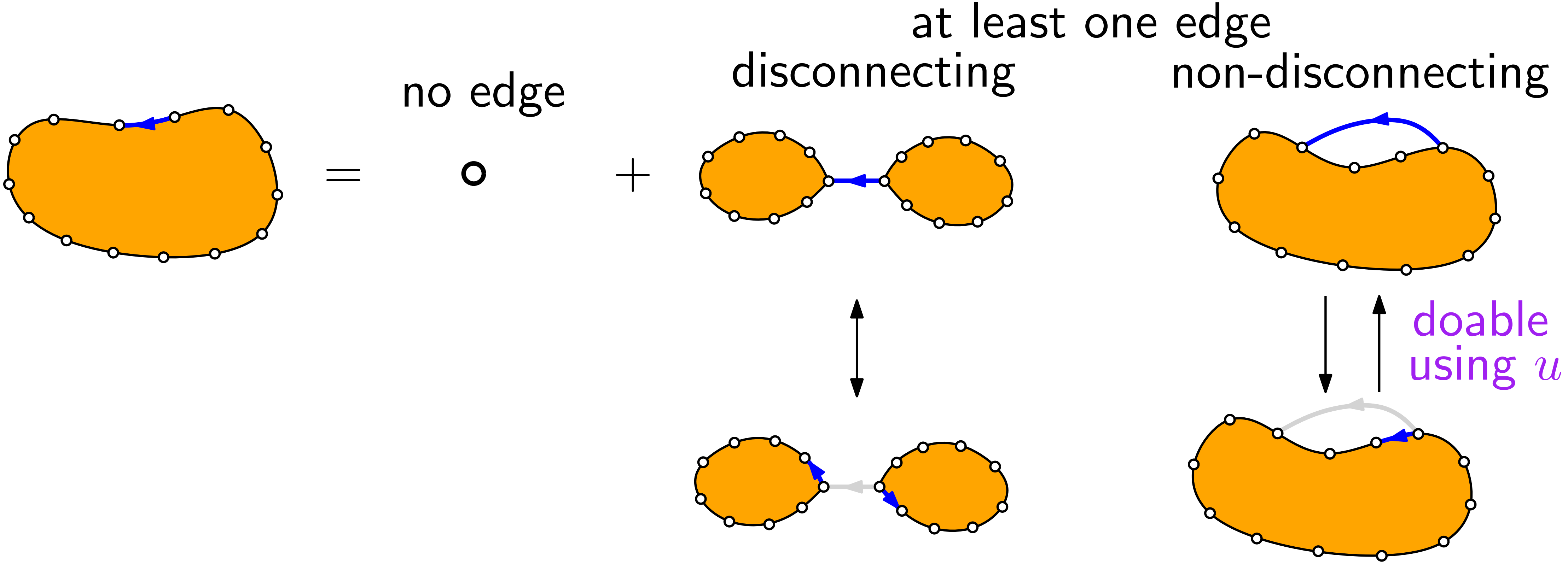
Adding a secondary variable

Let $m_{n,k}$ be the number of rooted maps with n edges and outer degree k

Let $M(z,u) = \sum_{n,k\geq 0} m_{n,k} z^n u^k$ be the associated generating function

$$= 1 + z(u+u^2) + z^2(2u+2u^2+3u^3+2u^4) + \cdots$$

Decomposition by deleting the root:



$$M(z,u) \quad = \quad 1 \quad + \quad zu^2 \cdot M(z,u)^2 \quad + \quad zu\frac{uM(z,u)-M(z,1)}{u-1}$$

Adding a secondary variable

Let $m_{n,k}$ be the number of rooted maps with n edges and outer degree k

Let $M(z,u) = \sum_{n,k\geq 0} m_{n,k} z^n u^k$ be the associated generating function

Functional equation obtained:

$$M(z,u) \quad = \quad 1 \quad + \quad zu^2 \cdot M(z,u)^2 \; + \; zu\frac{uM(z,u) - M(z,1)}{u-1}$$

of the form $P(M(z,u), M(z,1), z, u) = 0$

Adding a secondary variable

Let $m_{n,k}$ be the number of rooted maps with n edges and outer degree k

Let $M(z,u) = \sum_{n,k\geq 0} m_{n,k} z^n u^k$ be the associated generating function

Functional equation obtained:

$$M(z,u) \quad = \quad 1 \quad + \quad zu^2 \cdot M(z,u)^2 \; + \; zu\frac{uM(z,u) - M(z,1)}{u-1}$$

of the form $P(M(z,u), M(z,1), z, u) = 0$

One equation, two unknown: $M(z,u)$ and $M(z,1)$

But a unique solution (2 unknown are correlated)

Equation $\Rightarrow$ $M(z,u) = 1 + z(u+u^2) + z^2(2u + 2u^2 + 3u^3 + 2u^4) + \cdots$

Adding a secondary variable

Let $m_{n,k}$ be the number of rooted maps with n edges and outer degree k

Let $M(z,u) = \sum_{n,k\geq 0} m_{n,k} z^n u^k$ be the associated generating function

Functional equation obtained:

$$M(z,u) \quad = \quad 1 \quad + \quad zu^2 \cdot M(z,u)^2 \; + \; zu\frac{uM(z,u) - M(z,1)}{u-1}$$

of the form $P(M(z,u), M(z,1), z, u) = 0$

One equation, two unknown: $M(z,u)$ and $M(z,1)$

But a unique solution (2 unknown are correlated)

Equation $\Rightarrow$ $M(z,u) = 1 + z(u+u^2) + z^2(2u + 2u^2 + 3u^3 + 2u^4) + \cdots$

Guess/and/check or explicit solution methods:

[Brown, Tutte'65, Bousquet-Mélou-Jehanne'06, Eynard'10]

$$\Rightarrow M(z,1) = \frac{1}{54z^2}(-1 + 18z + (1-12z)^{3/2}) = \sum_{n\geq 0} \frac{2\cdot 3^n}{(n+2)(n+1)}\binom{2n}{n} z^n$$

Bijective proof: which formula to prove

Let $q_n = \#(\text{rooted quadrangulations with } n \text{ faces})$

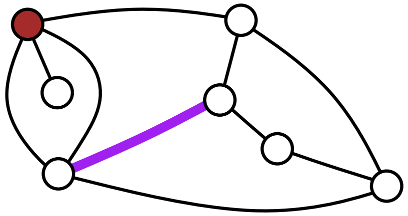
We want to show (bijectively) that $q_n = \frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n} z^n$

Bijective proof: which formula to prove

Let $q_n = \#(\text{rooted quadrangulations with } n \text{ faces})$

We want to show (bijectively) that $$q_n = \frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n} z^n$$

Consider $b_n = \#(\text{quad. with } n \text{ faces, a marked vertex and a marked edge})$

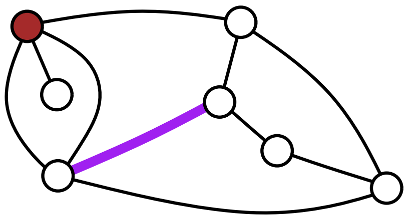


Bijective proof: which formula to prove

Let $q_n = \#$(rooted quadrangulations with n faces)

We want to show (bijectively) that $q_n = \frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n} z^n$

Consider $b_n = \#$(quad. with n faces, a marked vertex and a marked edge)

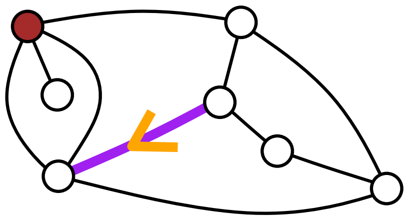


Bijective proof: which formula to prove

Let $q_n = \#$(rooted quadrangulations with n faces)

We want to show (bijectively) that $q_n = \frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n} z^n$

Consider $b_n = \#$(quad. with n faces, a marked vertex and a marked edge)



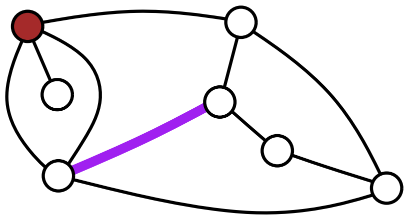
Simple relation between b_n and q_n: $\underbrace{(n+2)}_{\#\text{(vertices)}} \cdot q_n = 2 \cdot b_n$

Bijective proof: which formula to prove

Let $q_n = \#$(rooted quadrangulations with n faces)

We want to show (bijectively) that $q_n = \frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n} z^n$

Consider $b_n = \#$(quad. with n faces, a marked vertex and a marked edge)



Simple relation between b_n and q_n: $\underbrace{(n+2)}_{\#(\text{vertices})} \cdot q_n = 2 \cdot b_n$

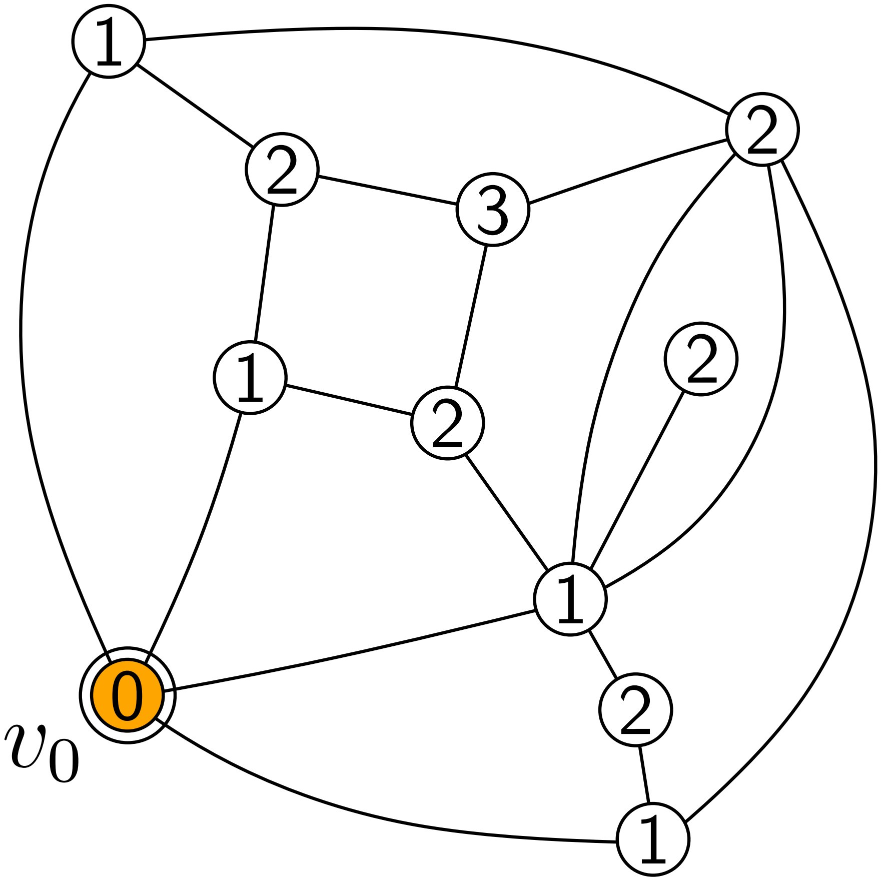
Hence showing $q_n = \frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n} z^n$

amounts to showing $$b_n = 3^n \frac{(2n)!}{n!(n+1)!} = 3^n \mathrm{Cat}_n$$

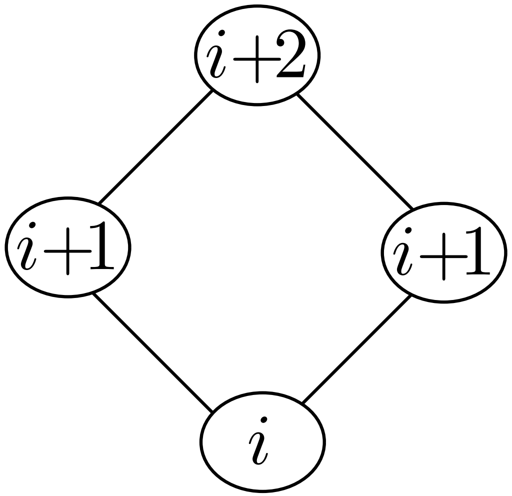
Pointed quadrangulations, geodesic labelling

Pointed quadrangulation = quadrangulation with a marked vertex v_0

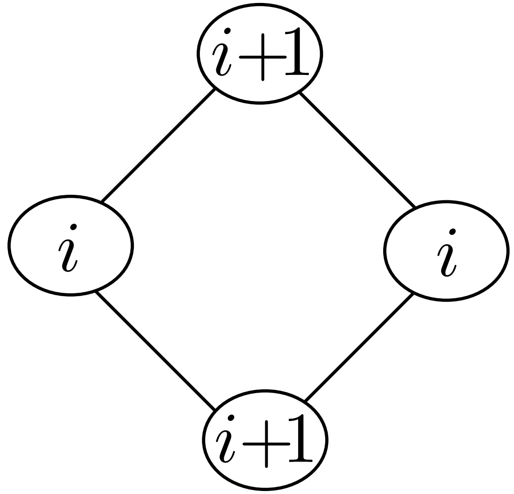
Geodesic labelling with respect to v_0: $\ell(v) = \mathrm{dist}(v_0, v)$



Rk: two types of faces



stretched

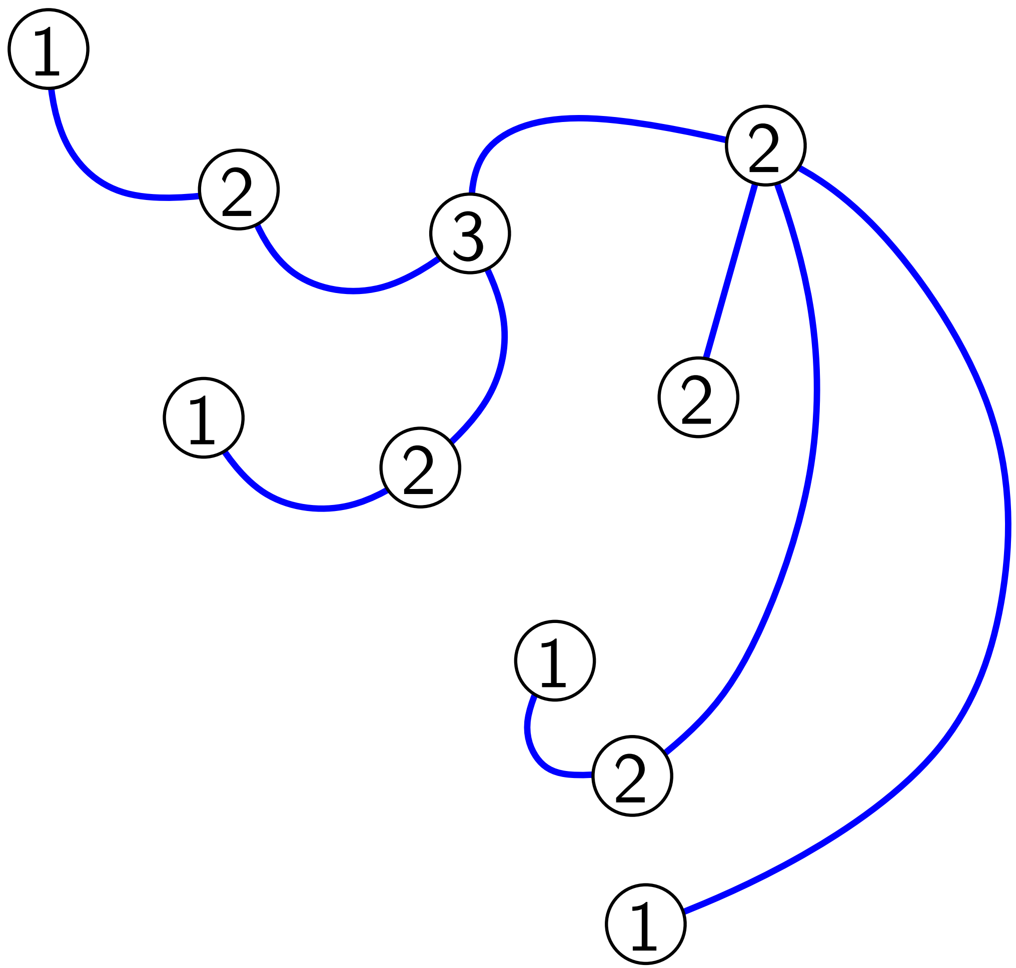


confluent

Well-labelled trees

Well-labelled tree = plane tree where

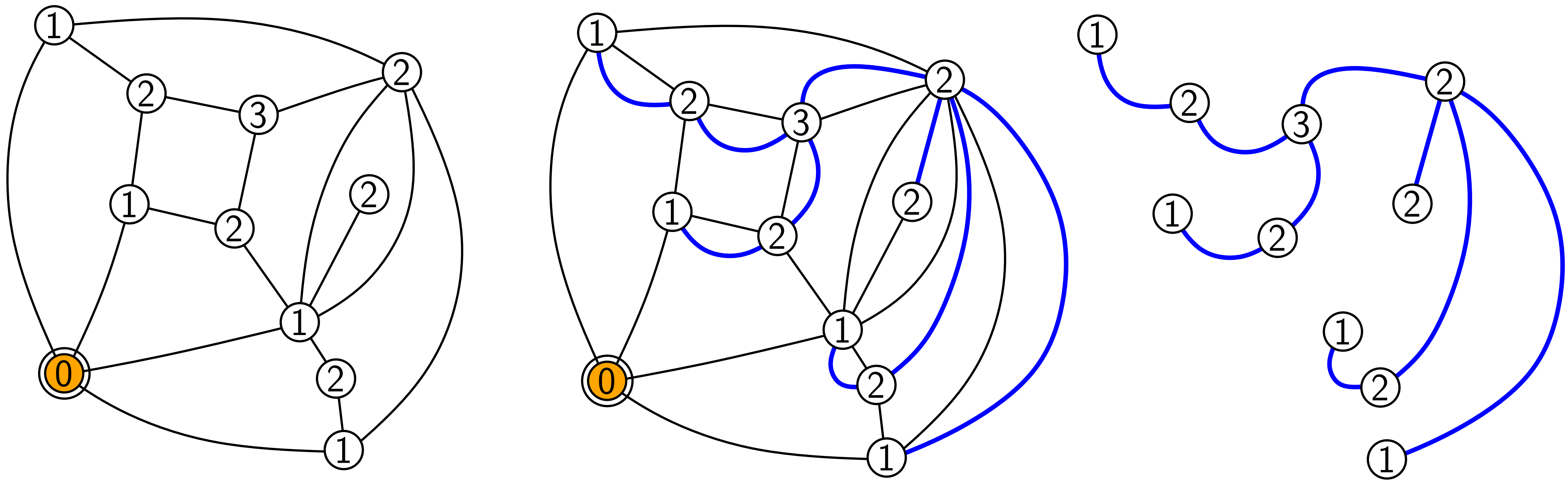
- each vertex v has a label $\ell(v) \in \mathbb{Z}$
- each edge $e = \{u, v\}$ satisfies $|\ell(u) - \ell(v)| \leq 1$
- the minimum label over all vertices is 1



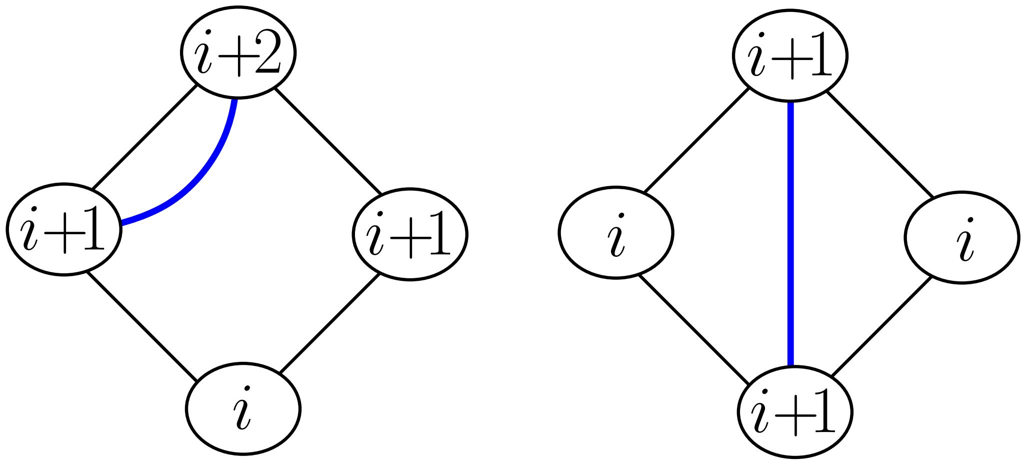
The Schaeffer bijection **[Schaeffer'99]**, also **[Cori-Vauquelin'81]**

Pointed quadrangulation $\Rightarrow$ well-labelled tree with min-label=1

n faces $\qquad\qquad$ n edges

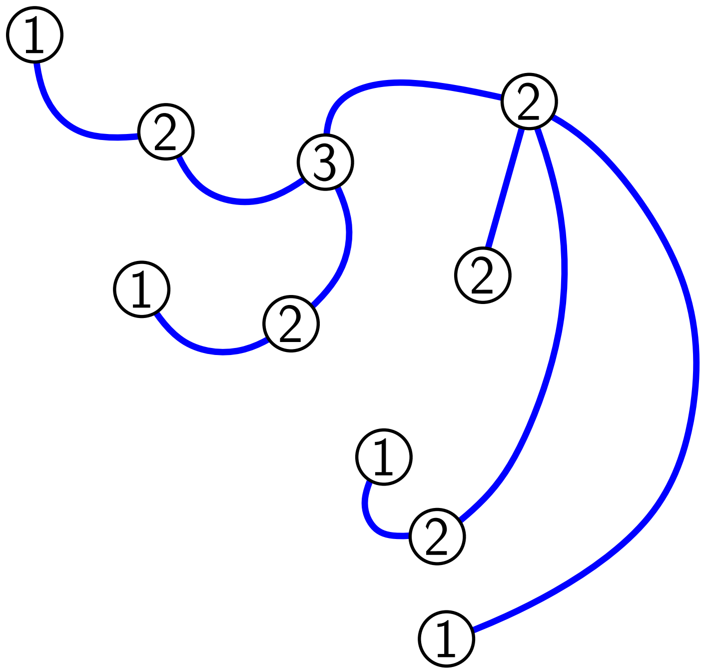


Local rule in each face:



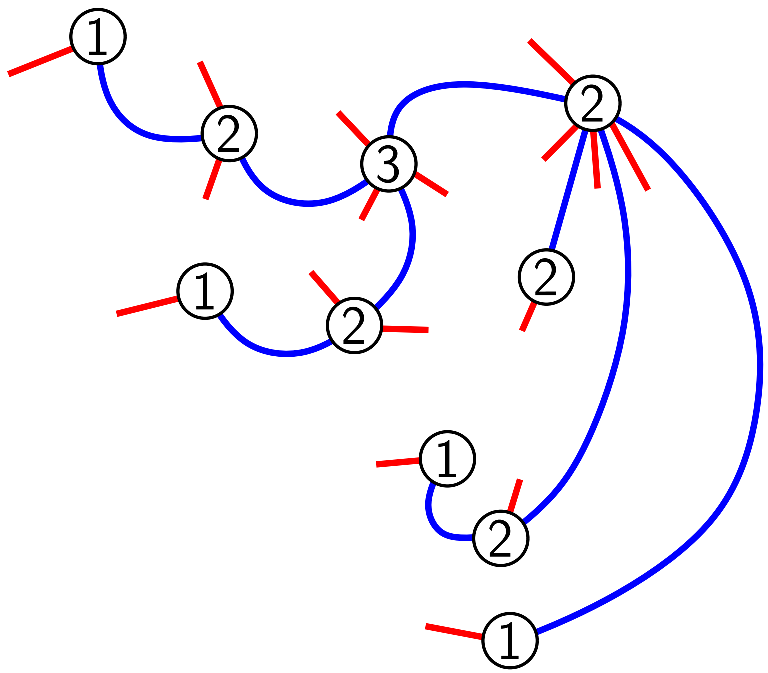
The Schaeffer bijection **[Schaeffer'99]**, also **[Cori-Vauquelin'81]**

From a well-labelled tree to a pointed quadrangulation



The Schaeffer bijection **[Schaeffer'99]**, also **[Cori-Vauquelin'81]**

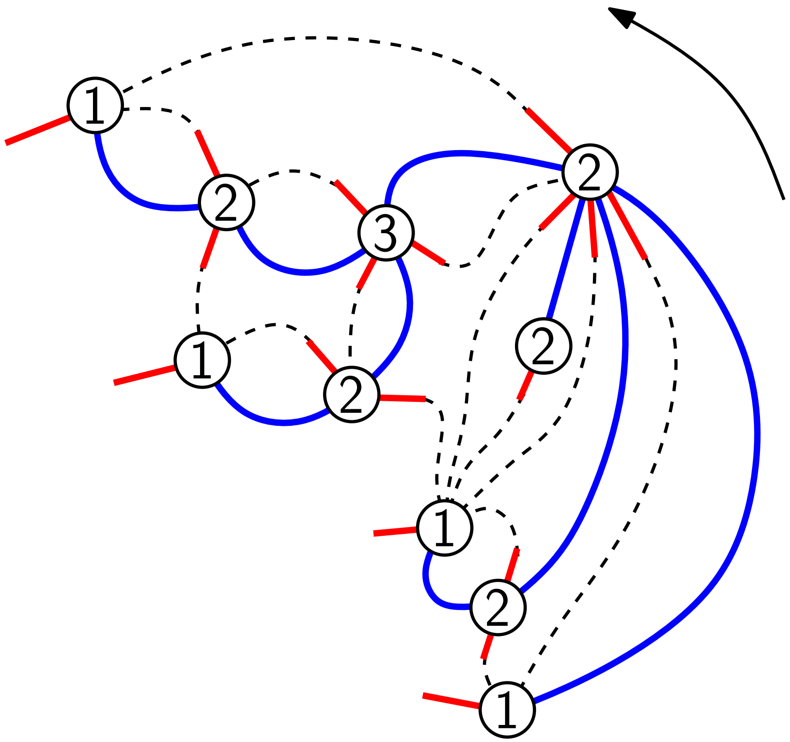
From a well-labelled tree to a pointed quadrangulation



1) insert a "leg" at each corner

The Schaeffer bijection **[Schaeffer'99]**, also **[Cori-Vauquelin'81]**

From a well-labelled tree to a pointed quadrangulation

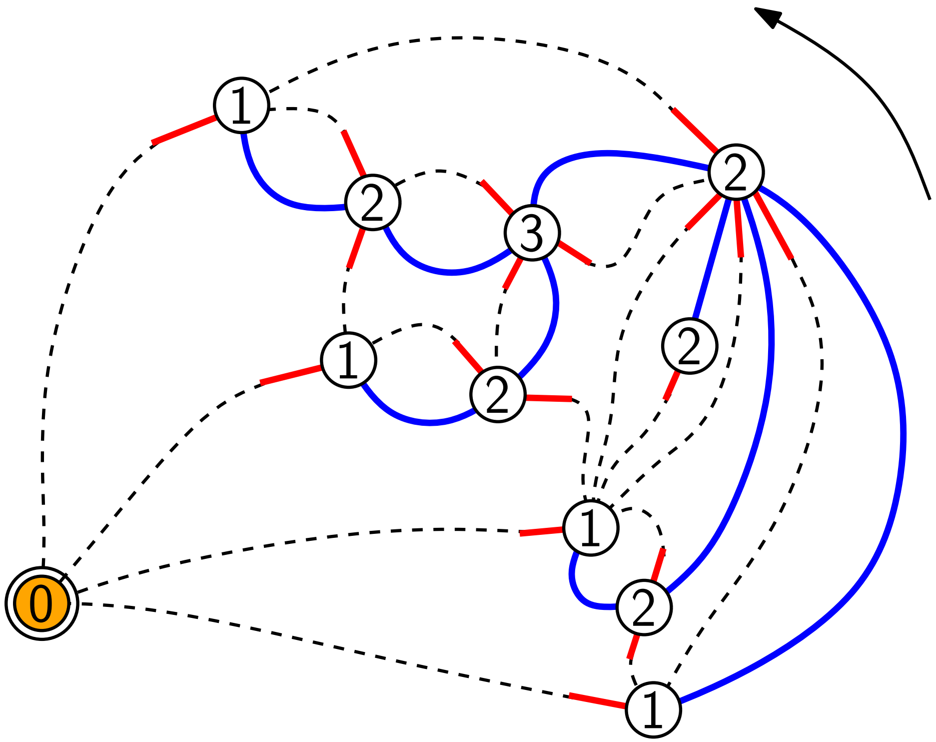


1) insert a "leg" at each corner

2) connect each leg of label $i \geq 2$ to the next corner of label $i-1$ in ccw order around the tree

The Schaeffer bijection **[Schaeffer'99]**, also **[Cori-Vauquelin'81]**

From a well-labelled tree to a pointed quadrangulation



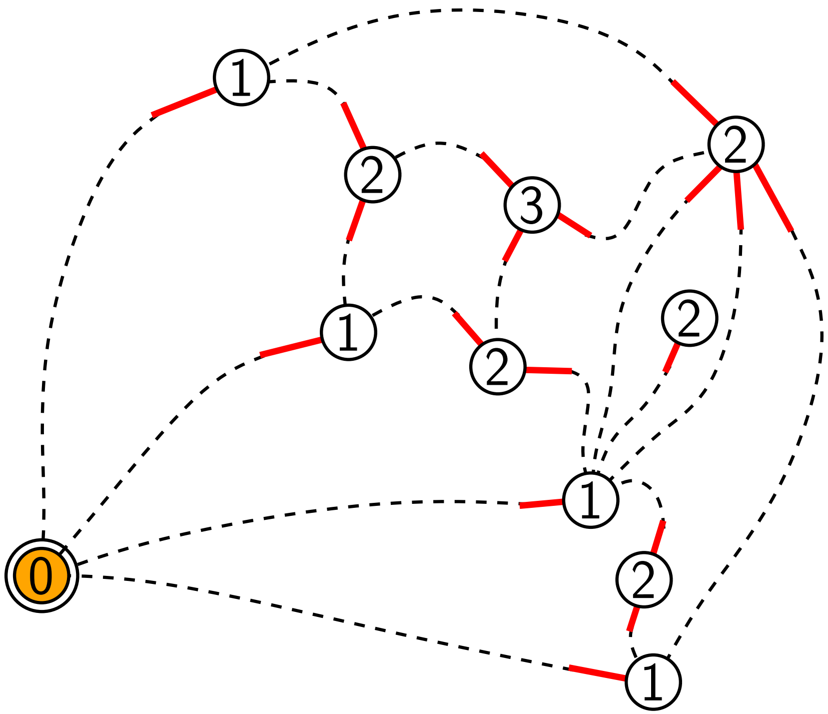
1) insert a "leg" at each corner

2) connect each leg of label $i \geq 2$ to the next corner of label $i-1$ in ccw order around the tree

3) create a new vertex v_0 outside and connect legs of label 1 to it

The Schaeffer bijection **[Schaeffer'99]**, also **[Cori-Vauquelin'81]**

From a well-labelled tree to a pointed quadrangulation



1) insert a "leg" at each corner

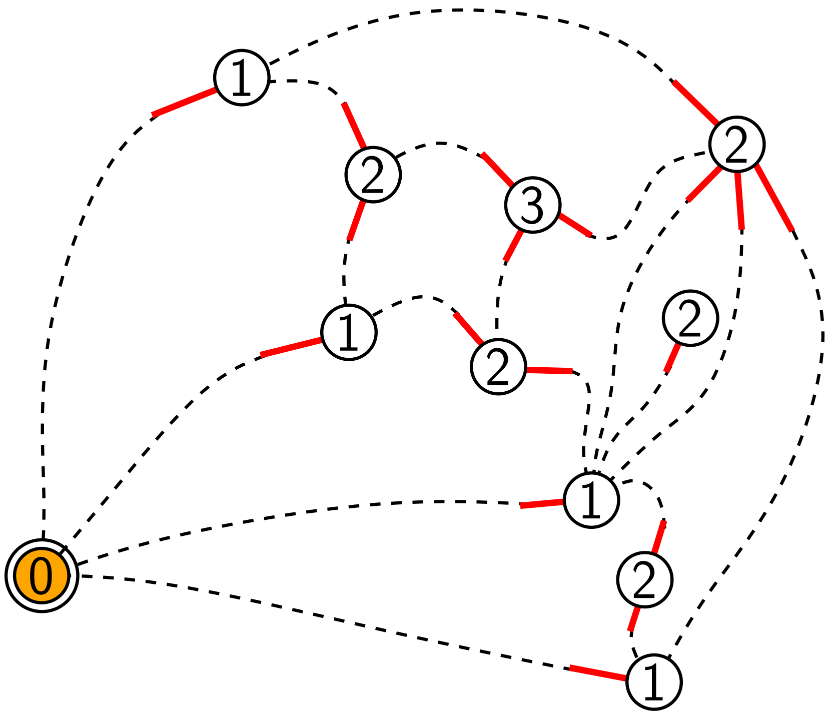
2) connect each leg of label $i \geq 2$ to the next corner of label $i-1$ in ccw order around the tree

3) create a new vertex v_0 outside and connect legs of label 1 to it

4) erase the tree-edges

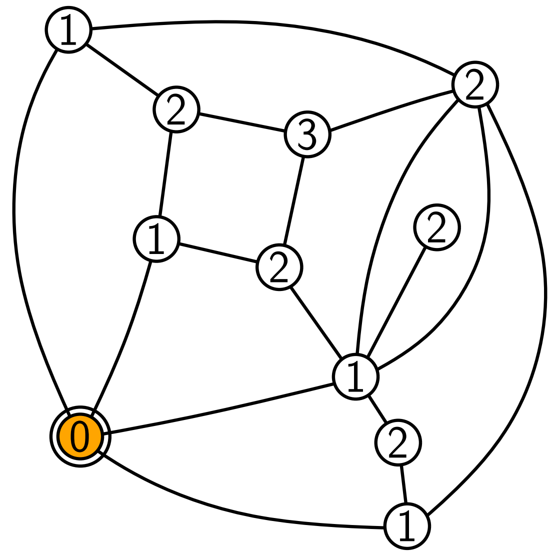
The Schaeffer bijection **[Schaeffer'99]**, also **[Cori-Vauquelin'81]**

From a well-labelled tree to a pointed quadrangulation

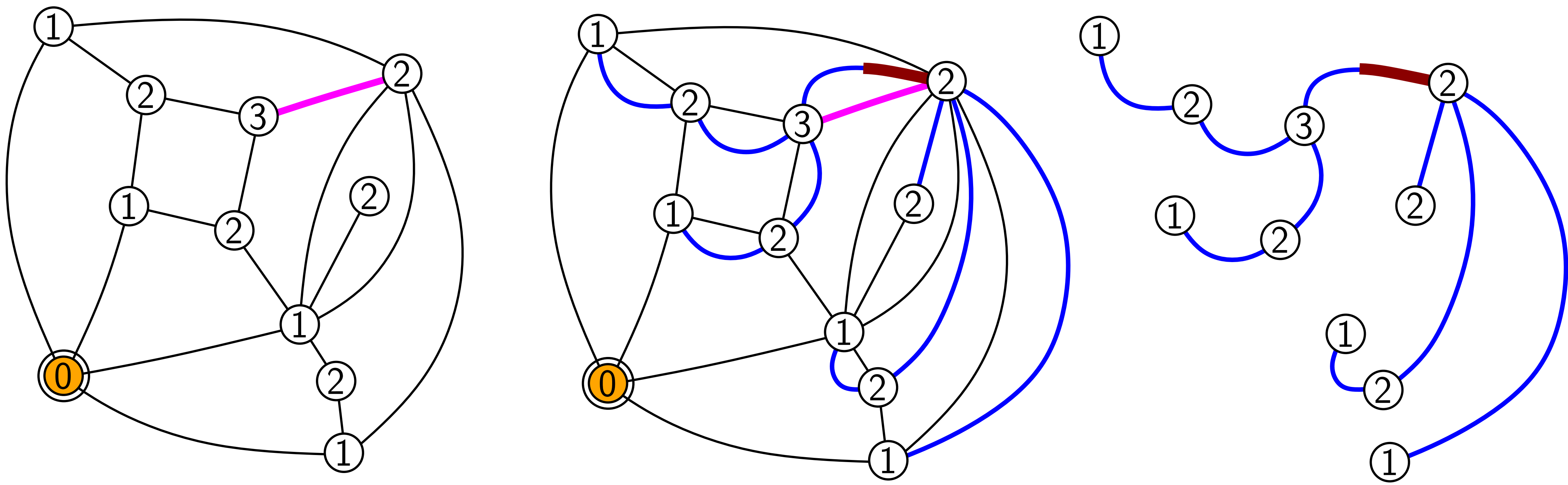


1) insert a "leg" at each corner
2) connect each leg of label $i \geq 2$ to the next corner of label $i-1$ in ccw order around the tree
3) create a new vertex v_0 outside and connect legs of label 1 to it
4) erase the tree-edges

recover the original pointed quadrangulation

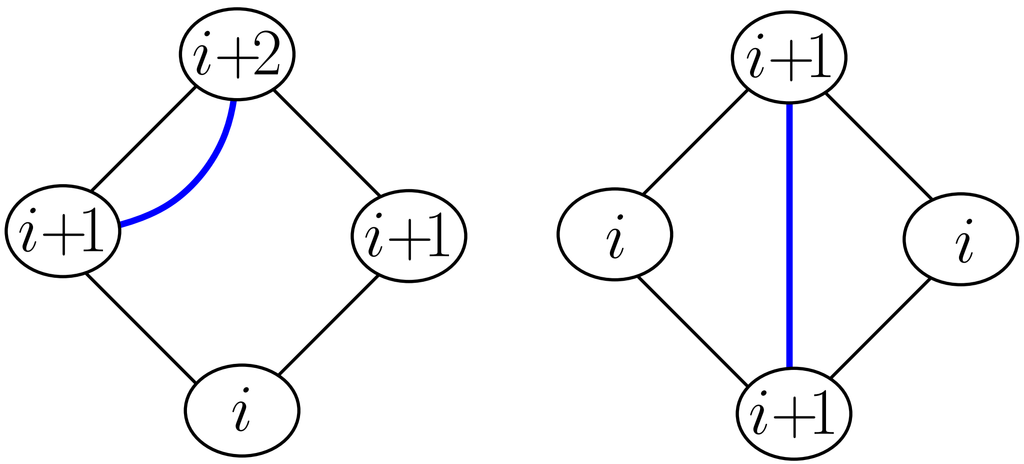


The effect of marking an edge



Local rule in each face:

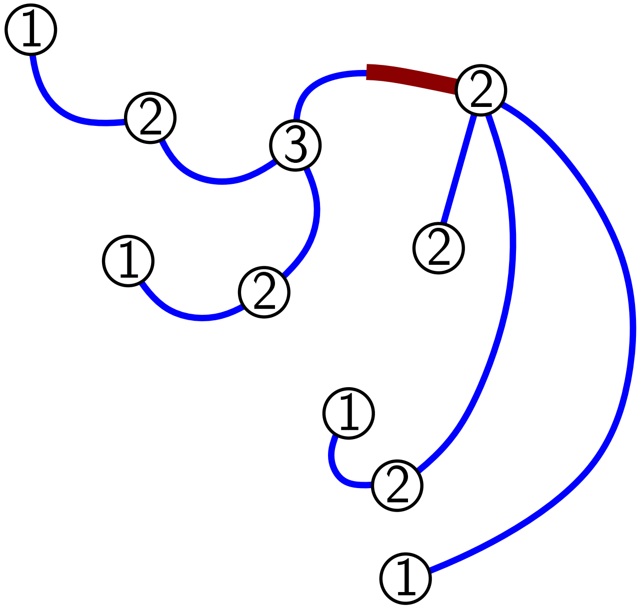
marked edge



marked half-edge

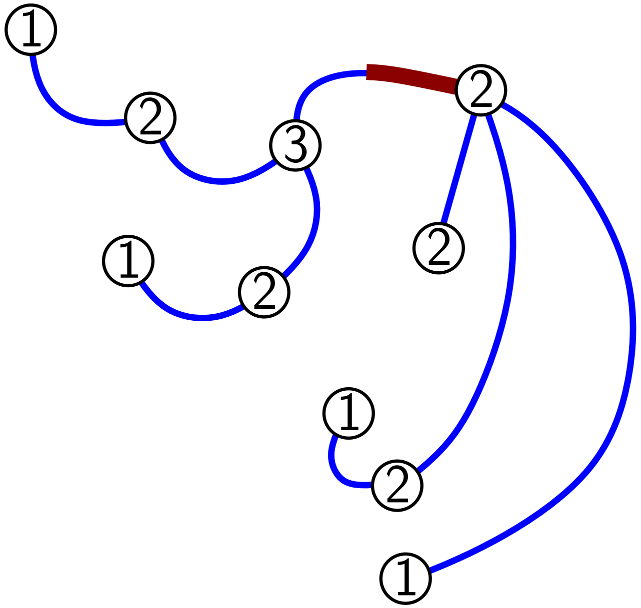
Bijective proof of counting formula

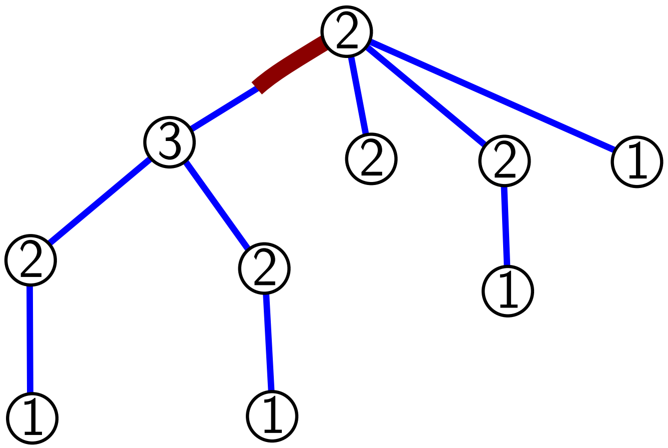
Schaeffer's bijection $\Rightarrow b_n = \#$(rooted well-labelled trees with n edges)

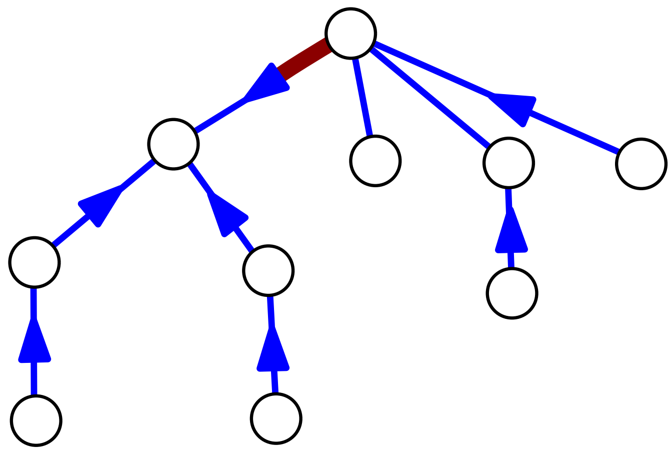


Bijective proof of counting formula

Schaeffer's bijection $\Rightarrow b_n = \#$(rooted well-labelled trees with n edges)







$$b_n = 3^n \mathrm{Cat}_n = 3^n \frac{(2n)!}{n!(n+1)!}$$

Application to study distances in random maps

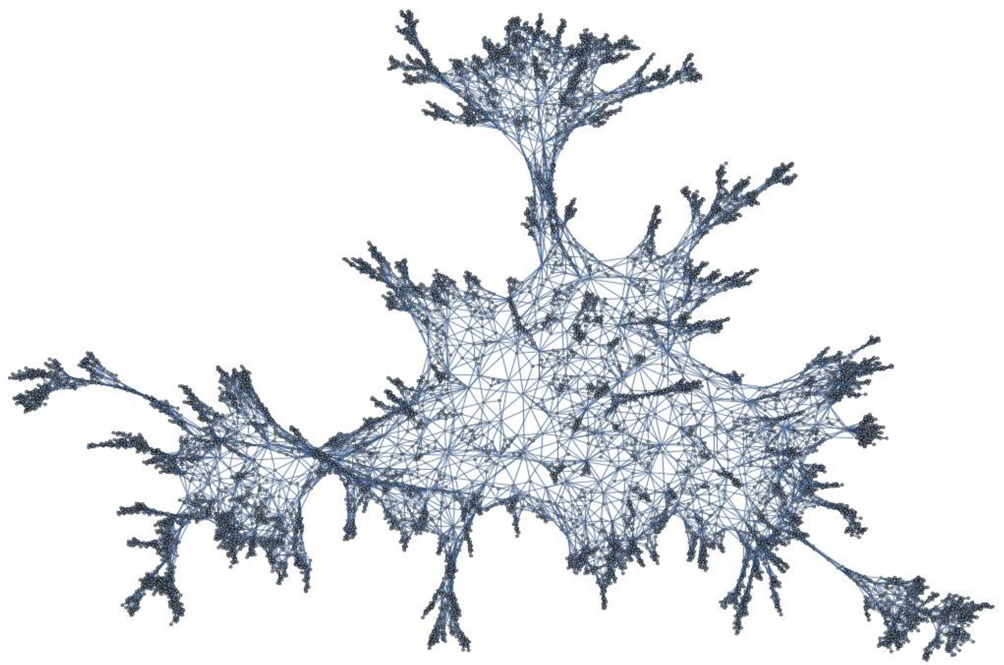
- **Typical distance** between (random) vertices in random maps
  the order of magnitude is $n^{1/4}$ ($\neq n^{1/2}$ **in random trees**)

random quadrang. { - [Chassaing-Schaeffer'04] probabilistic
- [Bouttier Di Francesco Guitter'03] exact GF expressions

- How does a random map (rescaled by $n^{1/4}$) "look like" ?

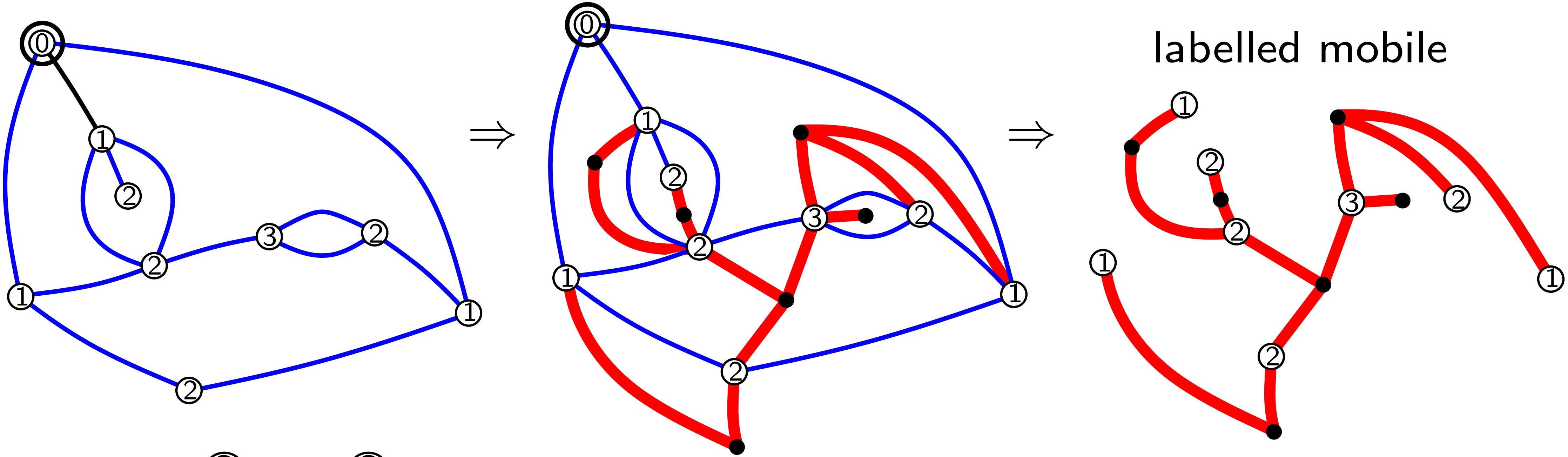
as a (rescaled) **discrete metric space**
convergence to the "Brownian map"

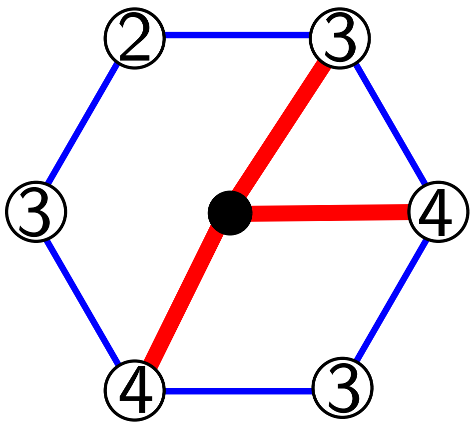
[Le Gall'13, Miermont'13]

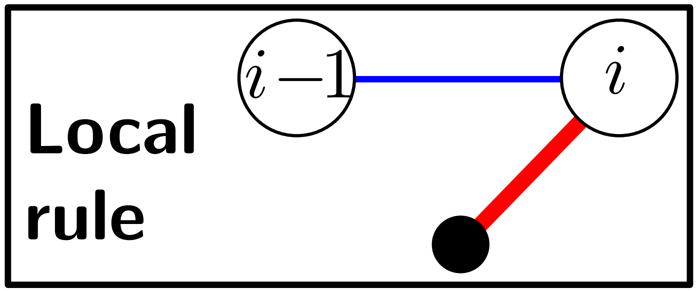


Extension to pointed bipartite maps

[Bouttier, Di Francesco, Guitter'04]



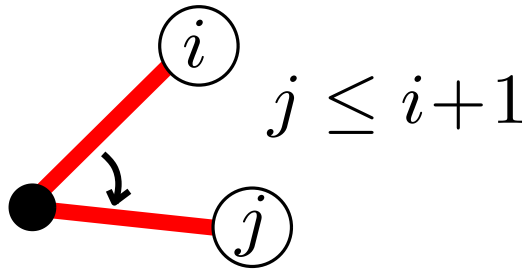




Conditions:

(i) $\exists$ vertex of label 1

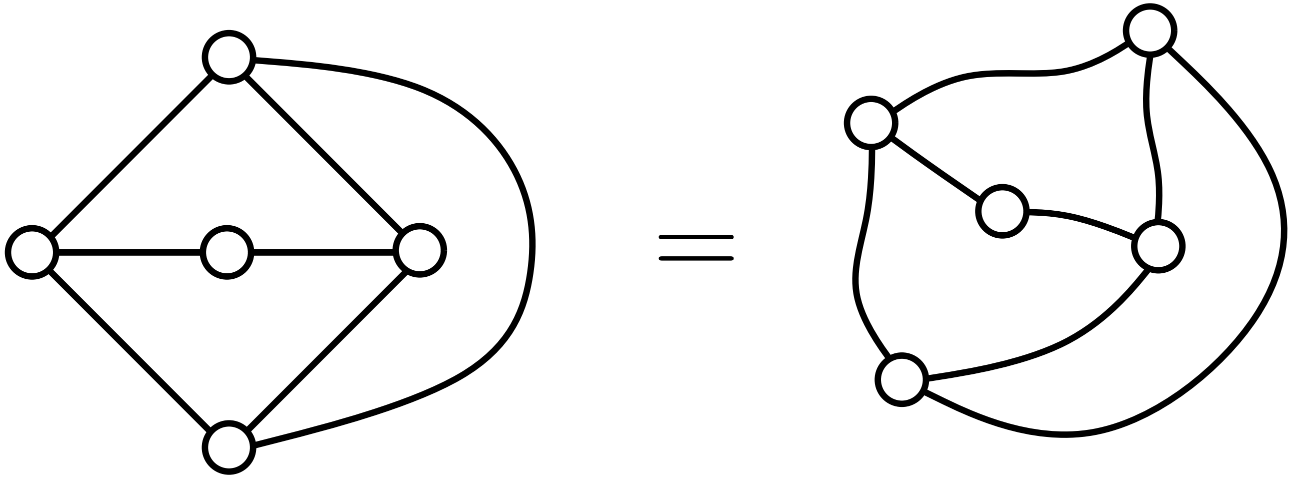
(ii)



Geometric representations of planar maps:
I. Straight-line drawings

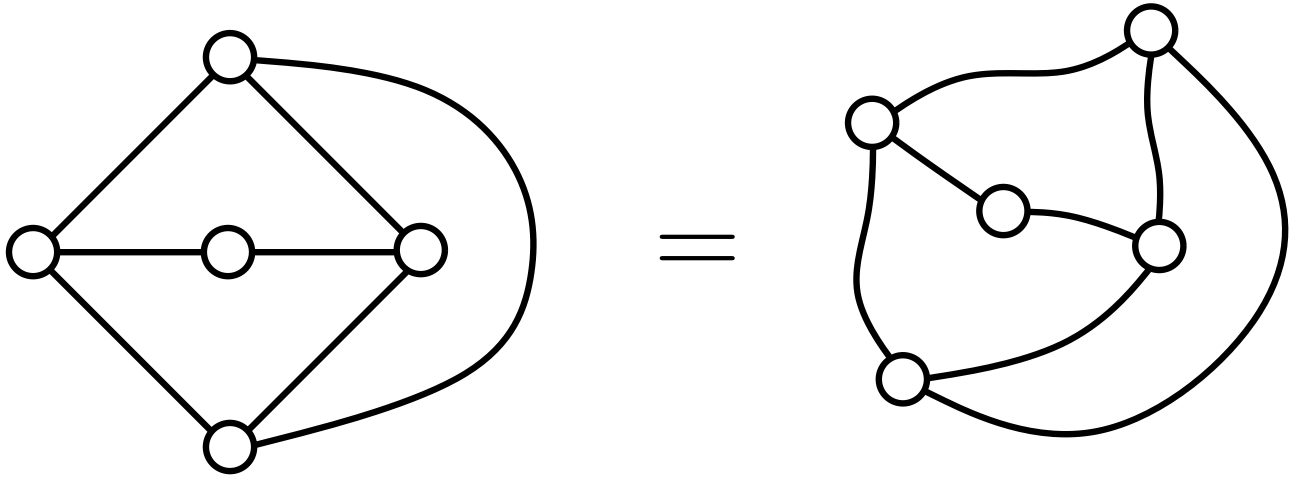
Existence question

planar map (with outer face) = equivalence class of planar drawings of graphs up to continuous deformation



Existence question

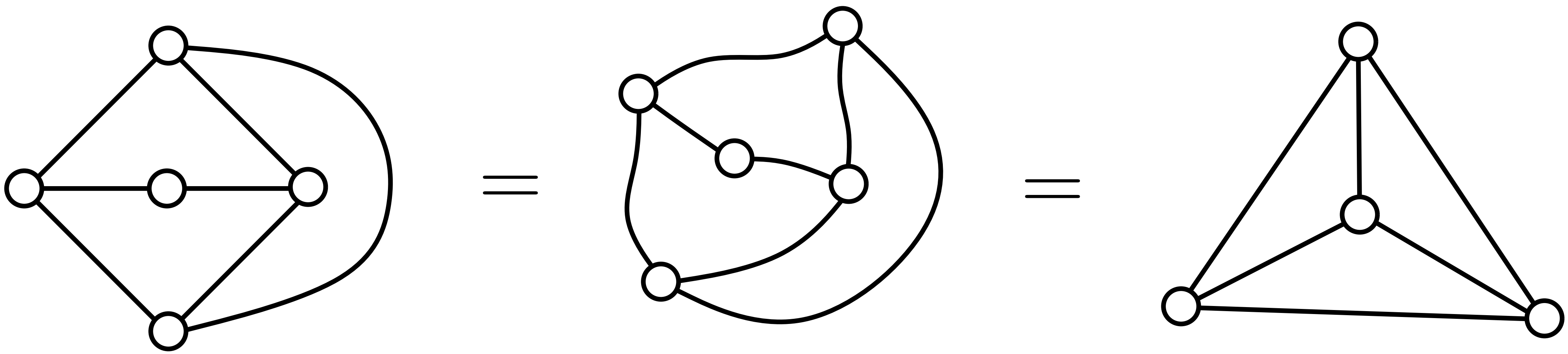
planar map (with outer face) = equivalence class of planar drawings of graphs up to continuous deformation



Question: Does there always exist an equivalent planar drawing such that all edges are drawn as segments ?

Existence question

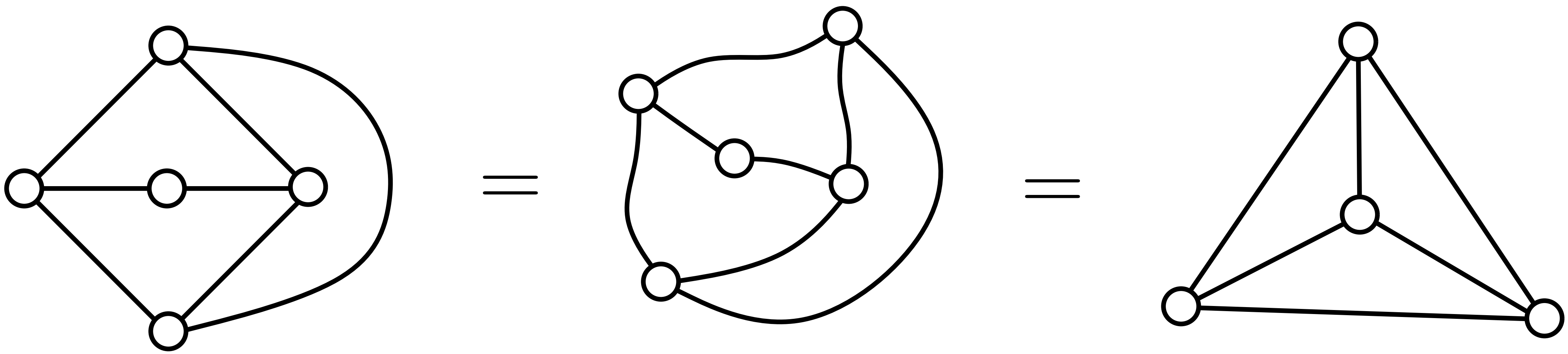
planar map (with outer face) = equivalence class of planar drawings of graphs up to continuous deformation



Question: Does there always exist an equivalent planar drawing such that all edges are drawn as segments ?

Existence question

planar map (with outer face) = equivalence class of planar drawings of graphs up to continuous deformation

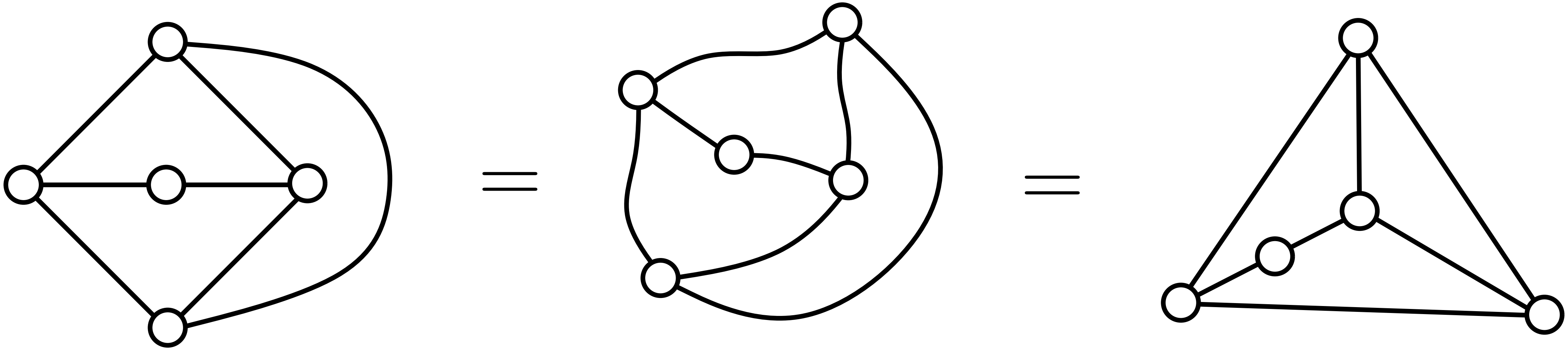


Question: Does there always exist an equivalent planar drawing such that all edges are drawn as segments ?

(such as drawing is called a (planar) **straight-line drawing**)

Existence question

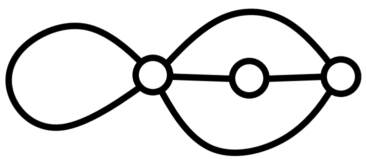
planar map (with outer face) = equivalence class of planar drawings of graphs up to continuous deformation



Question: Does there always exist an equivalent planar drawing such that all edges are drawn as segments ?

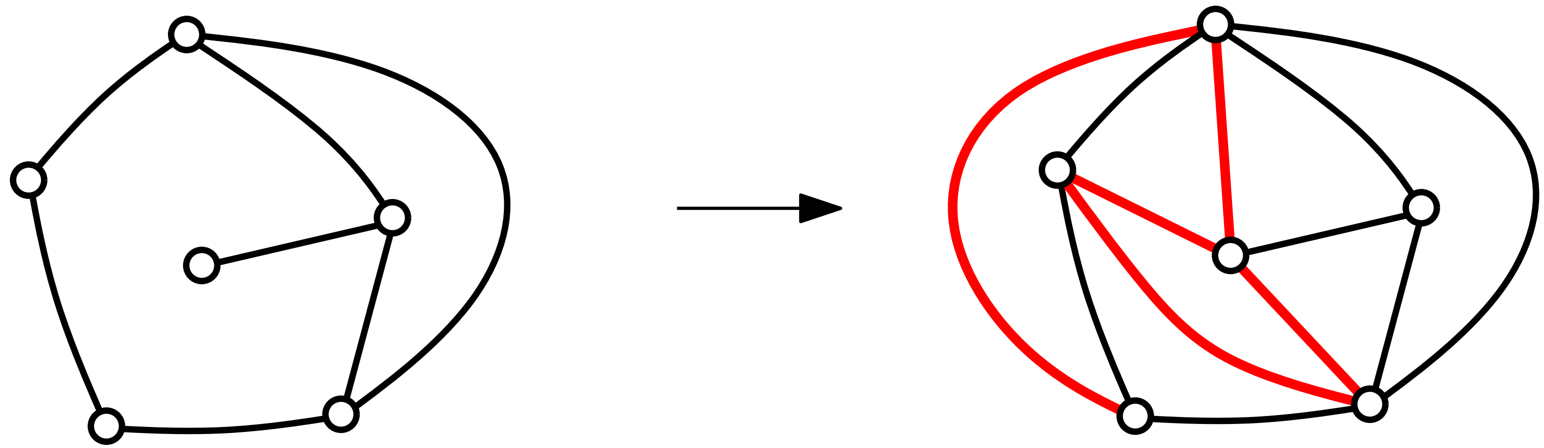
(such as drawing is called a (planar) **straight-line drawing**)

Remark: For such a drawing to exist, the map needs to be simple



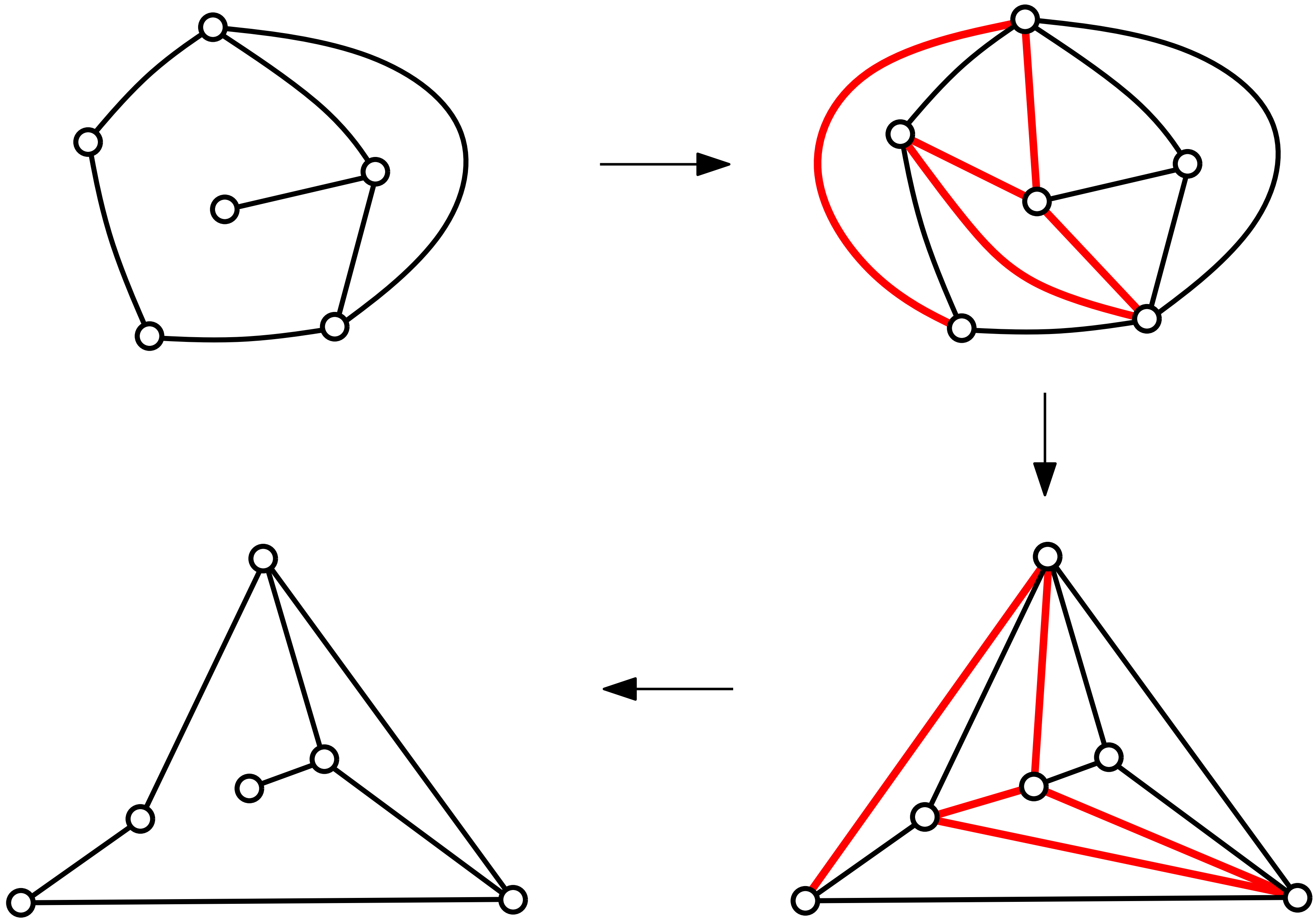
Existence proof (reduction to triangulations)

- Any simple planar map M can be completed to a simple triangulation T (**Exercise**: can be done without creating new vertices, only edges)



Existence proof (reduction to triangulations)

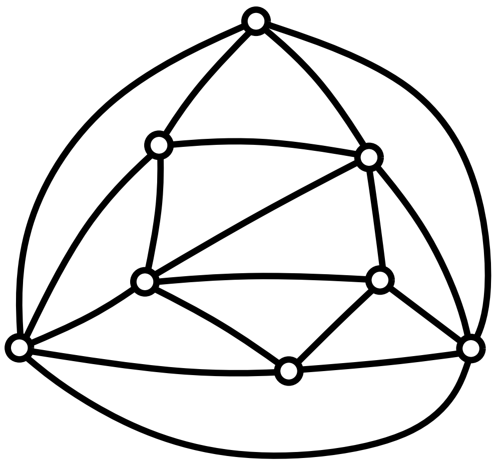
- Any simple planar map M can be completed to a simple triangulation T
  (**Exercise**: can be done without creating new vertices, only edges)
- A straight-line drawing of T yields a straight-line drawing of M



Existence proof (for triangulations)

First proof: induction on the number of vertices

Let T be a triangulation with n vertices

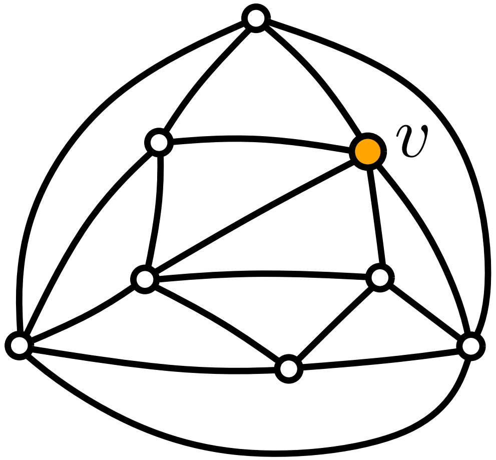


Existence proof (for triangulations)

First proof: induction on the number of vertices

Let T be a triangulation with n vertices

Exercise: T has at least one inner vertex v of degree ≤ 5

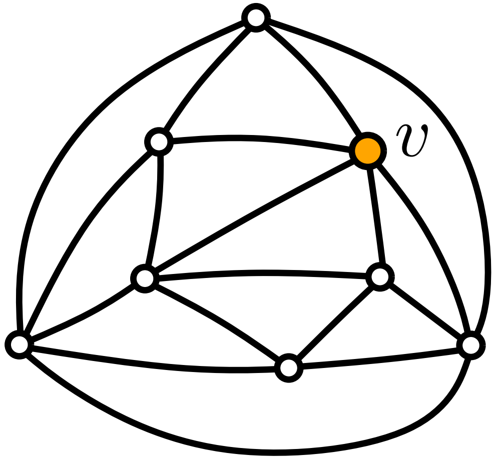


Existence proof (for triangulations)

First proof: induction on the number of vertices

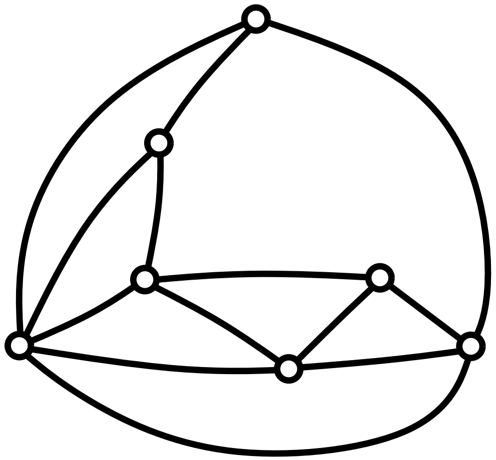
Let T be a triangulation with n vertices

Exercise: T has at least one inner vertex v of degree ≤ 5

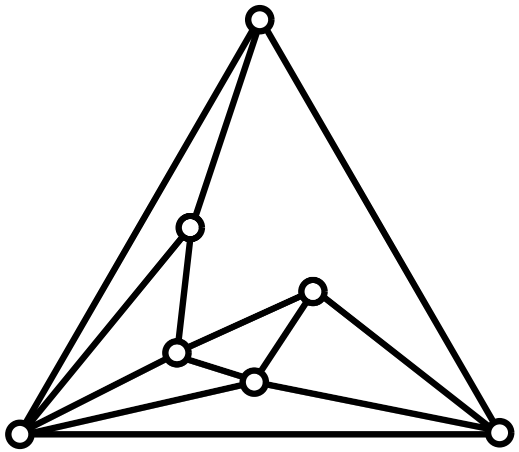


$\Downarrow$

$T \backslash v$



$\Rightarrow$



induction

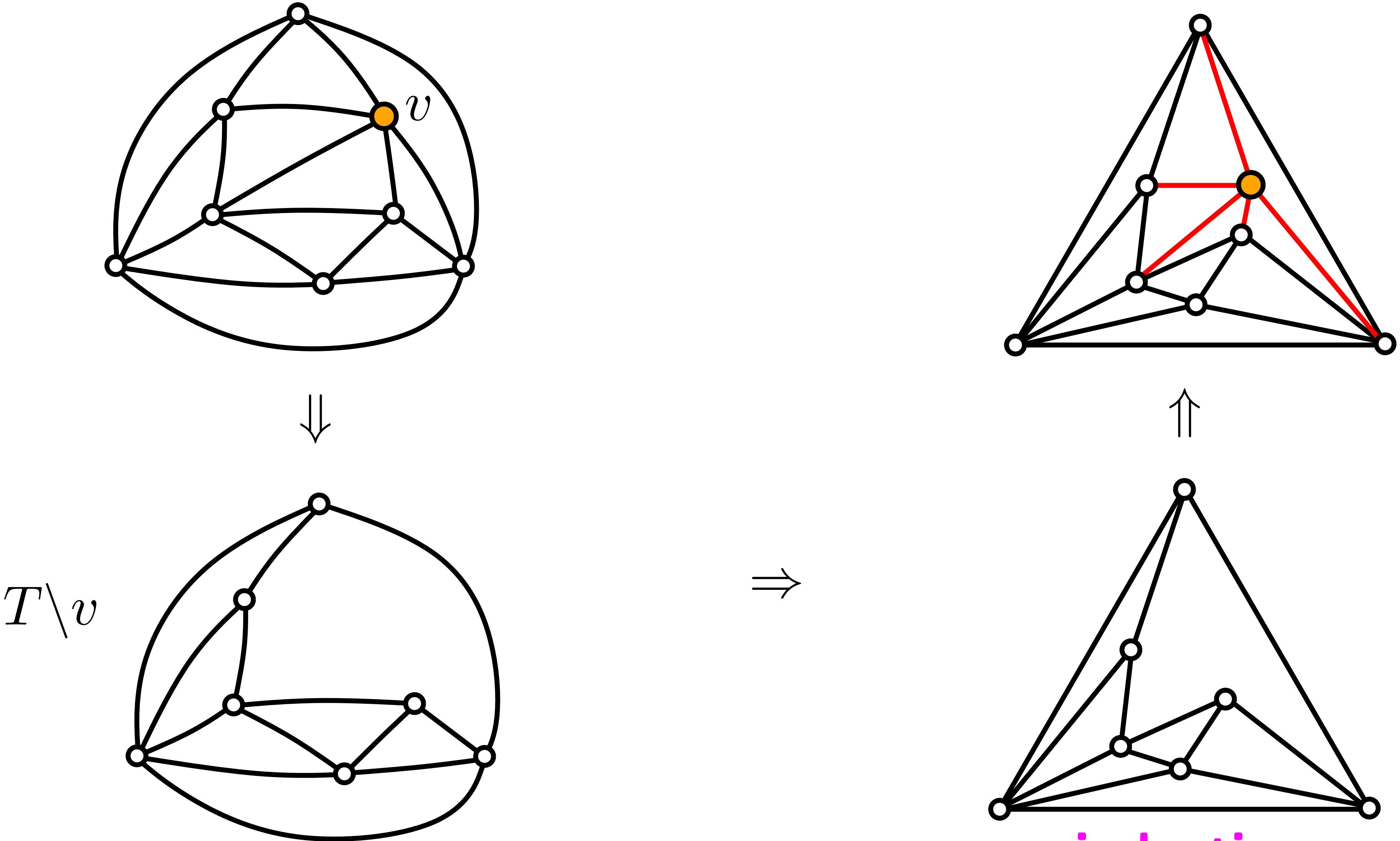
$T \backslash v$ has a straight-line drawing

Existence proof (for triangulations)

First proof: induction on the number of vertices

Let T be a triangulation with n vertices

Exercise: T has at least one inner vertex v of degree ≤ 5



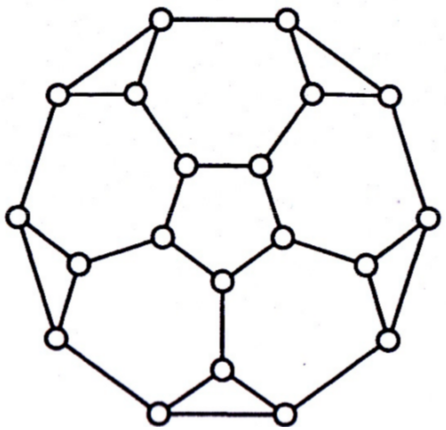
induction

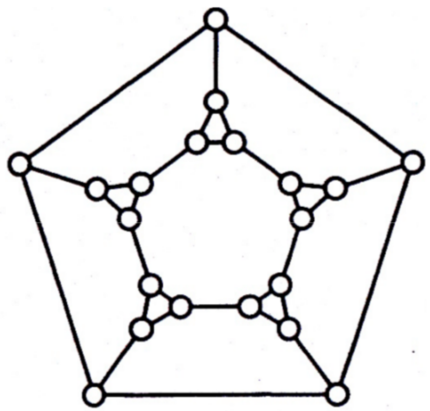
$T \backslash v$ has a straight-line drawing

Straight-line drawing algorithms

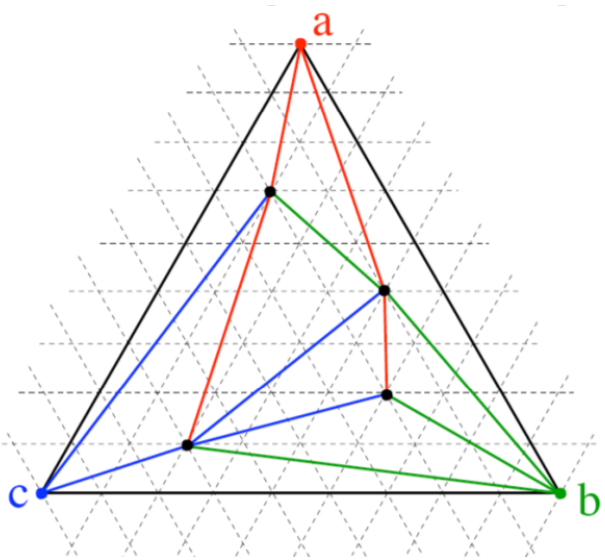
We present two famous algorithms (each with its advantages)

- Tutte's barycentric method

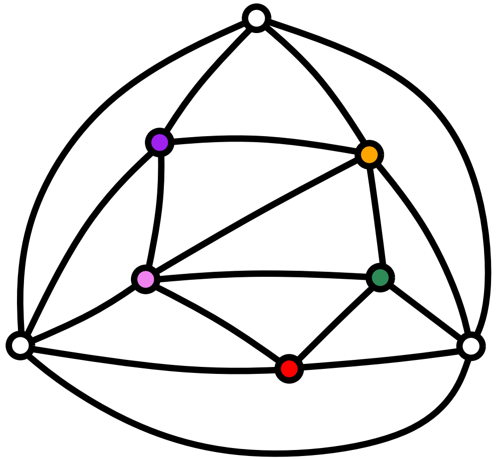


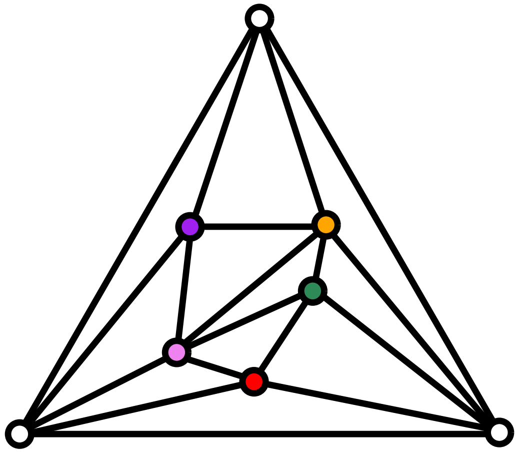


- Schnyder's face-counting algorithm

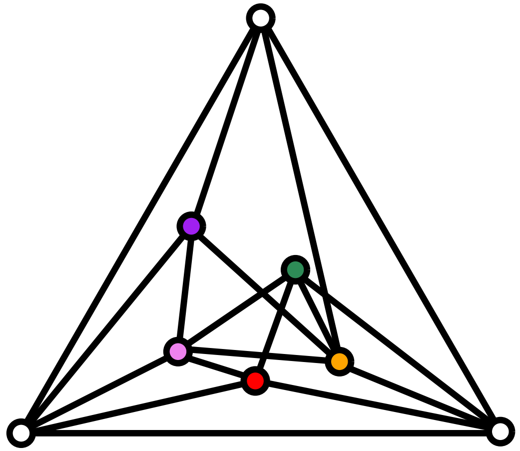


Planarity criterion for straight-line drawings



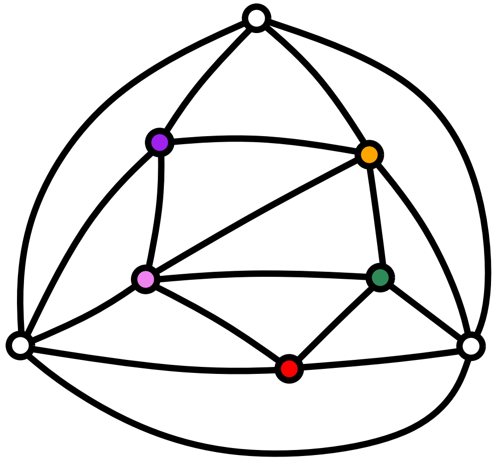


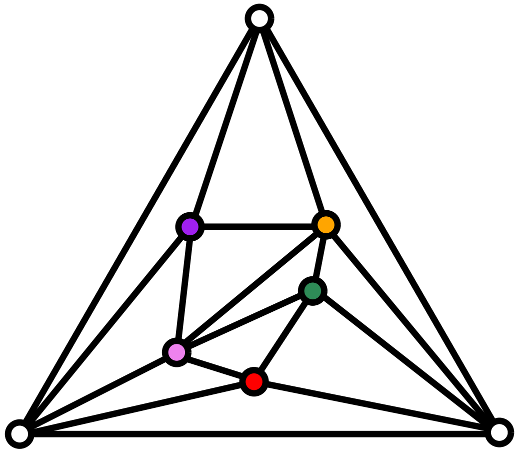
planar



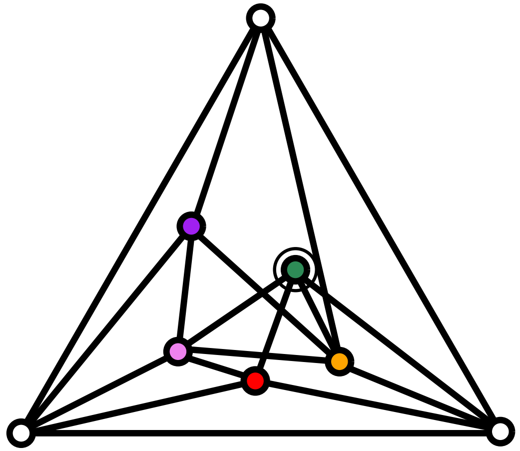
non-planar

Planarity criterion for straight-line drawings





planar



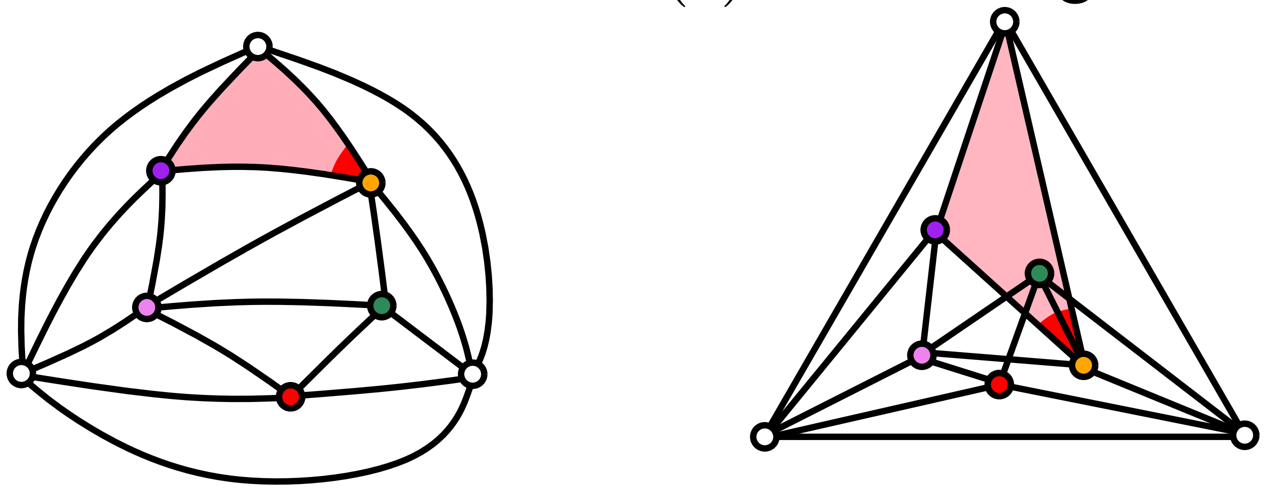
non-planar

Theorem: a straight-line drawing is planar iff every inner vertex is inside the **convex hull** of its neighbours

(works for triangulations and more generally for 3-connected planar graphs)

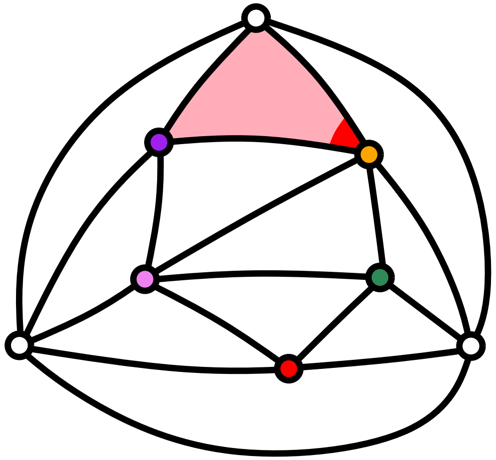
Proof idea

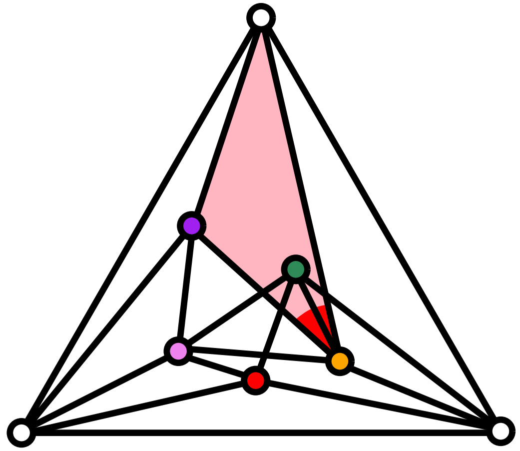
- For each corner $c \in T$ let $\theta(c)$ be the angle of c in the drawing



Proof idea

- For each corner $c \in T$ let $\theta(c)$ be the angle of c in the drawing

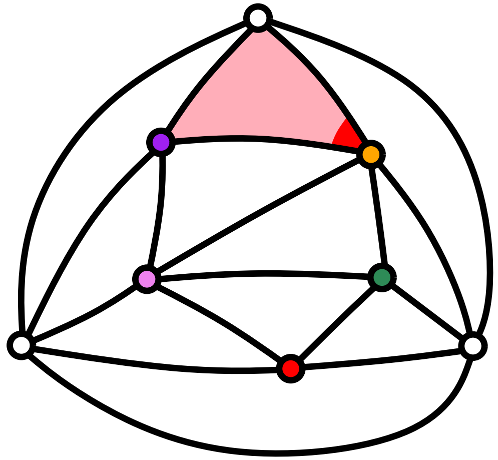


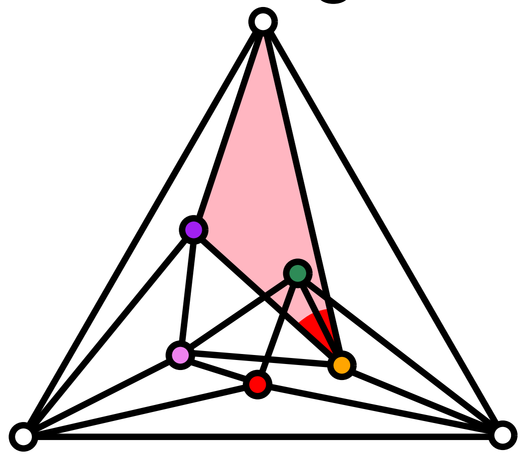


- For each vertex v, let $\Theta(v) = \sum_{c \in v} \theta(c)$

Proof idea

- For each corner $c \in T$ let $\theta(c)$ be the angle of c in the drawing

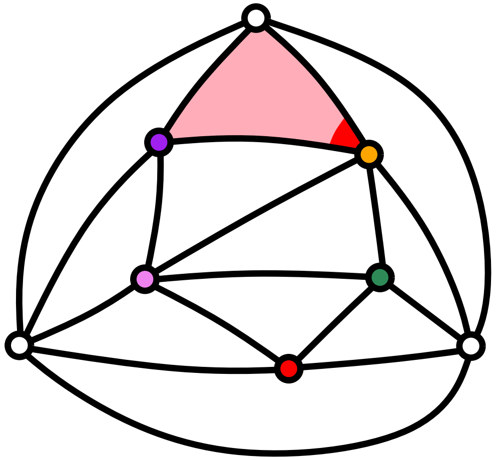


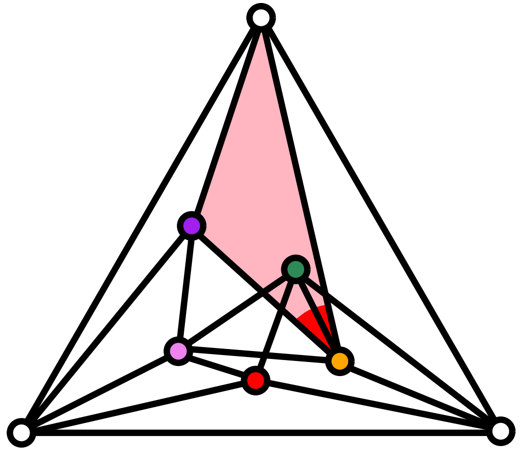


- For each vertex v, let $\Theta(v) = \sum_{c \in v} \theta(c)$
- Whatever the drawing we always have $\boxed{\sum_v \Theta(v) = 2\pi|V|}$

Proof idea

- For each corner $c \in T$ let $\theta(c)$ be the angle of c in the drawing

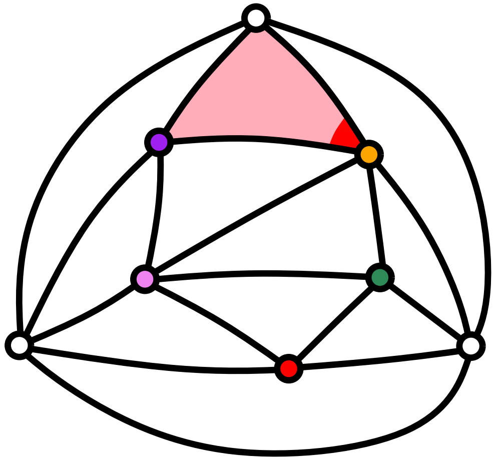


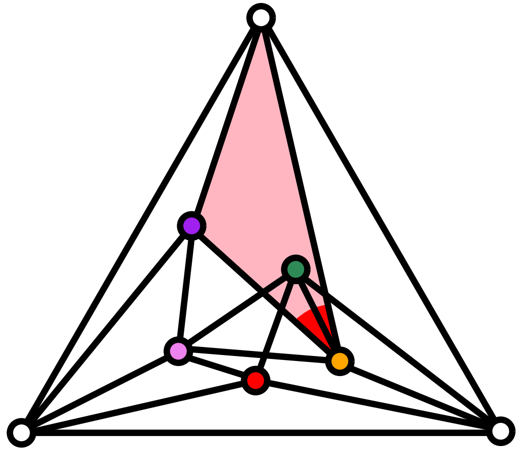


- For each vertex v, let $\Theta(v) = \sum_{c \in v} \theta(c)$
- Whatever the drawing we always have $\boxed{\sum_v \Theta(v) = 2\pi |V|}$
- If convex hull condition holds, then $\Theta(v) \geq 2\pi$ for each v

Proof idea

- For each corner $c \in T$ let $\theta(c)$ be the angle of c in the drawing

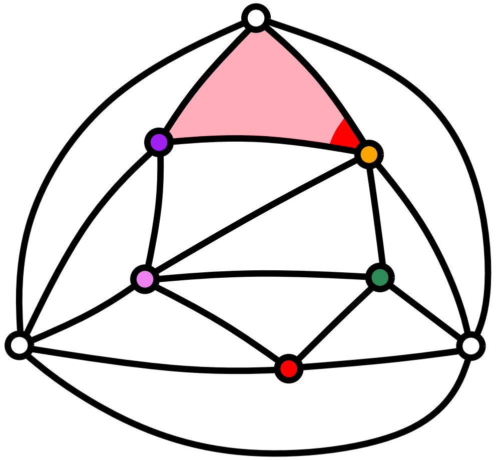


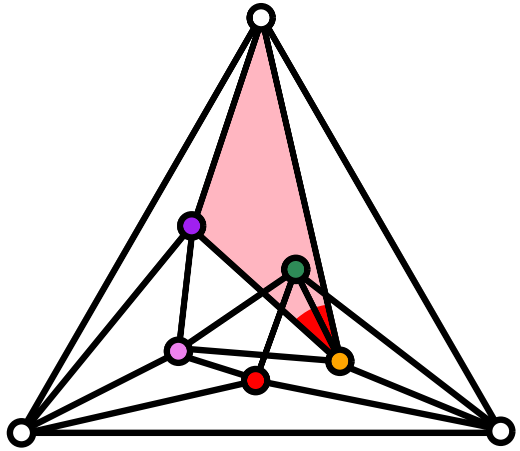


- For each vertex v, let $\Theta(v) = \sum_{c \in v} \theta(c)$
- Whatever the drawing we always have $\boxed{\sum_v \Theta(v) = 2\pi|V|}$
- If convex hull condition holds, then $\Theta(v) \geq 2\pi$ for each v and since $\sum_v \Theta(v) = 2\pi|V|$, must have $\Theta(v) = 2\pi$ for each v

Proof idea

- For each corner $c \in T$ let $\theta(c)$ be the angle of c in the drawing

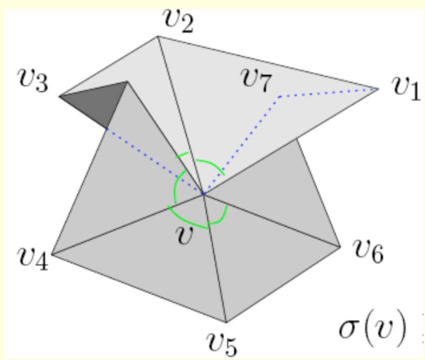




- For each vertex v, let $\Theta(v) = \sum_{c \in v} \theta(c)$
- Whatever the drawing we always have $\sum_v \Theta(v) = 2\pi|V|$
- If convex hull condition holds, then $\Theta(v) \geq 2\pi$ for each v and since $\sum_v \Theta(v) = 2\pi|V|$, must have $\Theta(v) = 2\pi$ for each v

Hence locally planar at each vertex
(no "folding" of triangles at a vertex)

$\Rightarrow$ the drawing is planar



Tutte's barycentric method

- Outer vertices v_1, v_2, v_3 are fixed at fixed positions (nailed)

- Each inner vertex is at the **barycenter of its neighbours**

$$x_i = \frac{1}{\Delta_i} \sum_{j \sim i} x_j \qquad y_i = \frac{1}{\Delta_i} \sum_{j \sim i} y_j \qquad \text{for } i \geq 4$$

Tutte's barycentric method

- Outer vertices v_1, v_2, v_3 are fixed at fixed positions (nailed)

- Each inner vertex is at the **barycenter of its neighbours**

$$x_i = \frac{1}{\Delta_i} \sum_{j \sim i} x_j \qquad y_i = \frac{1}{\Delta_i} \sum_{j \sim i} y_j \qquad \text{for } i \geq 4$$

$\Leftrightarrow \quad \sum_{j \sim i} x_i - x_j = 0$ and $\sum_{j \sim i} x_i - x_j = 0$ for each $i \geq 4$

Tutte's barycentric method

- Outer vertices v_1, v_2, v_3 are fixed at fixed positions (nailed)

- Each inner vertex is at the **barycenter of its neighbours**

$$x_i = \frac{1}{\Delta_i} \sum_{j \sim i} x_j \qquad y_i = \frac{1}{\Delta_i} \sum_{j \sim i} y_j \qquad \text{for } i \geq 4$$

$$\Leftrightarrow \quad \sum_{j \sim i} x_i - x_j = 0 \quad \text{and} \quad \sum_{j \sim i} x_i - x_j = 0 \quad \text{for each } i \geq 4$$

- This drawing **exists and is unique**. It minimizes the energy

$$\mathcal{P} = \sum_e \ell(e)^2 = \sum_{\{i,j\} \in T} (x_i - x_j)^2 + (y_i - y_j)^2$$

under the constraint of fixed $x_1, x_2, x_3, y_1, y_2, y_3$

Tutte's barycentric method

- Outer vertices v_1, v_2, v_3 are fixed at fixed positions (nailed)

- Each inner vertex is at the **barycenter of its neighbours**

$$x_i = \frac{1}{\Delta_i} \sum_{j \sim i} x_j \qquad y_i = \frac{1}{\Delta_i} \sum_{j \sim i} y_j \qquad \text{for } i \geq 4$$

$$\Leftrightarrow \quad \sum_{j \sim i} x_i - x_j = 0 \quad \text{and} \quad \sum_{j \sim i} x_i - x_j = 0 \quad \text{for each } i \geq 4$$

- This drawing **exists and is unique**. It minimizes the energy

$$\mathcal{P} = \sum_e \ell(e)^2 = \sum_{\{i,j\} \in T} (x_i - x_j)^2 + (y_i - y_j)^2$$

under the constraint of fixed $x_1, x_2, x_3, y_1, y_2, y_3$

- also called **spring embedding** (each edge is a spring of energy $\ell(e)^2$)

Tutte's barycentric method

- Outer vertices v_1, v_2, v_3 are fixed at fixed positions (nailed)

- Each inner vertex is at the **barycenter of its neighbours**

$$x_i = \frac{1}{\Delta_i} \sum_{j \sim i} x_j \qquad y_i = \frac{1}{\Delta_i} \sum_{j \sim i} y_j \qquad \text{for } i \geq 4$$

$$\Leftrightarrow \quad \textstyle\sum_{j \sim i} x_i - x_j = 0 \quad \text{and} \quad \textstyle\sum_{j \sim i} x_i - x_j = 0 \quad \text{for each } i \geq 4$$

- This drawing **exists and is unique**. It minimizes the energy

$$\mathcal{P} = \textstyle\sum_e \ell(e)^2 = \textstyle\sum_{\{i,j\} \in T} (x_i - x_j)^2 + (y_i - y_j)^2$$

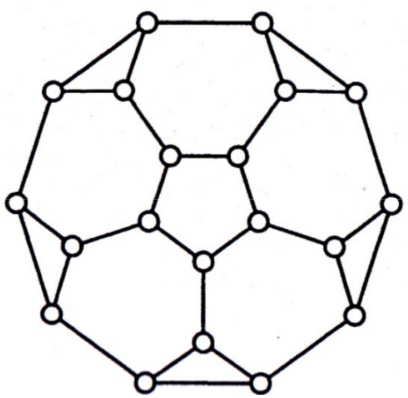
under the constraint of fixed $x_1, x_2, x_3, y_1, y_2, y_3$

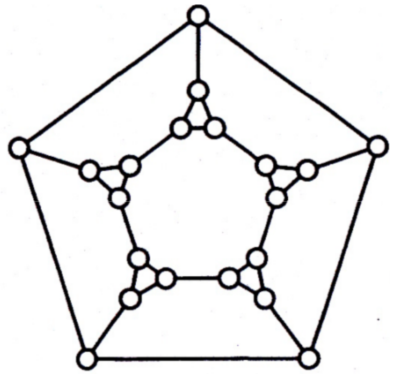
- also called **spring embedding** (each edge is a spring of energy $\ell(e)^2$)

Advantages/disadvantages

The good!

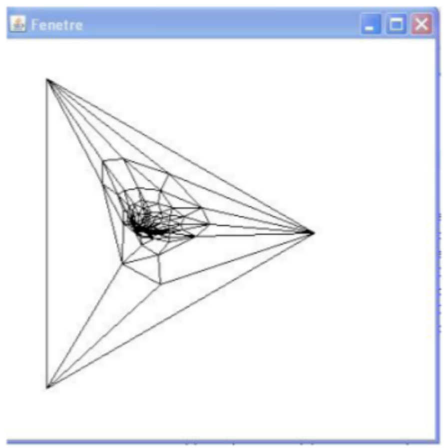
- displays the symmetries nicely
- easy to implement (solve a linear system)
- optimal for a certain energy criterion





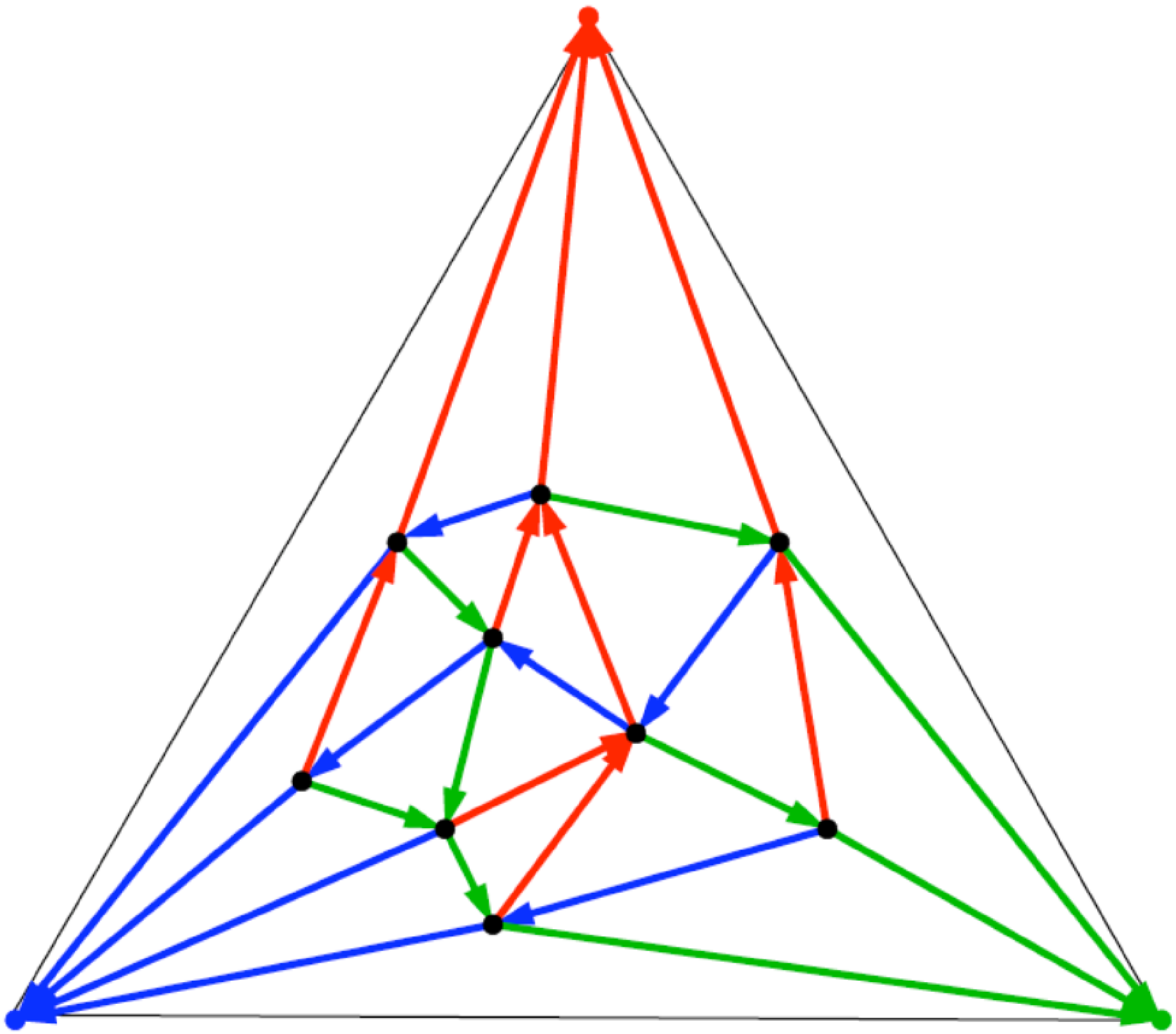
The less good:

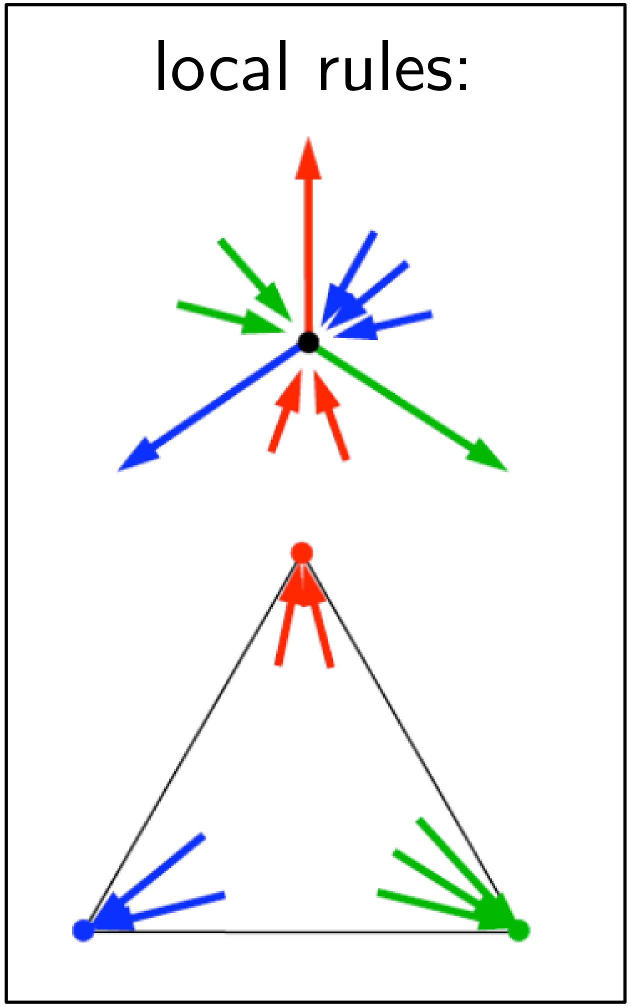
- a bit expensive computationally (solve linear system of size $|V|$)
- some very dense clusters (edges of length exponentially small in $|V|$)



Schnyder woods

Schnyder wood = each inner edge is given a direction and a color (red, green, blue) so as to satisfy local rules at each vertex:

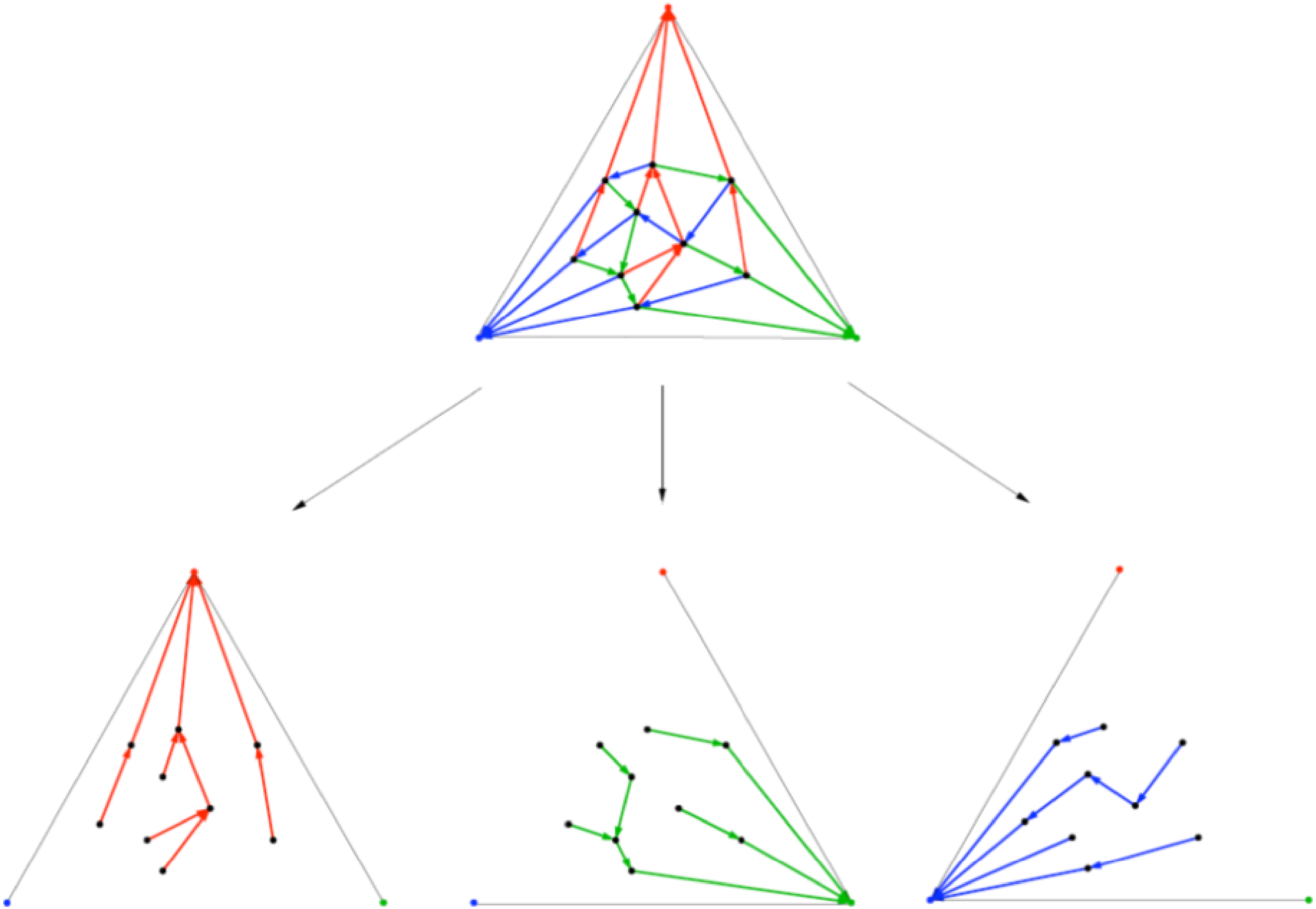




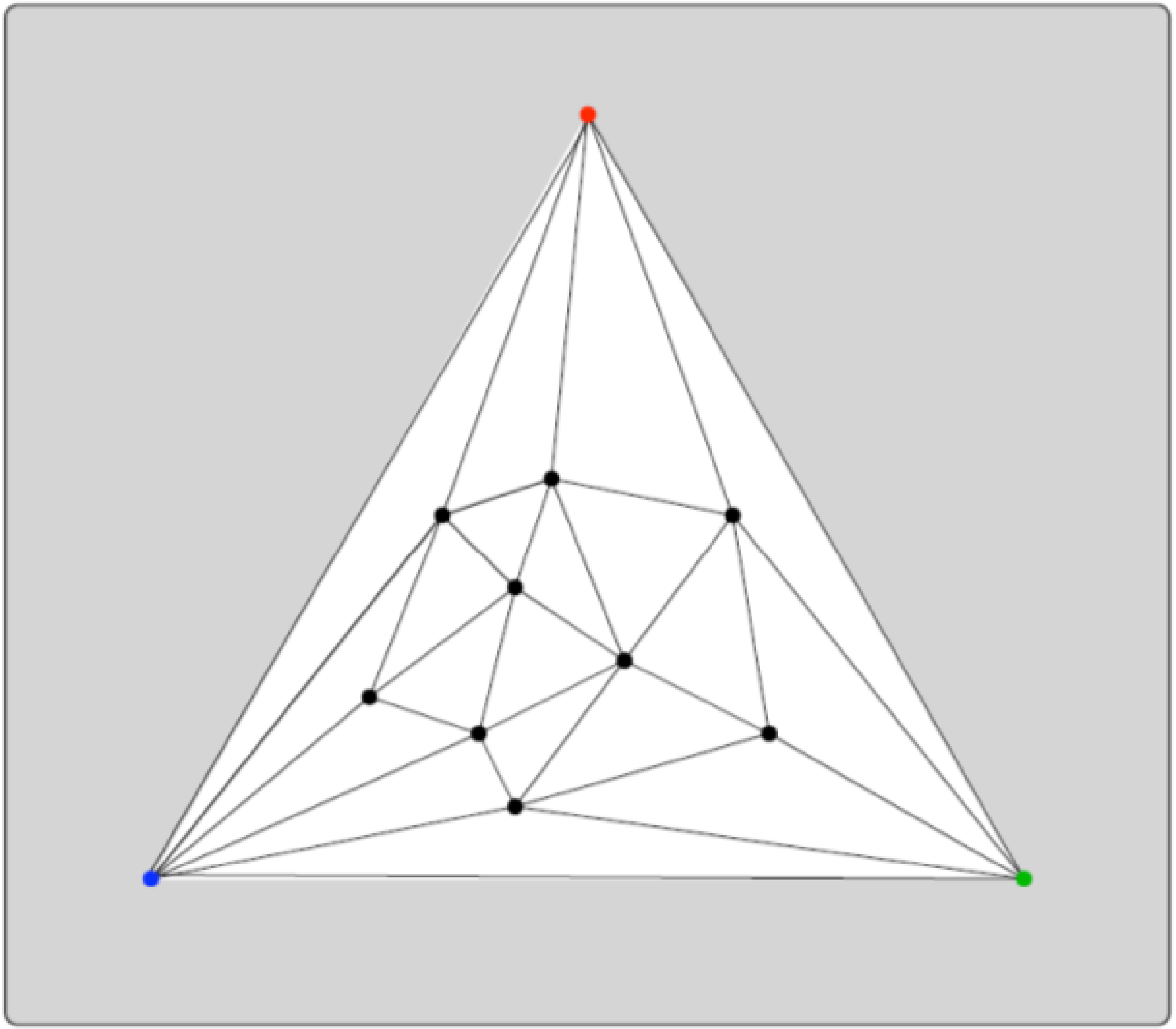
[Schnyder'89]: each (simple) triangulation admits a Schnyder wood

Fundamental property of Schnyder woods

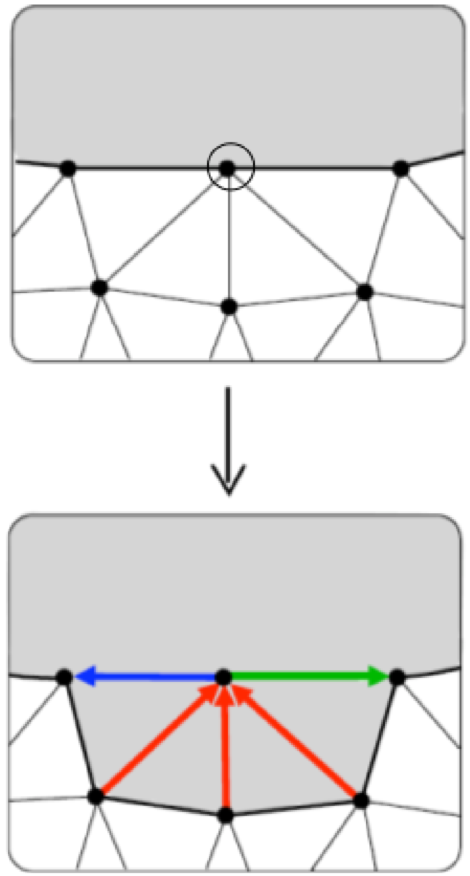
In each color the edges form a spanning tree (rooted at the 3 outer vertex)



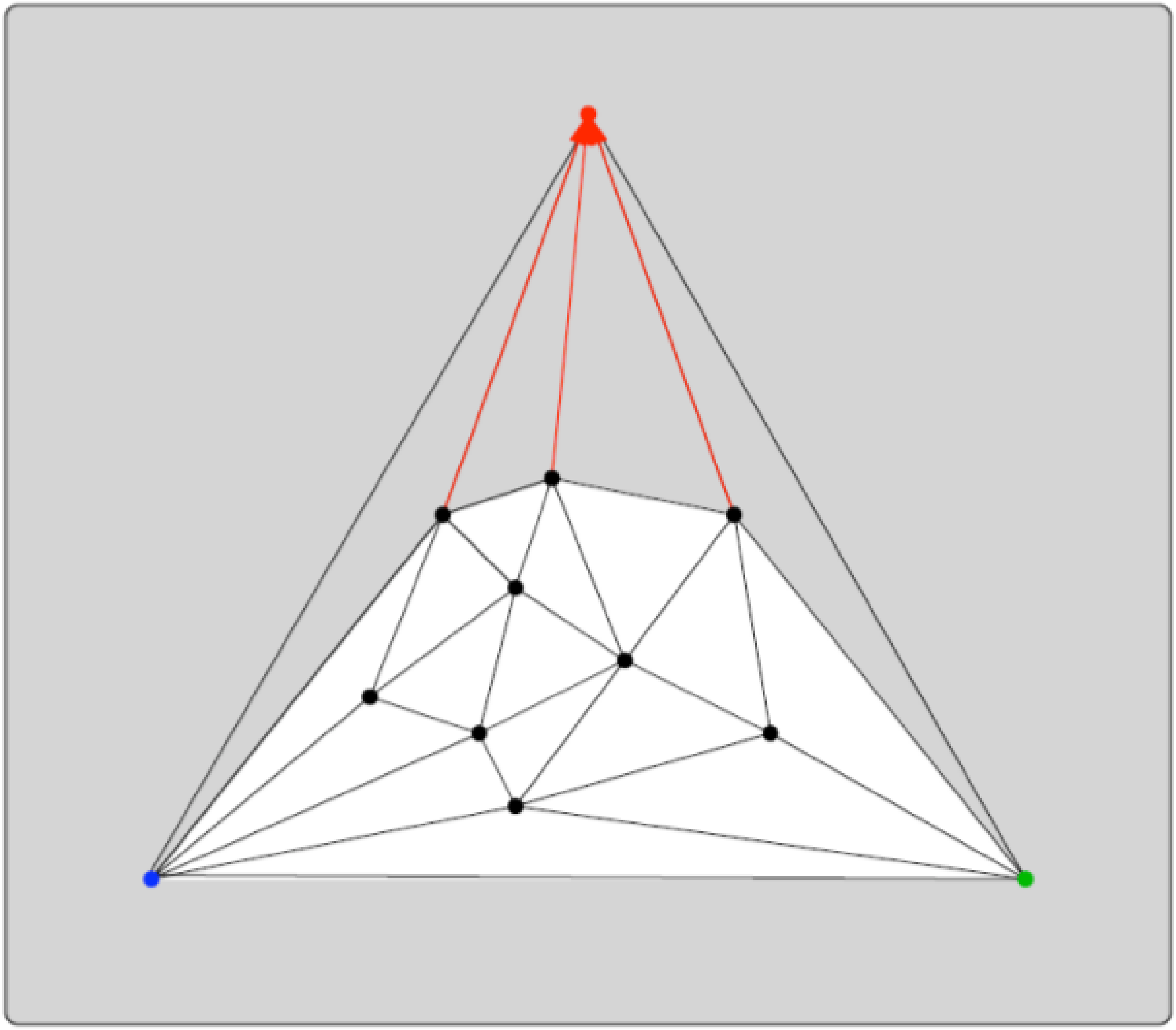
Shelling procedure to compute Schnyder woods



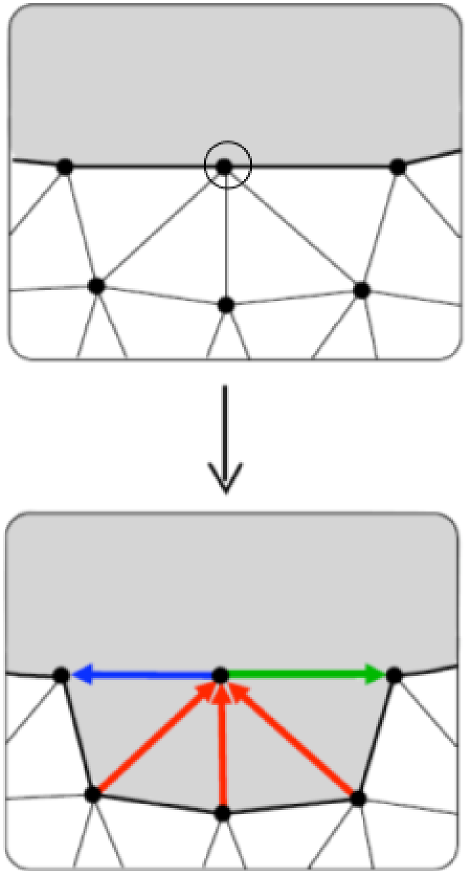
at each step:



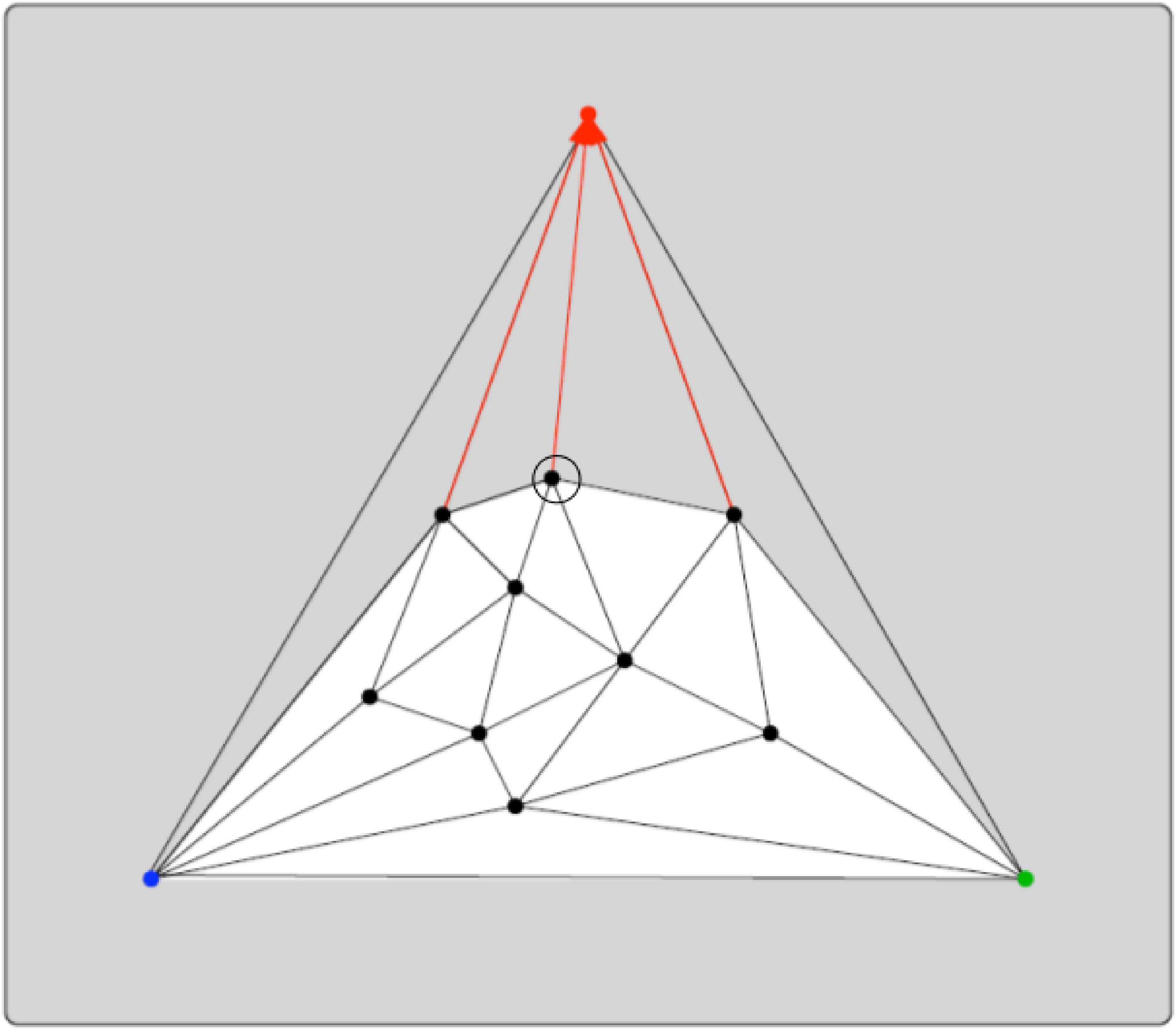
Shelling procedure to compute Schnyder woods



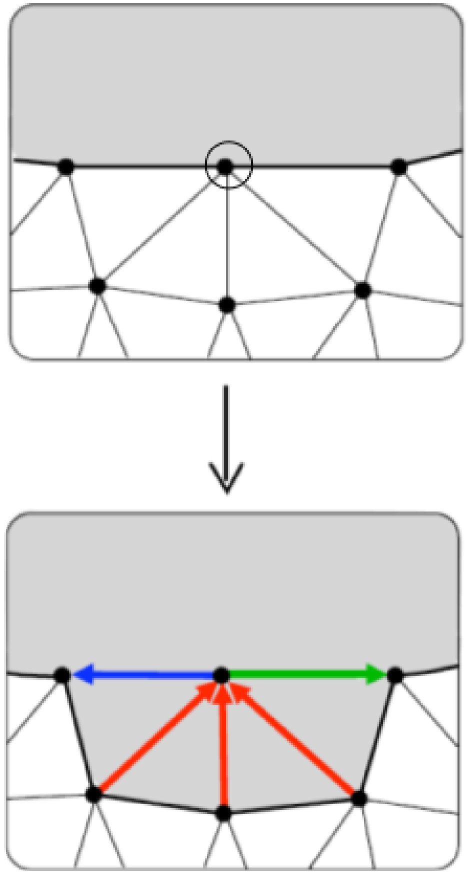
at each step:



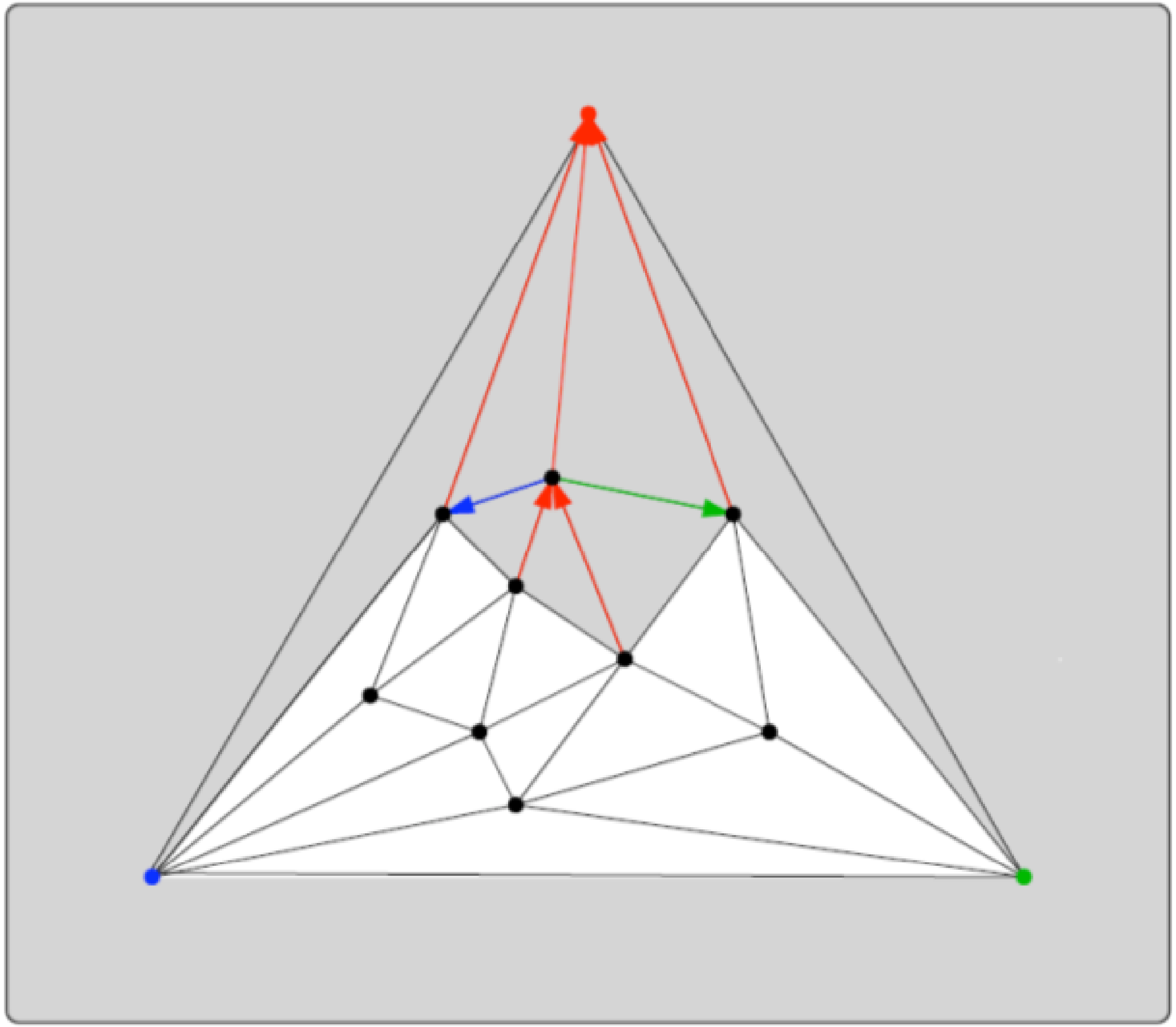
Shelling procedure to compute Schnyder woods



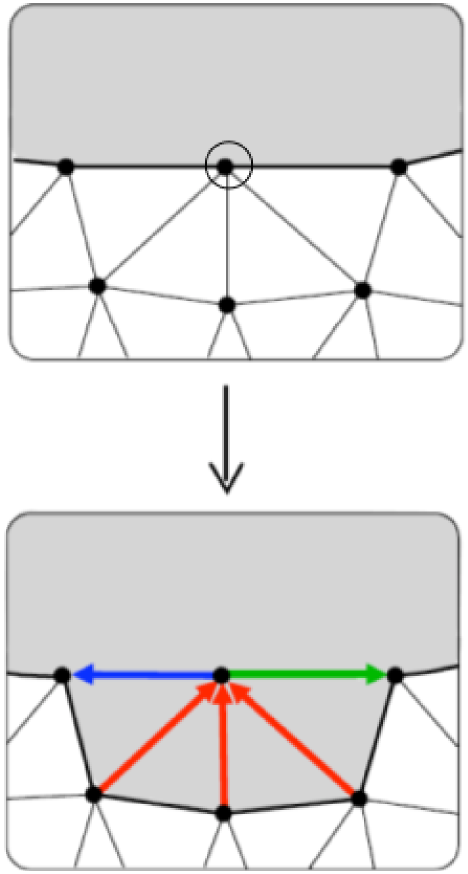
at each step:



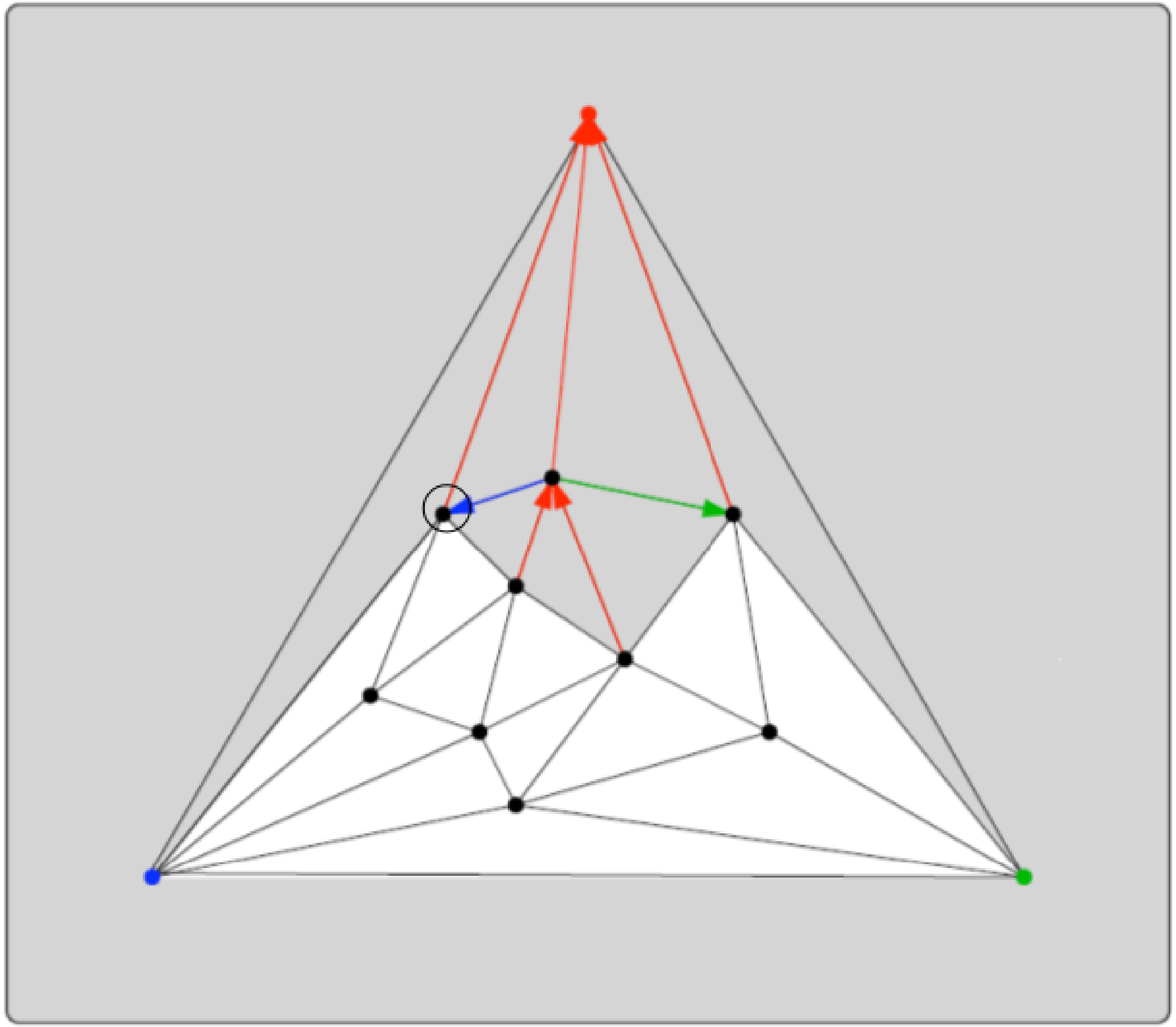
Shelling procedure to compute Schnyder woods



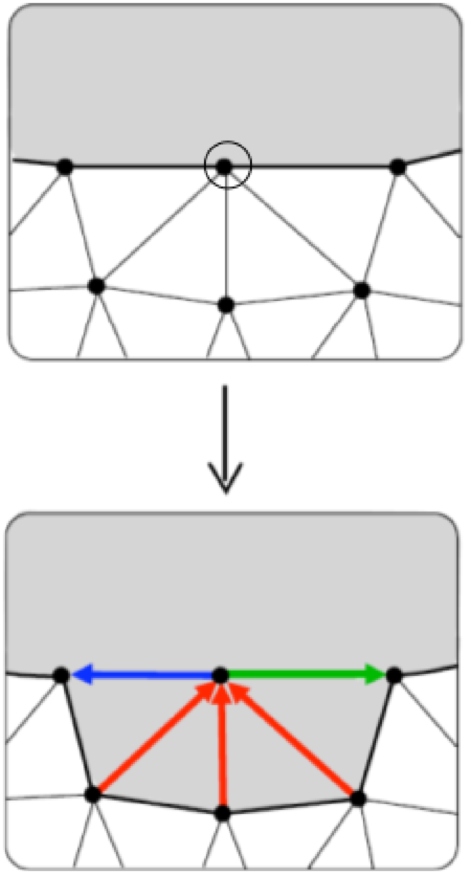
at each step:



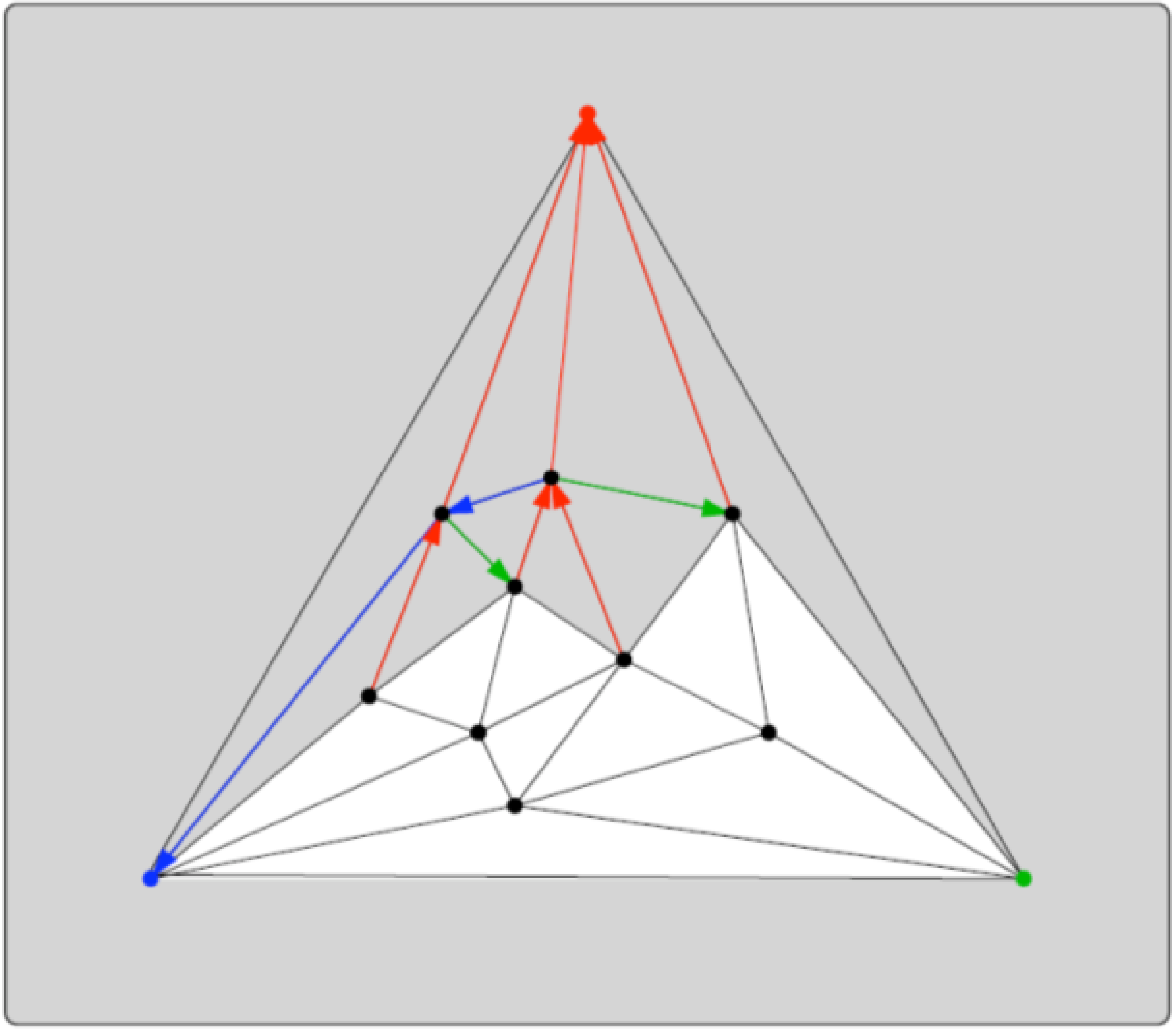
Shelling procedure to compute Schnyder woods



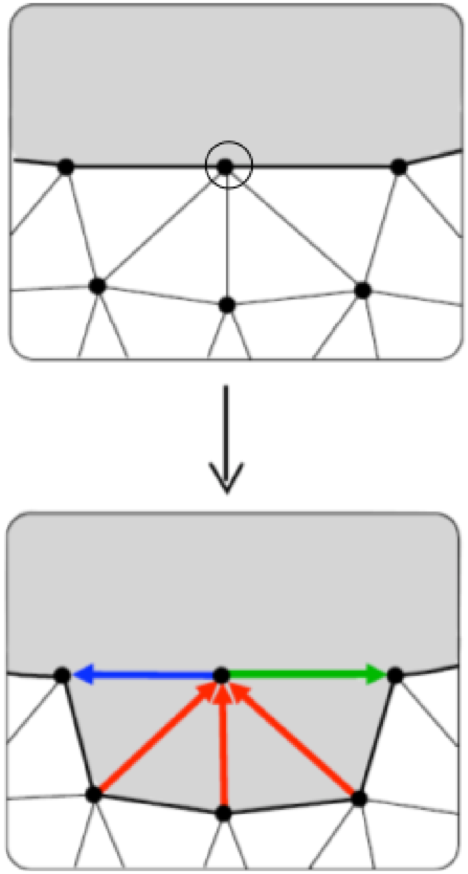
at each step:



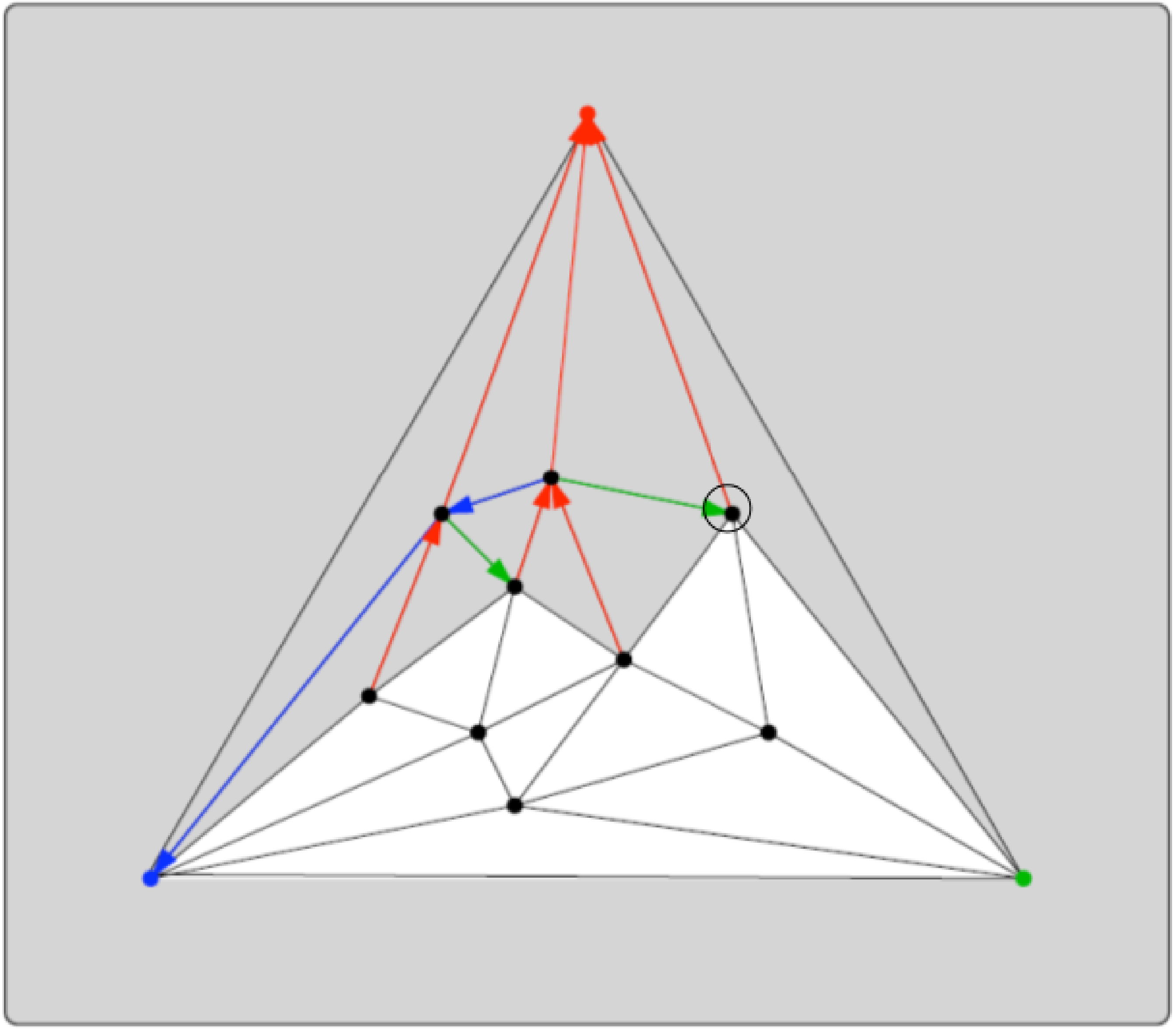
Shelling procedure to compute Schnyder woods



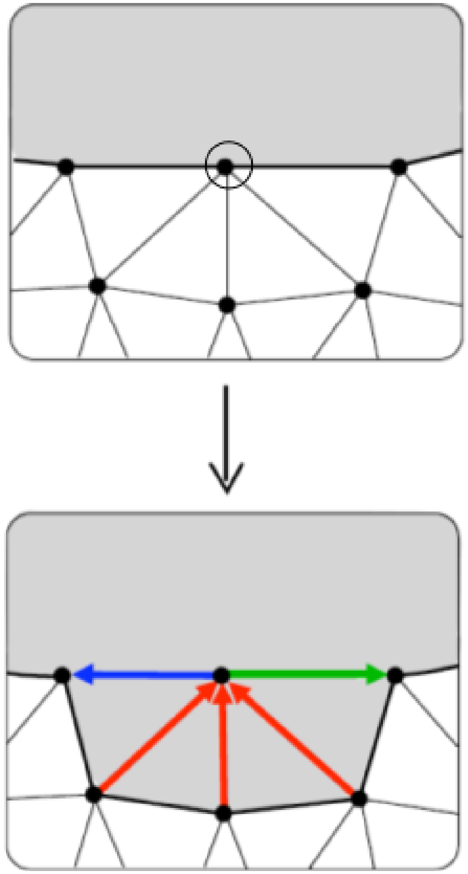
at each step:



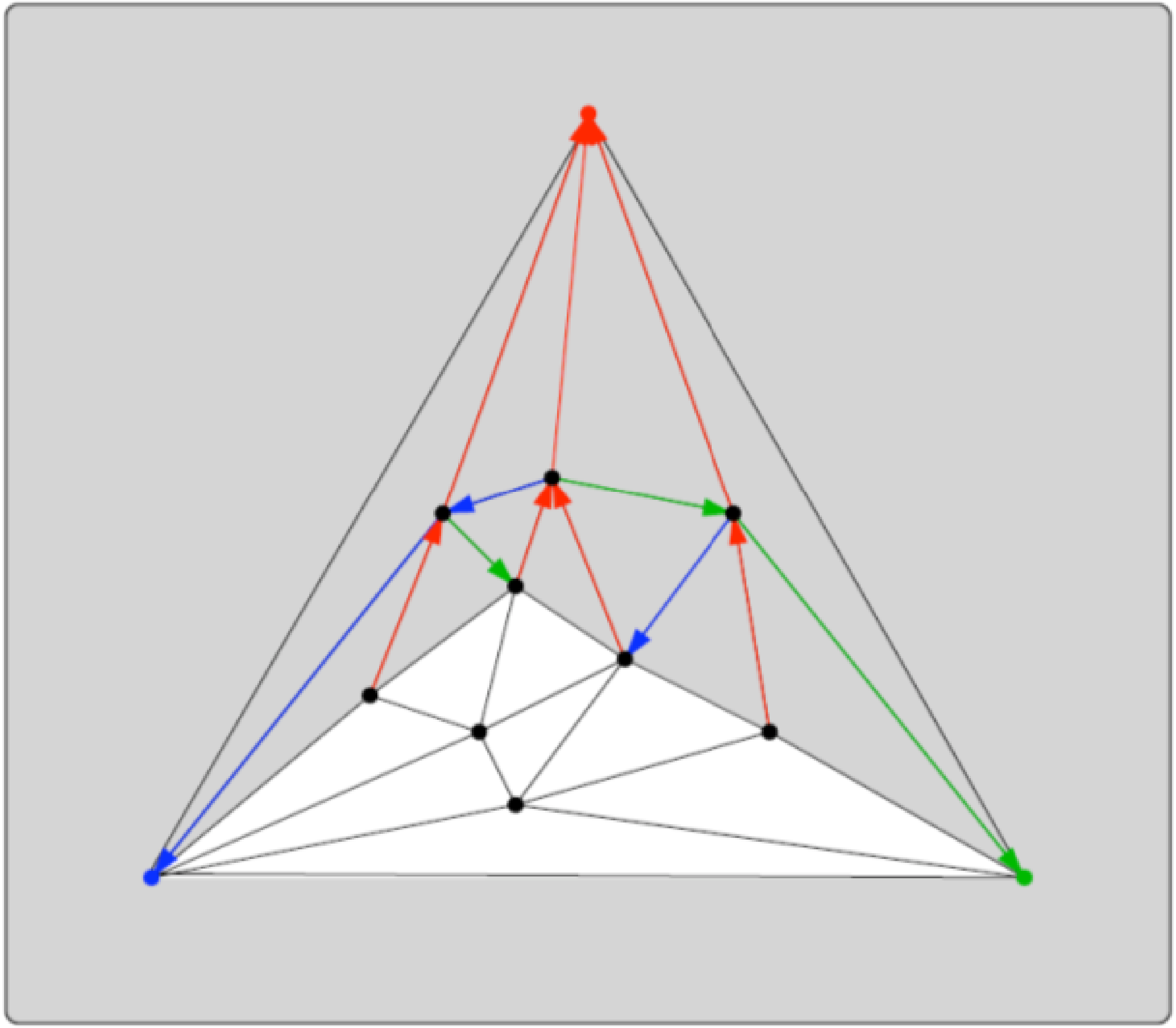
Shelling procedure to compute Schnyder woods



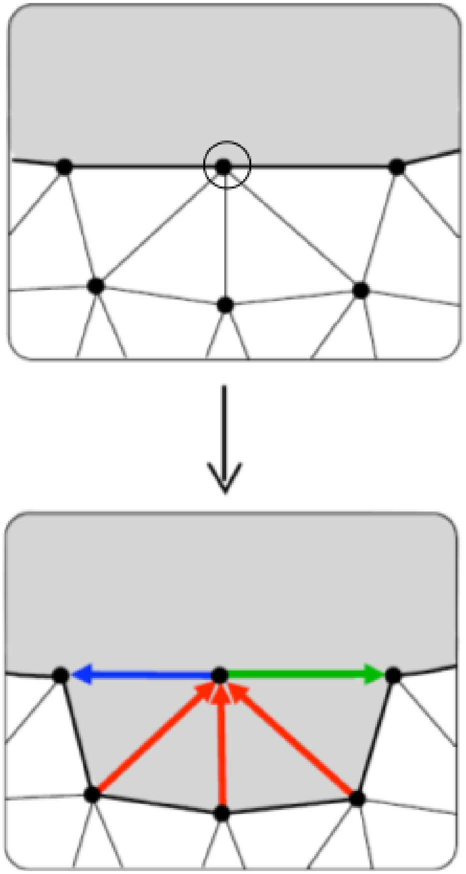
at each step:



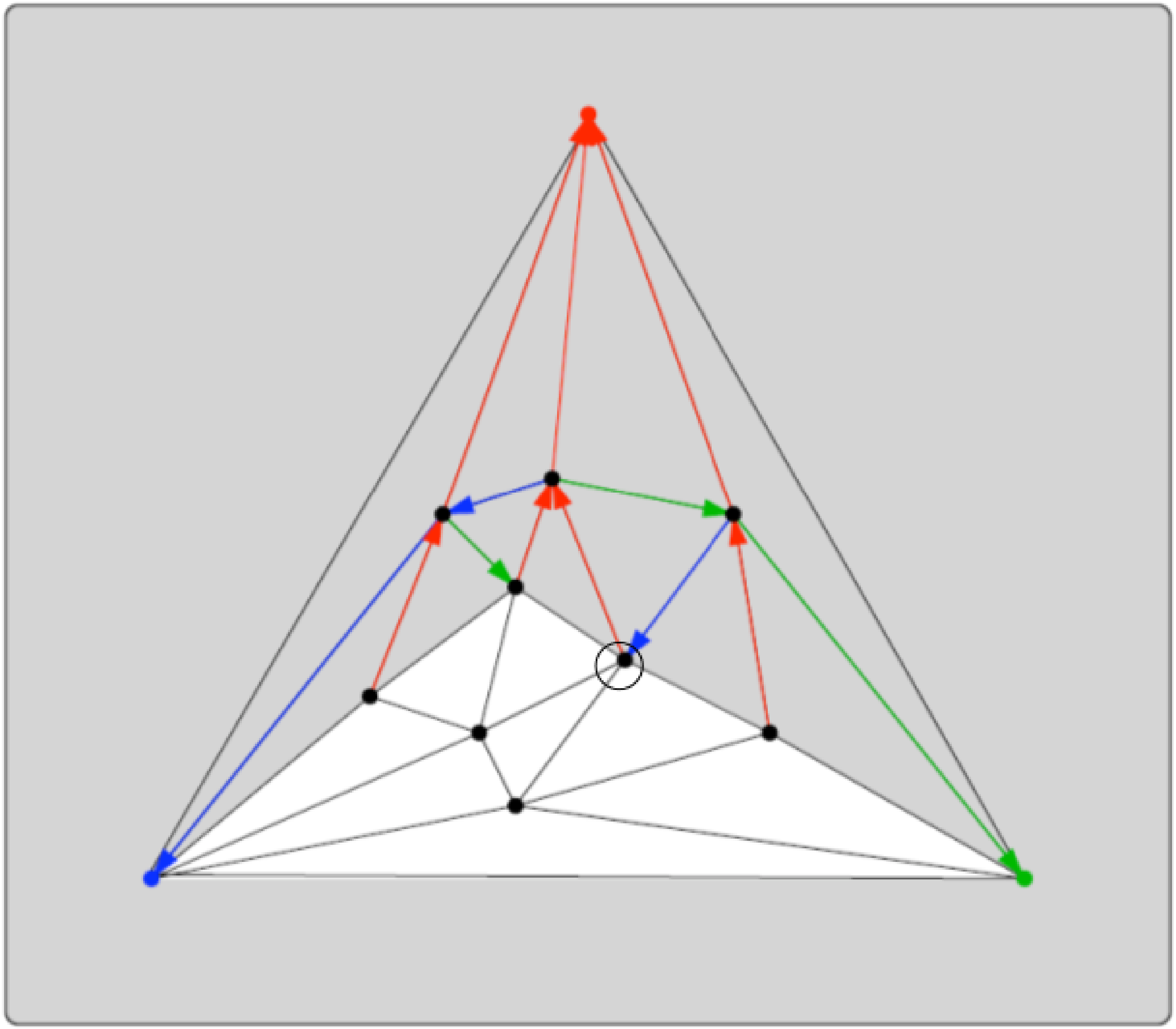
Shelling procedure to compute Schnyder woods



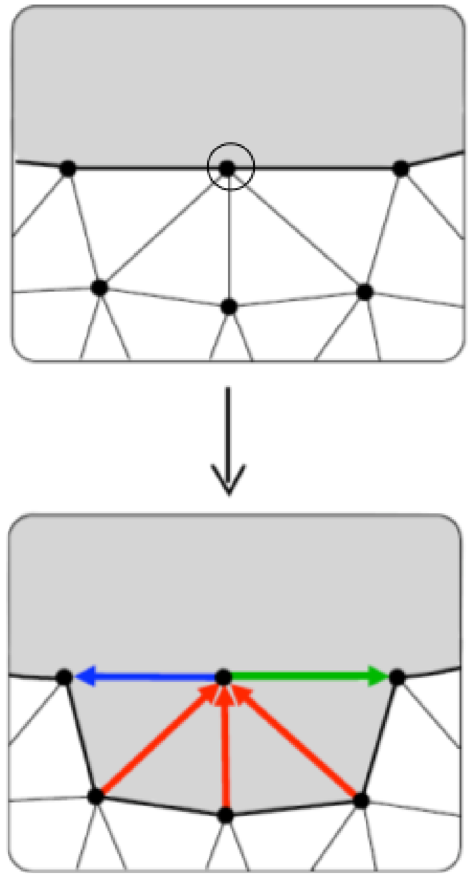
at each step:



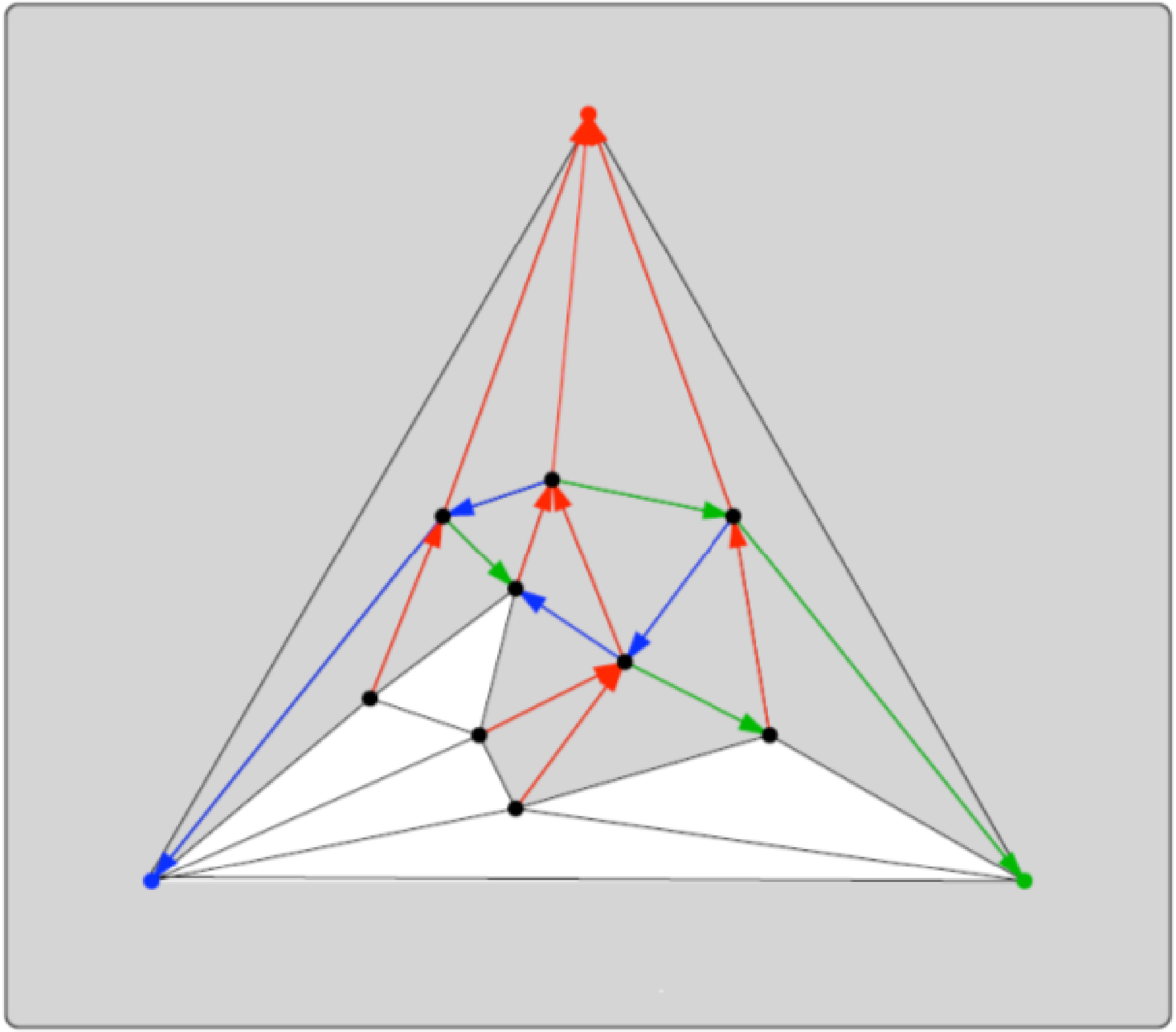
Shelling procedure to compute Schnyder woods



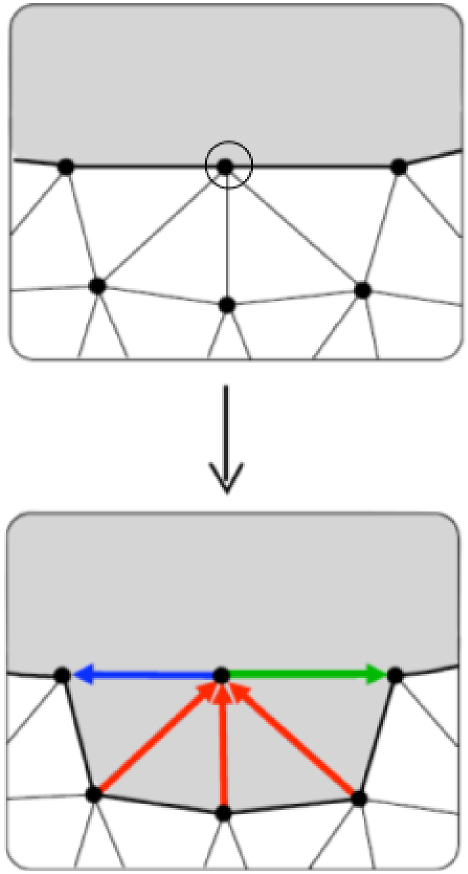
at each step:



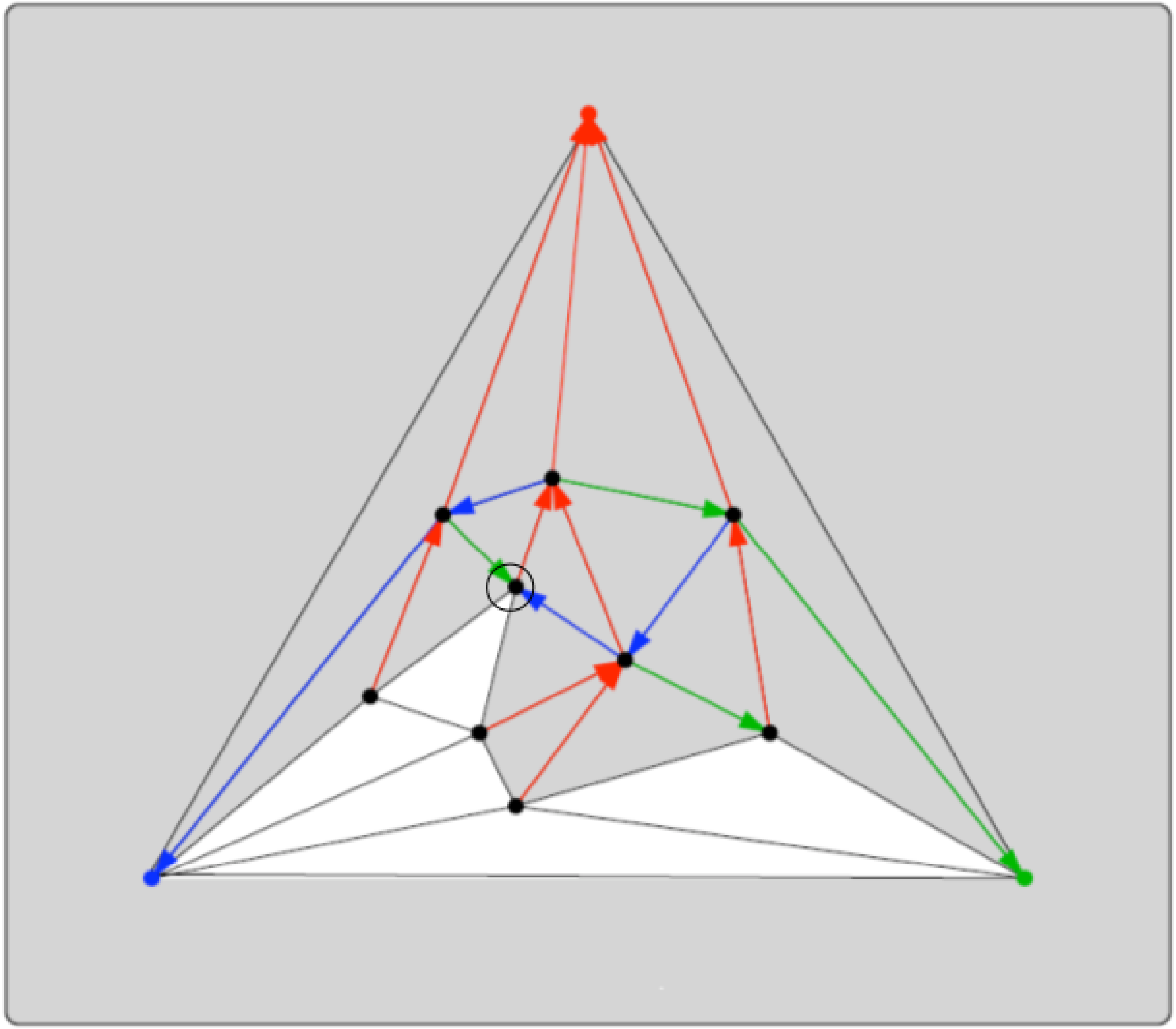
Shelling procedure to compute Schnyder woods



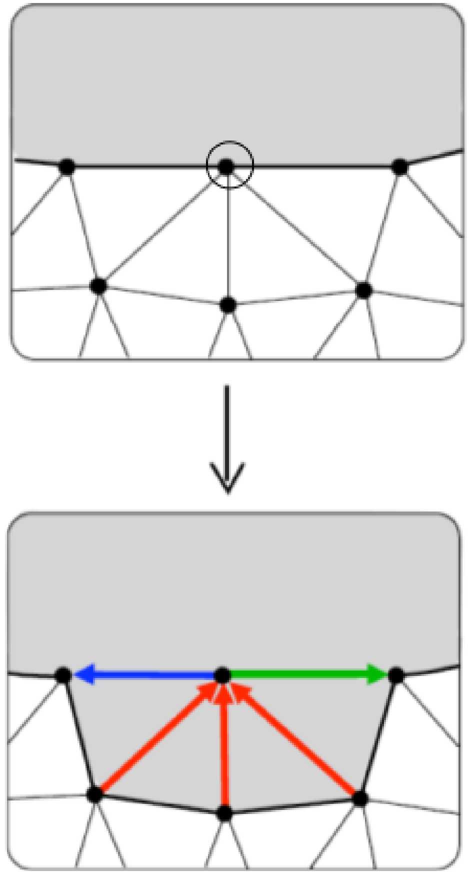
at each step:



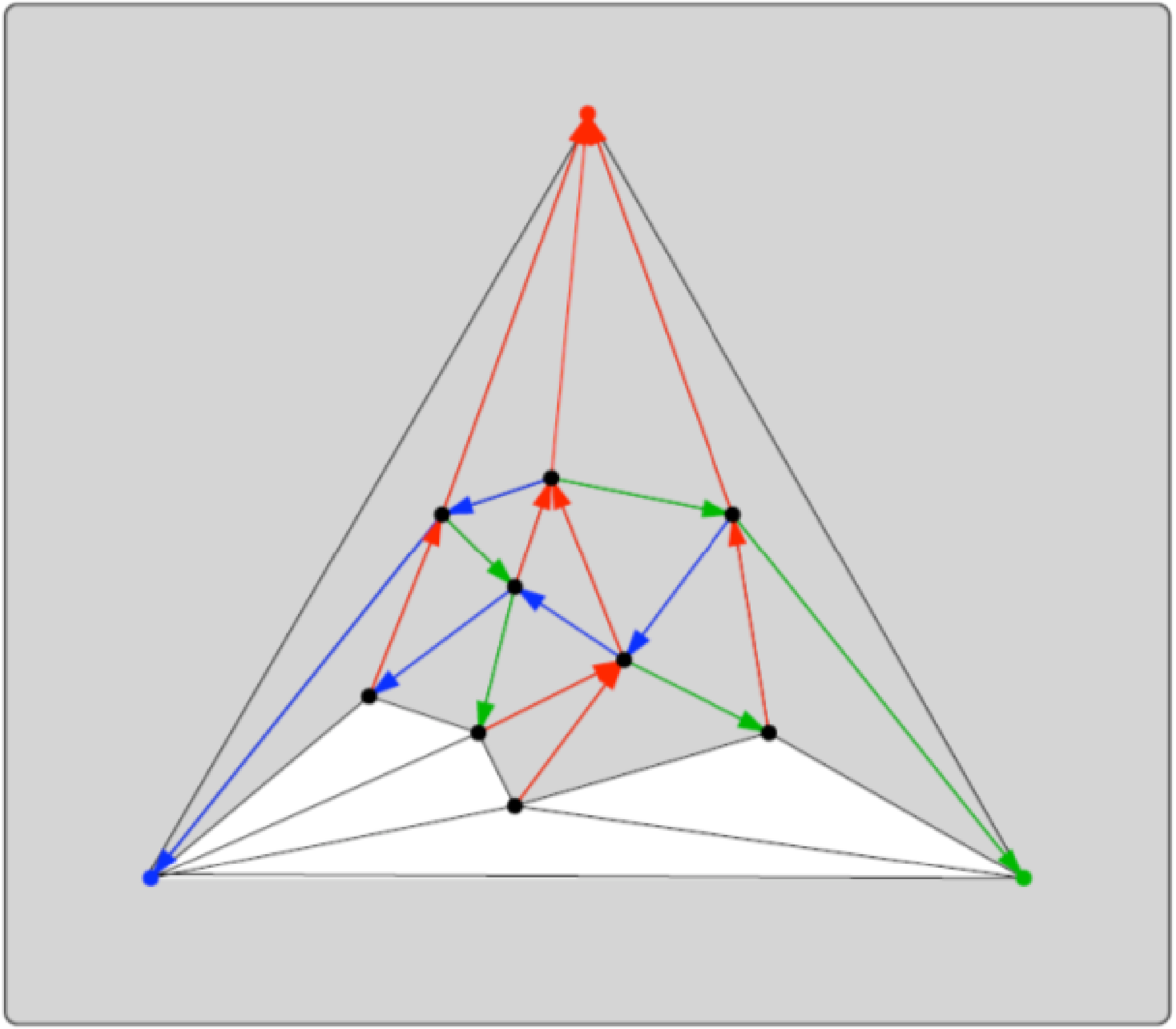
Shelling procedure to compute Schnyder woods



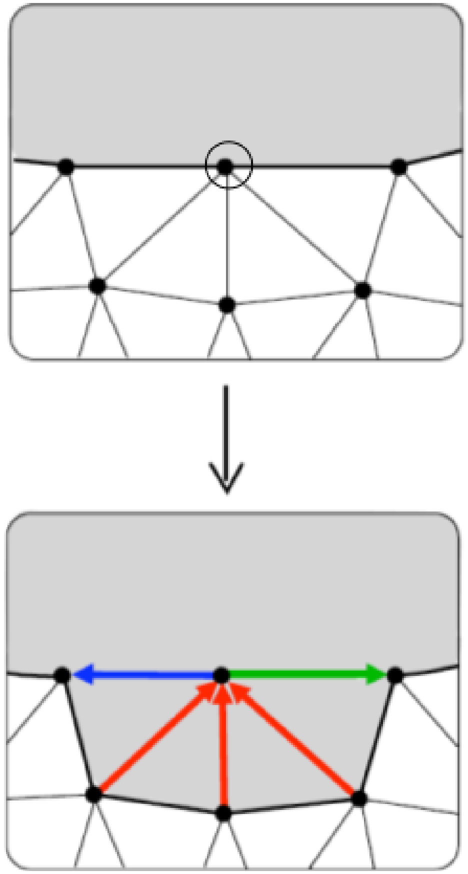
at each step:



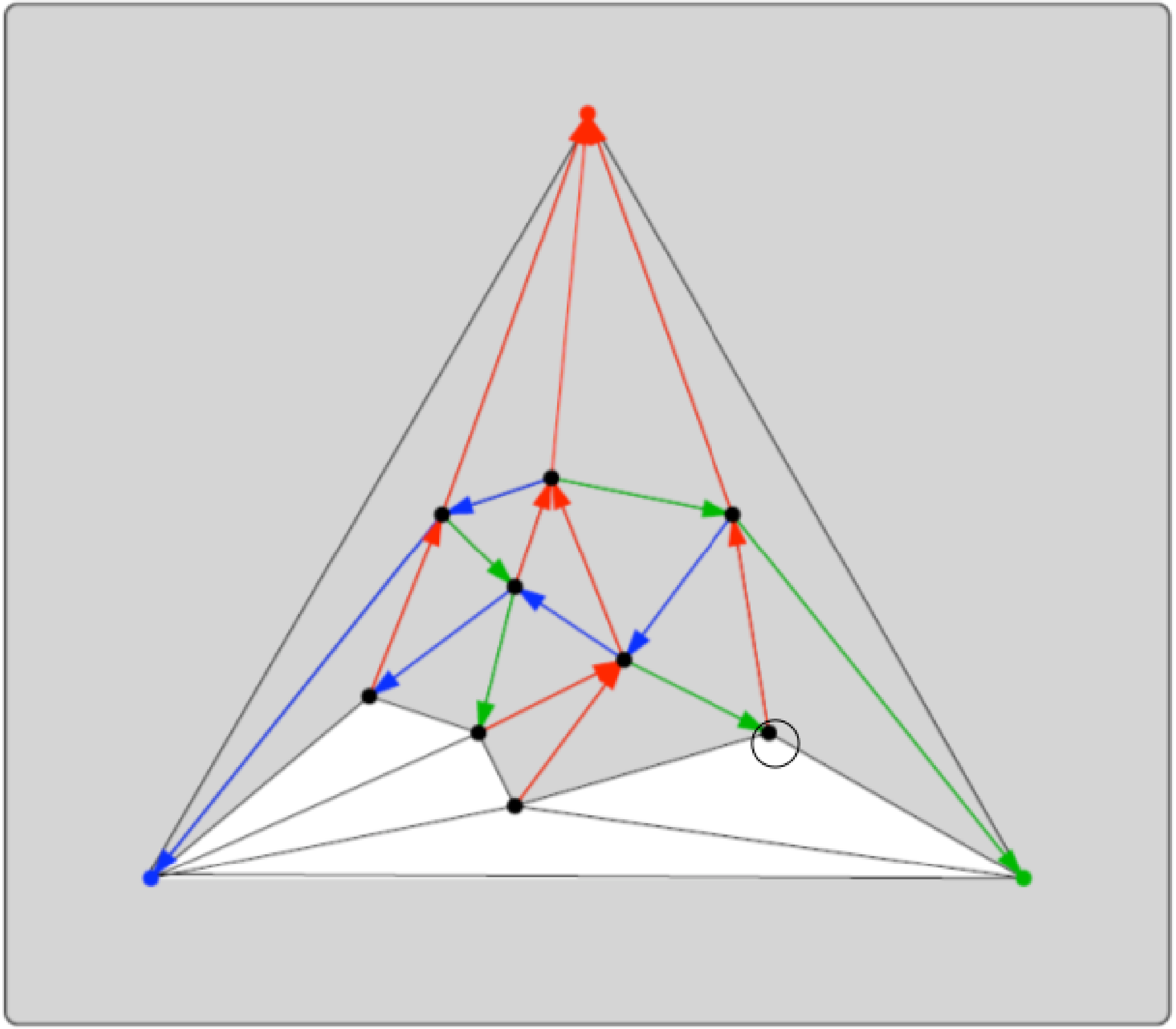
Shelling procedure to compute Schnyder woods



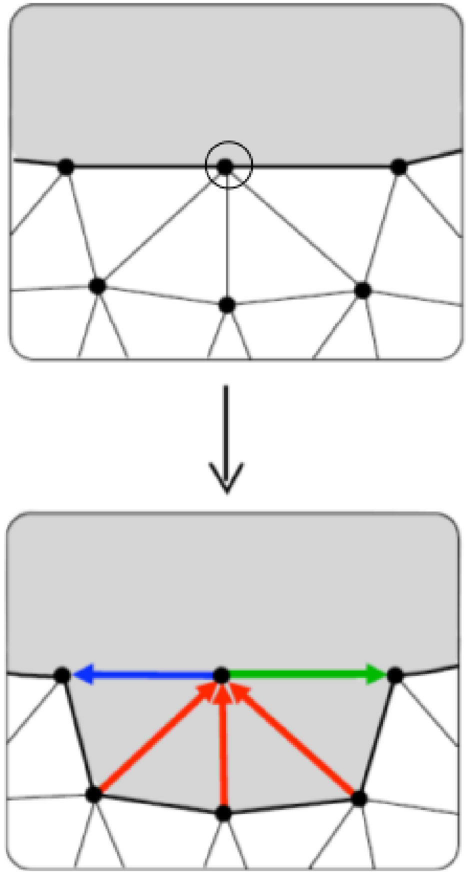
at each step:



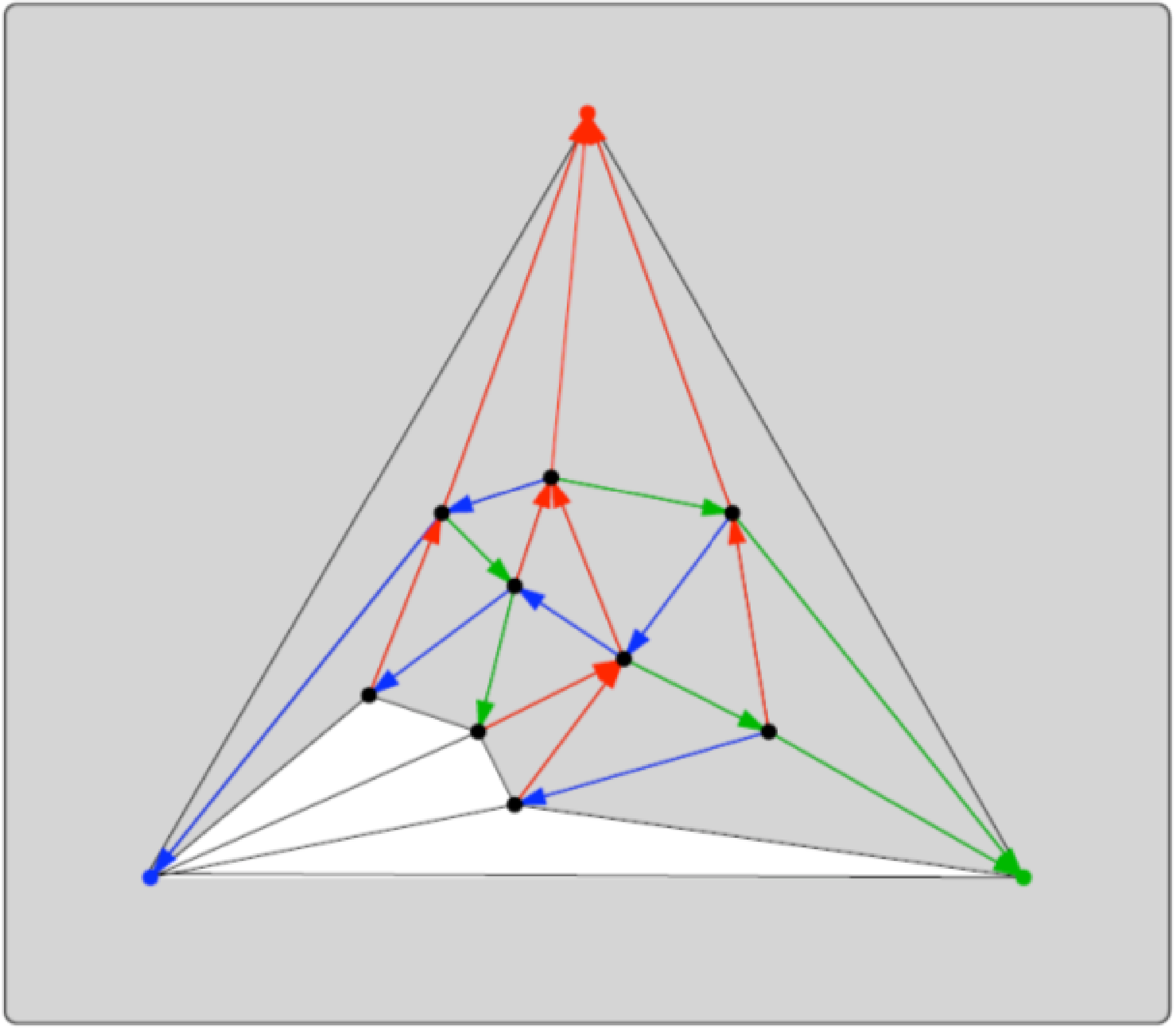
Shelling procedure to compute Schnyder woods



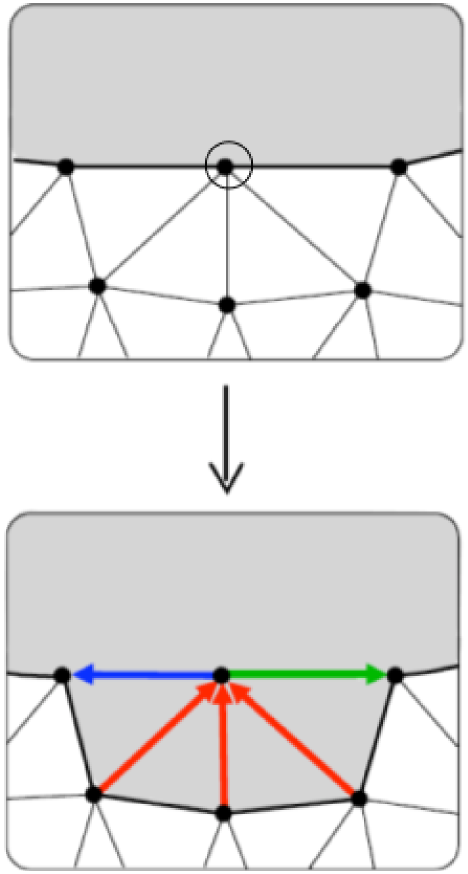
at each step:



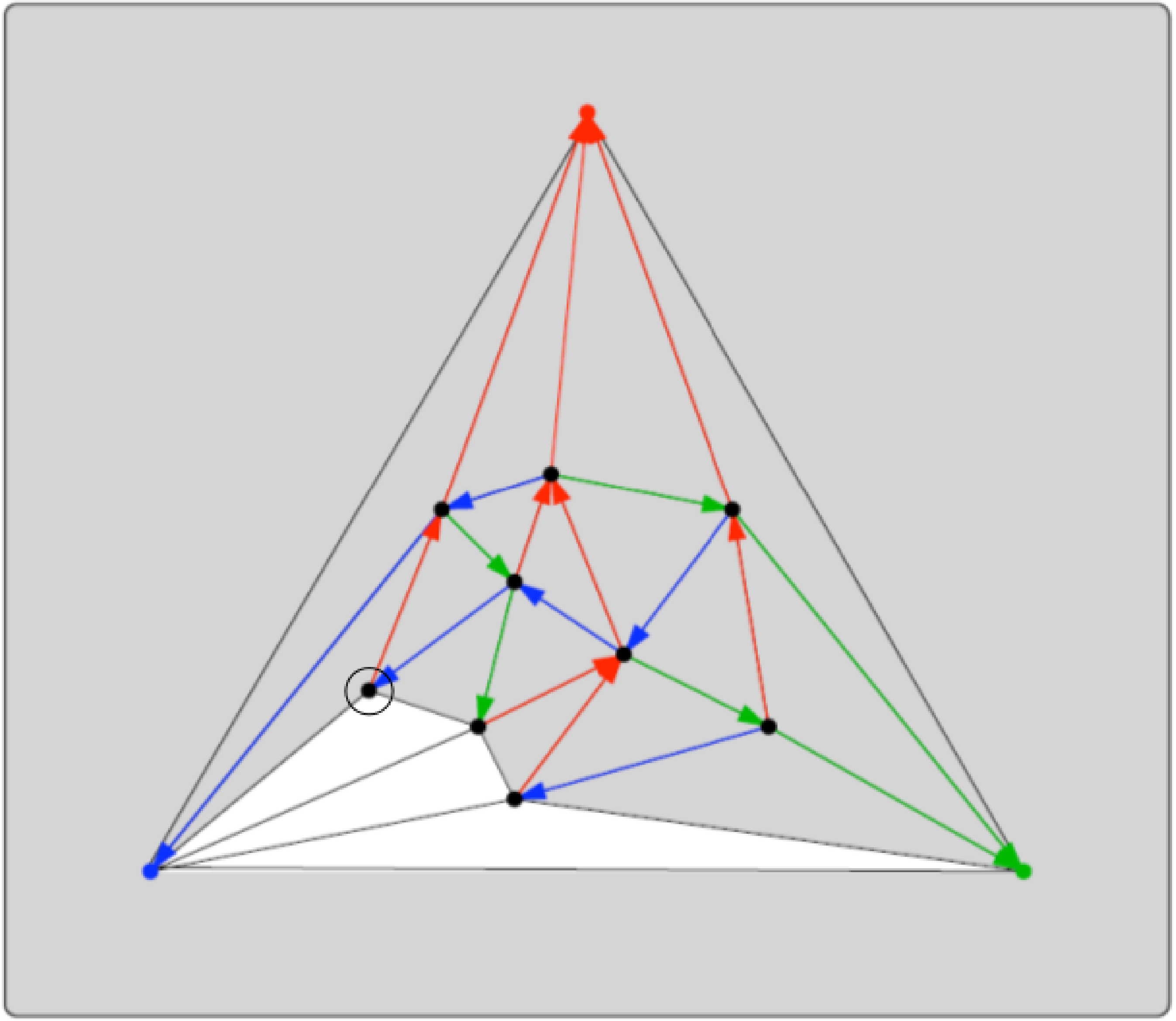
Shelling procedure to compute Schnyder woods



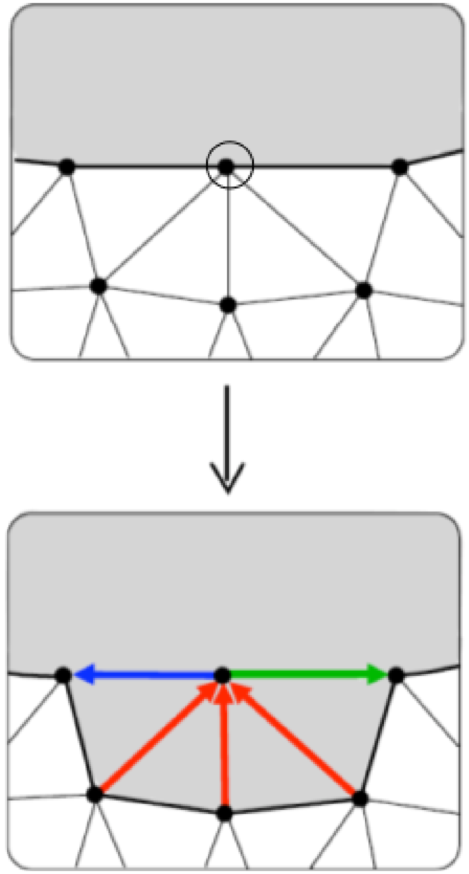
at each step:



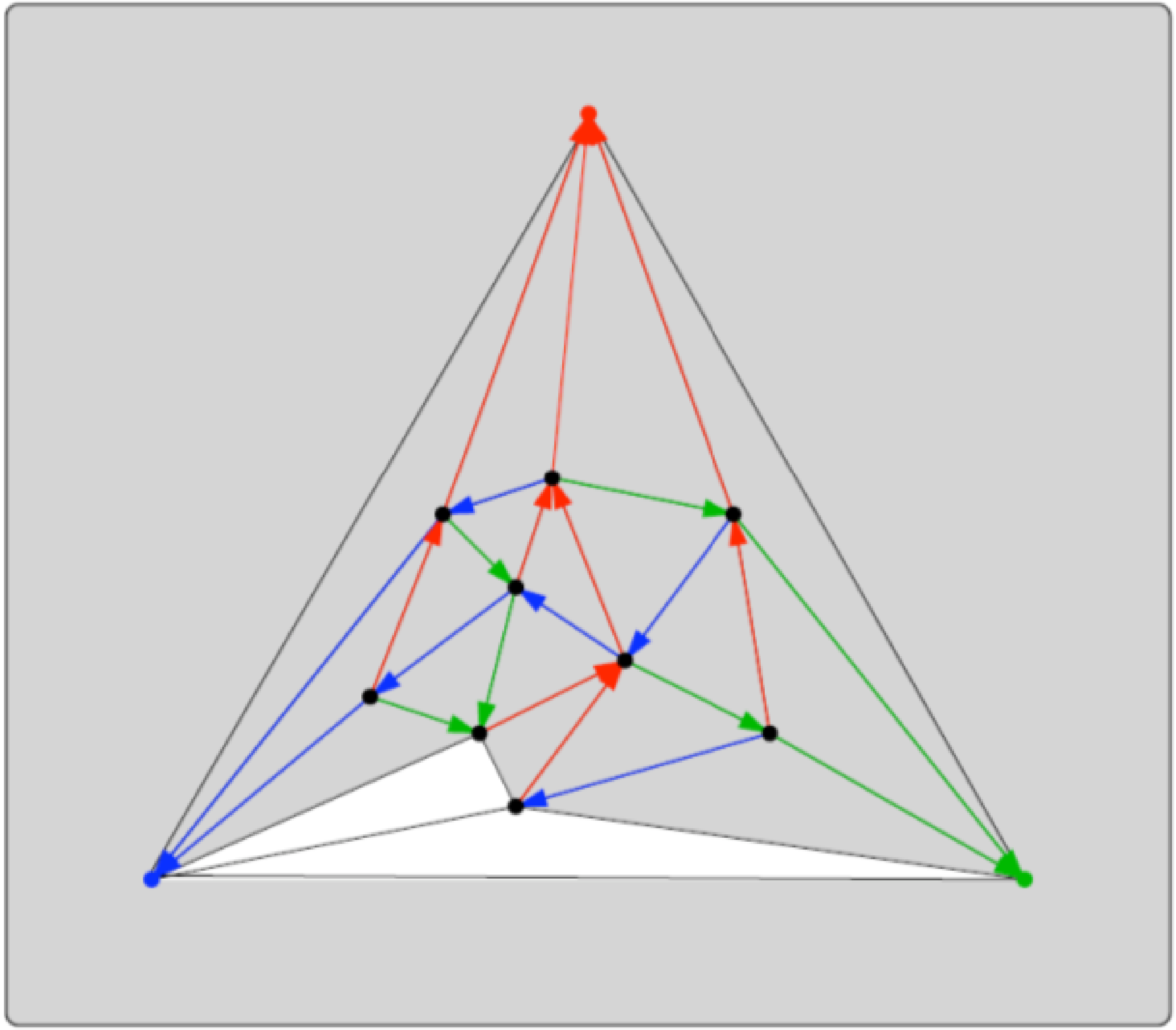
Shelling procedure to compute Schnyder woods



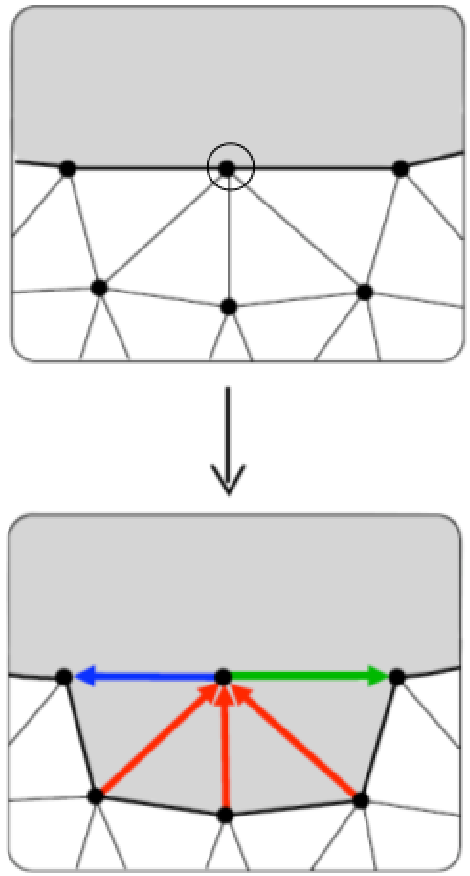
at each step:



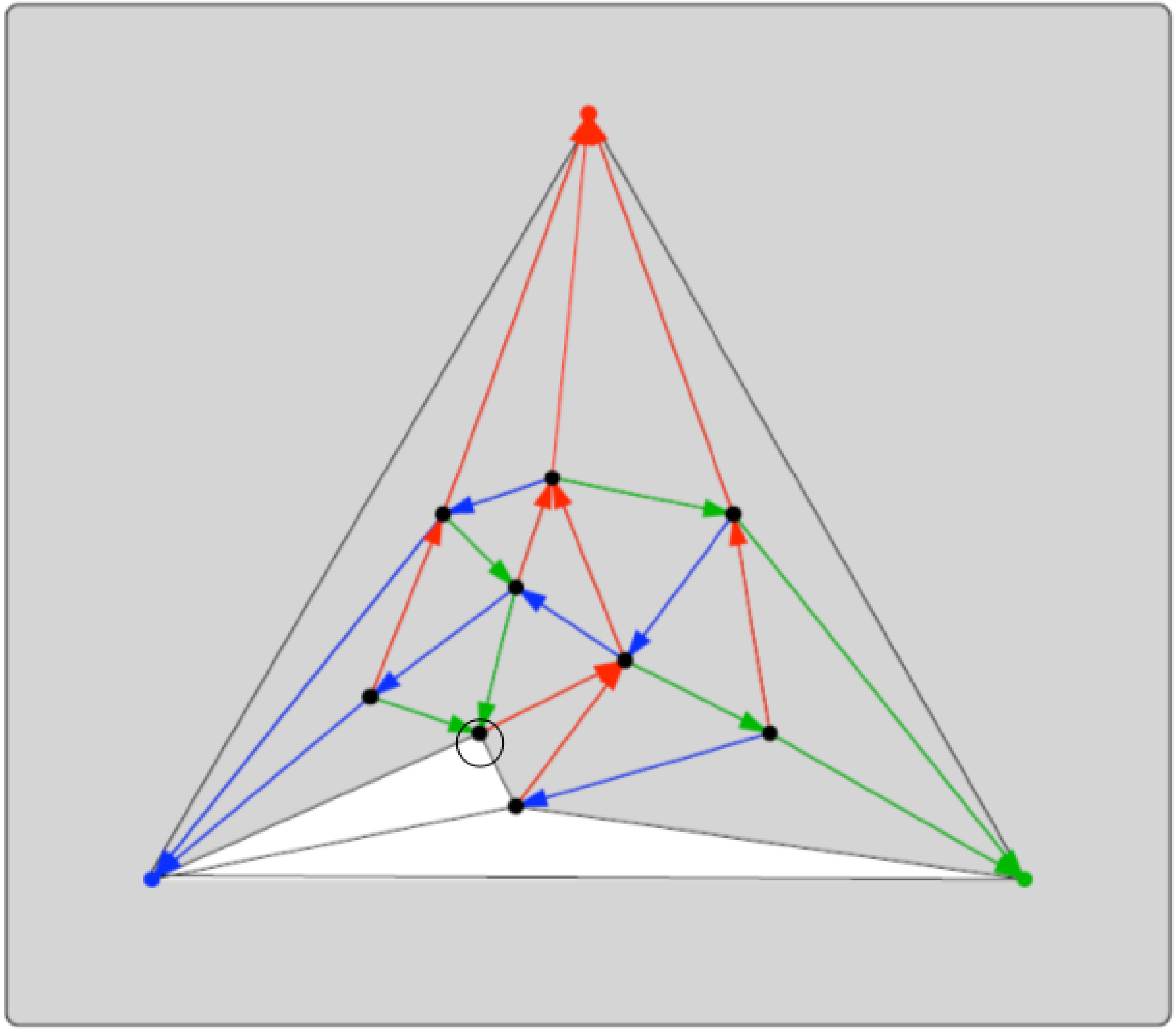
Shelling procedure to compute Schnyder woods



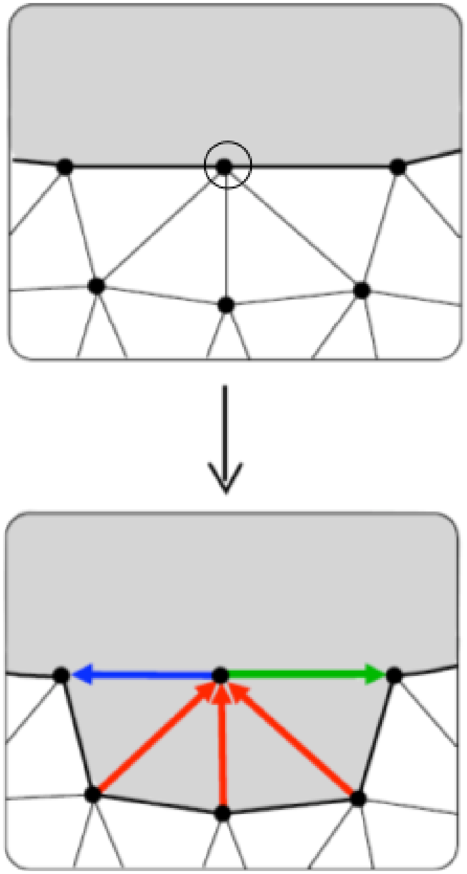
at each step:



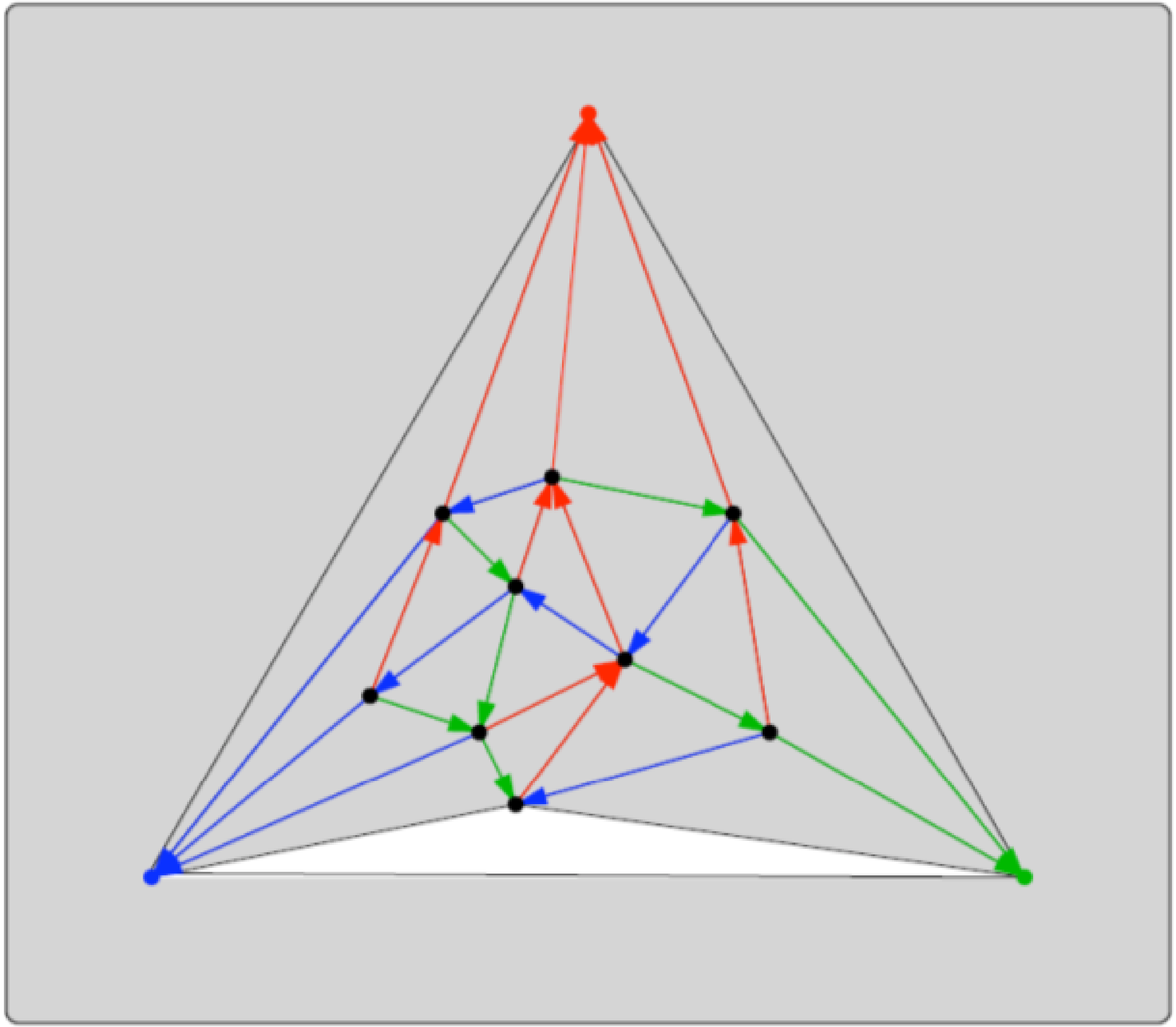
Shelling procedure to compute Schnyder woods



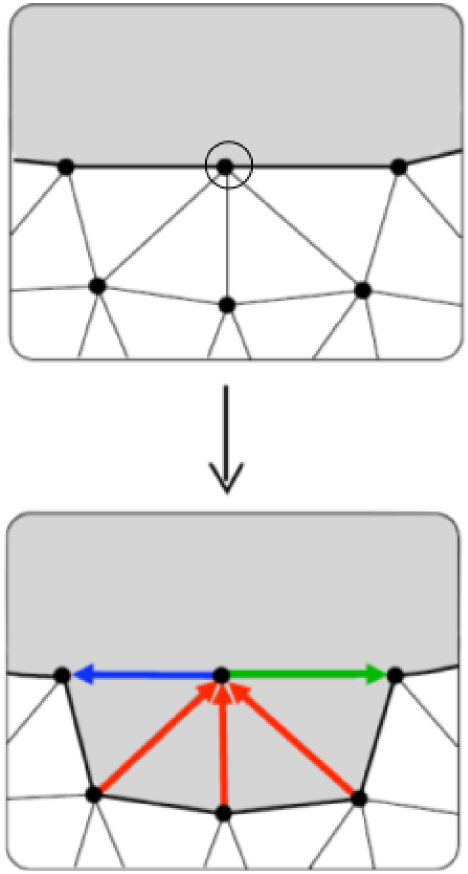
at each step:



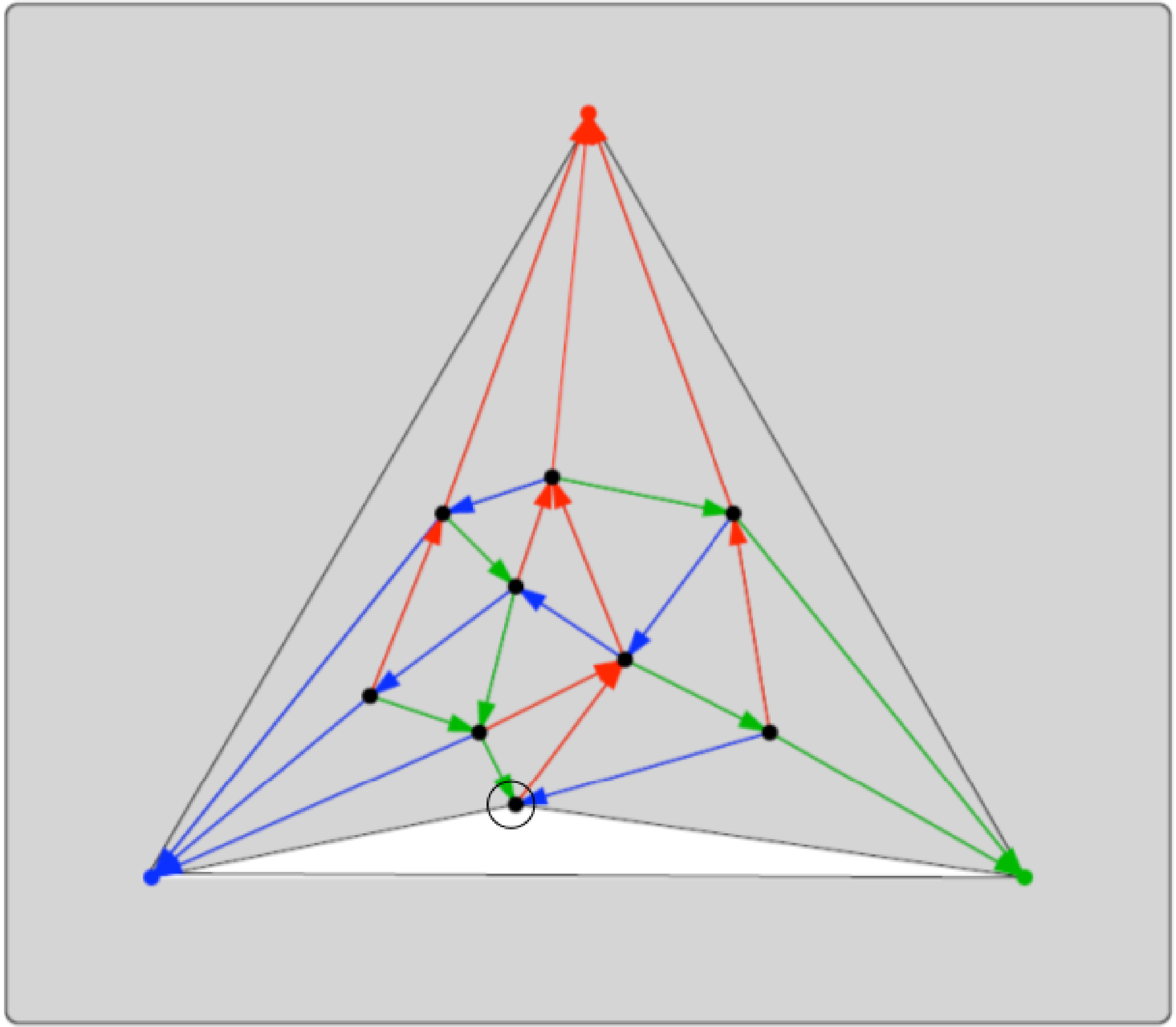
Shelling procedure to compute Schnyder woods



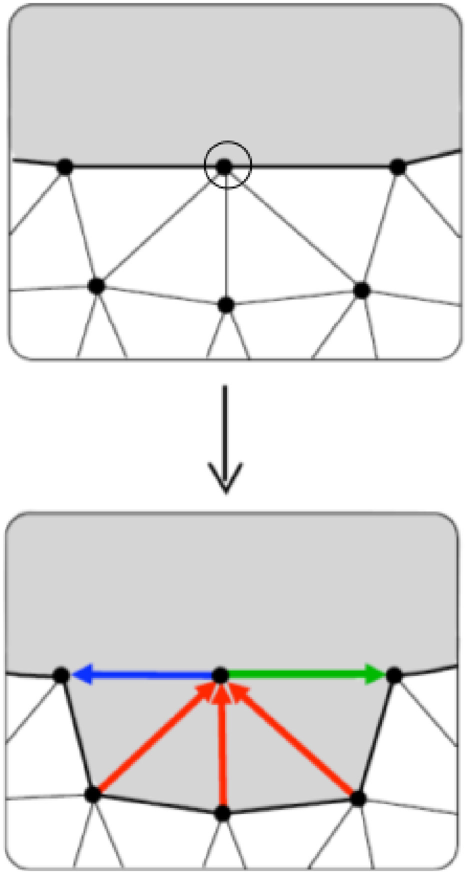
at each step:



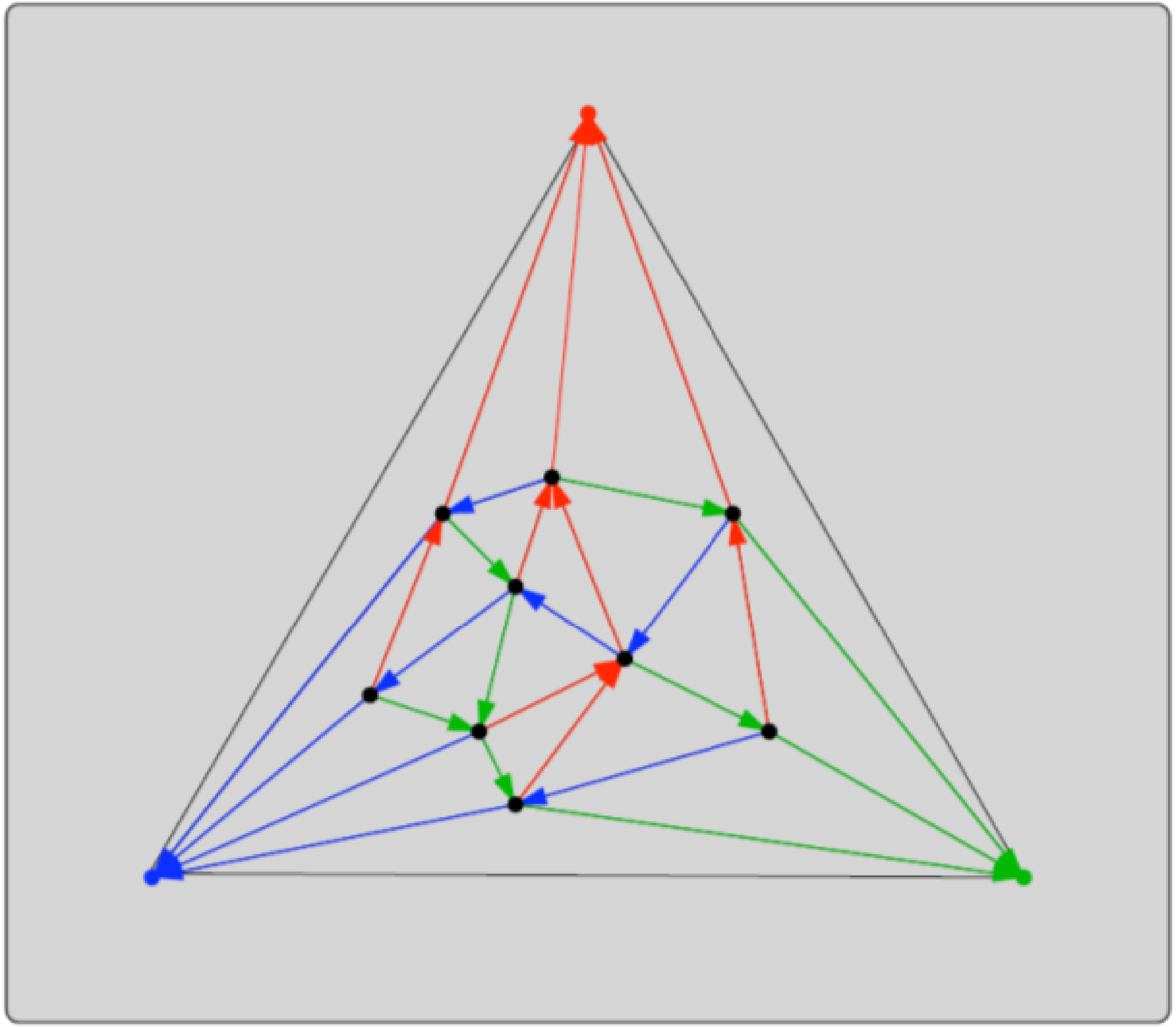
Shelling procedure to compute Schnyder woods



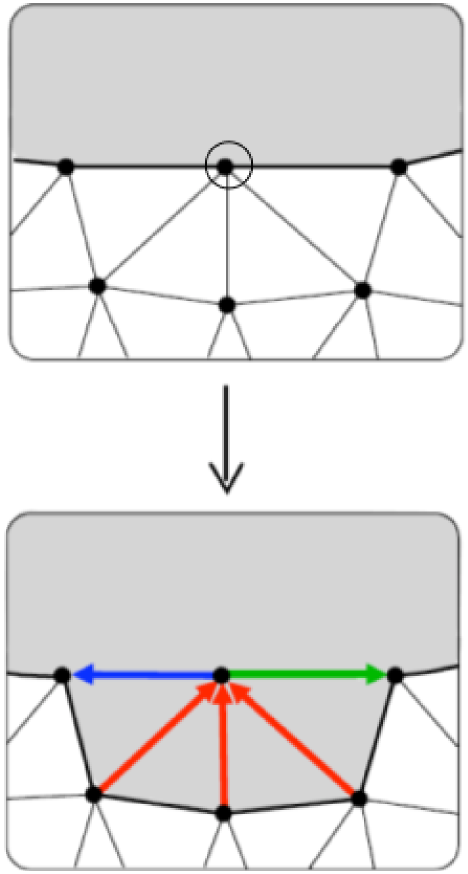
at each step:



Shelling procedure to compute Schnyder woods

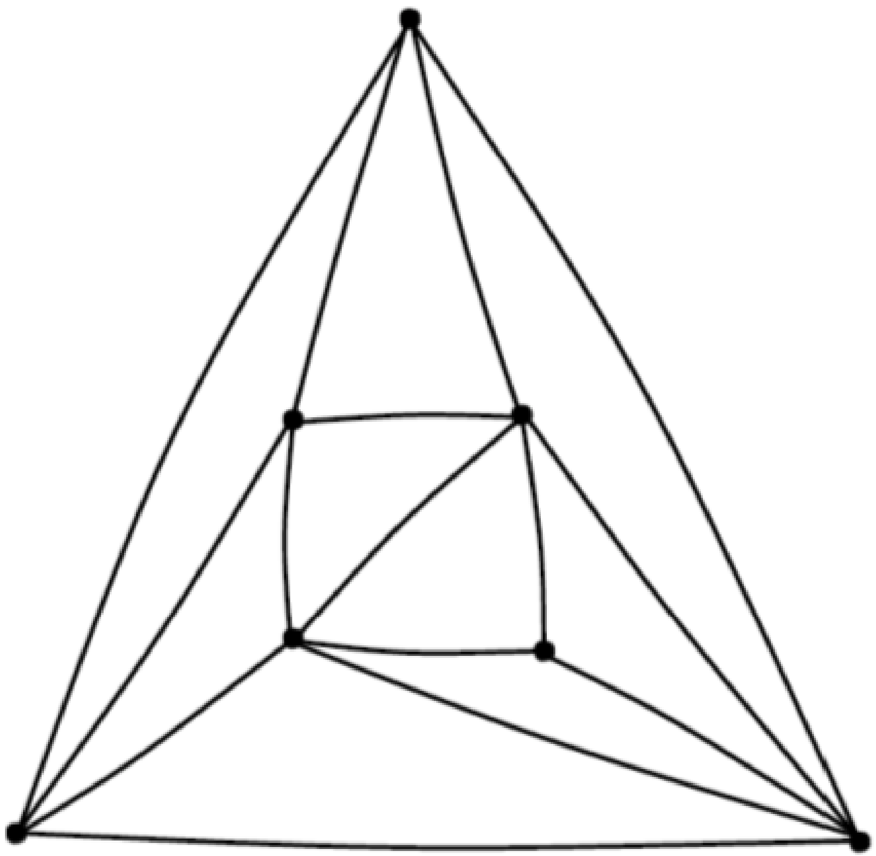


at each step:

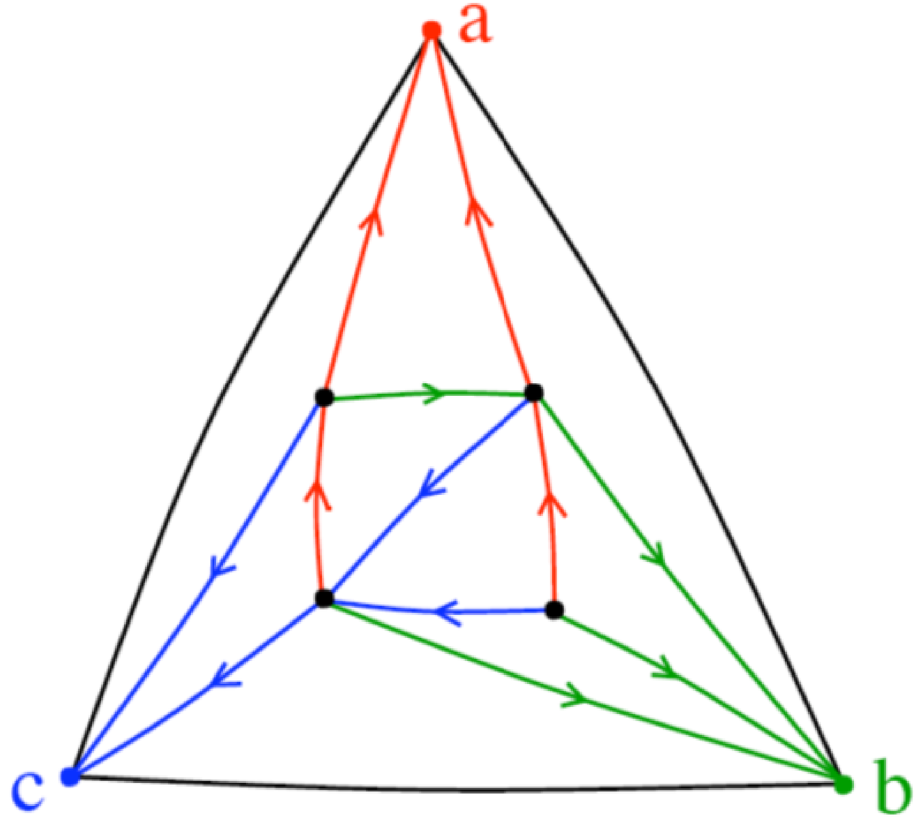


Face-counting drawing procedure

[Schnyder'90]

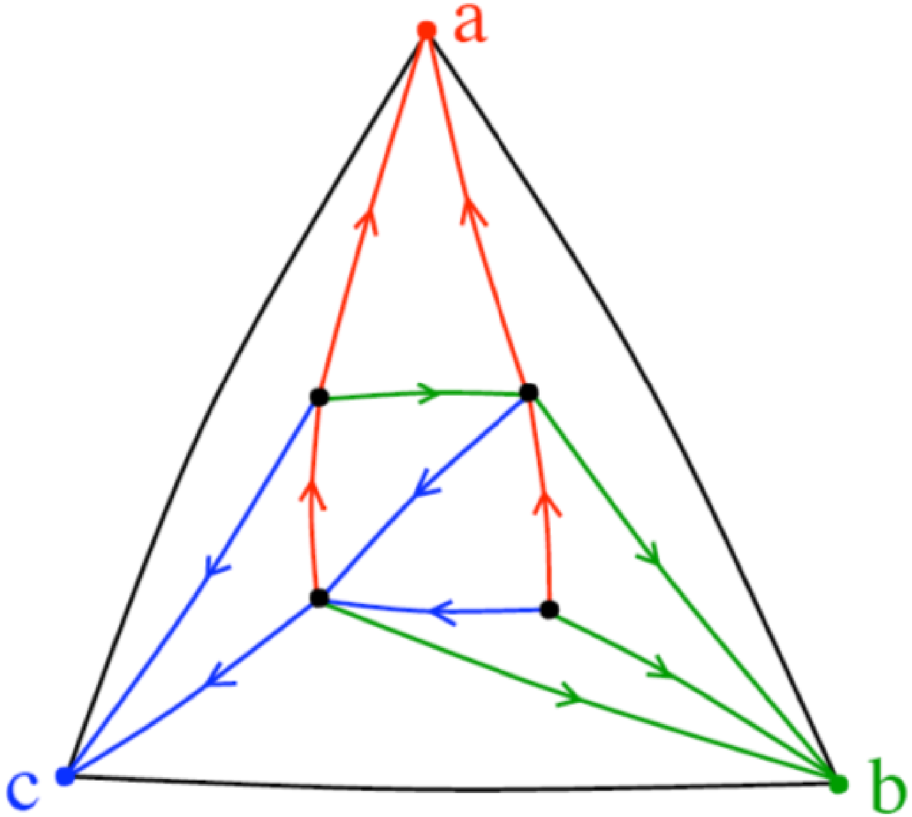


Face-counting drawing procedure
[Schnyder'90]

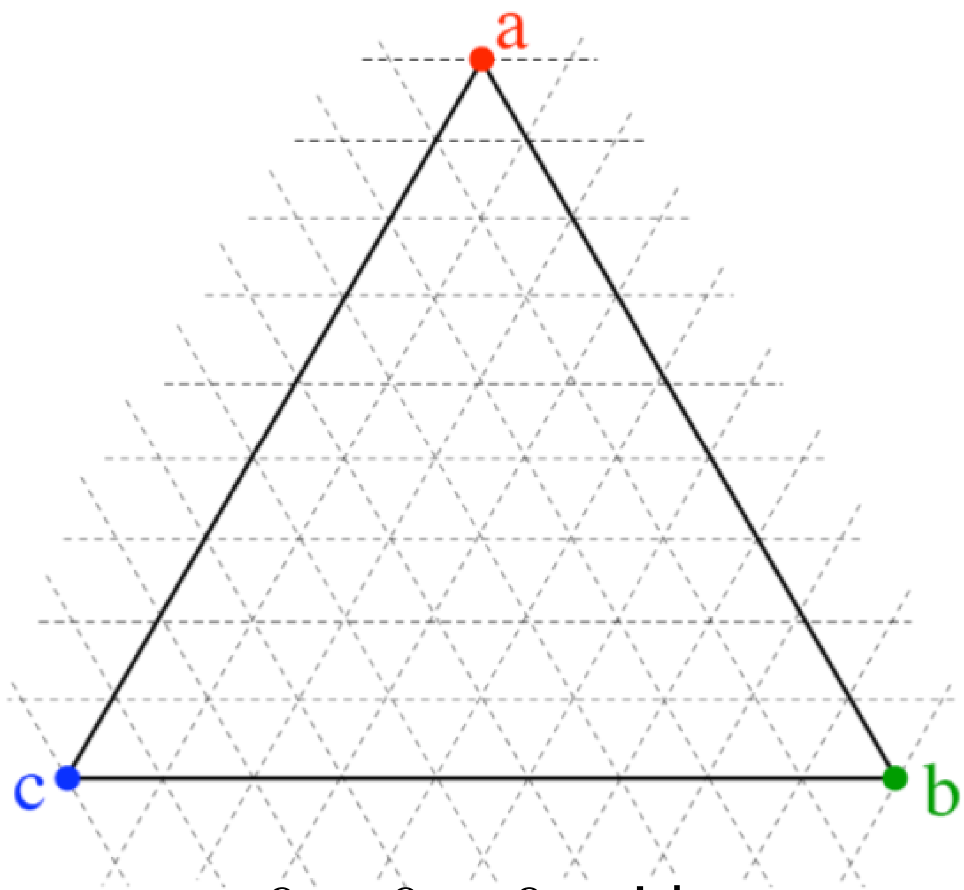


Face-counting drawing procedure

[Schnyder'90]



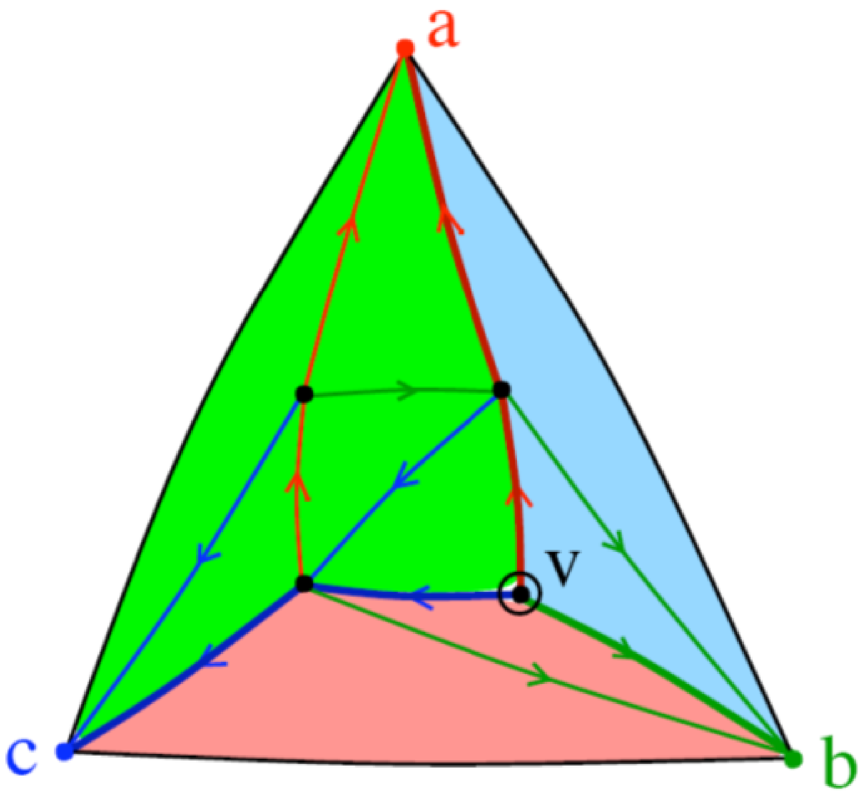
9 inner faces



$9 \times 9 \times 9$ grid

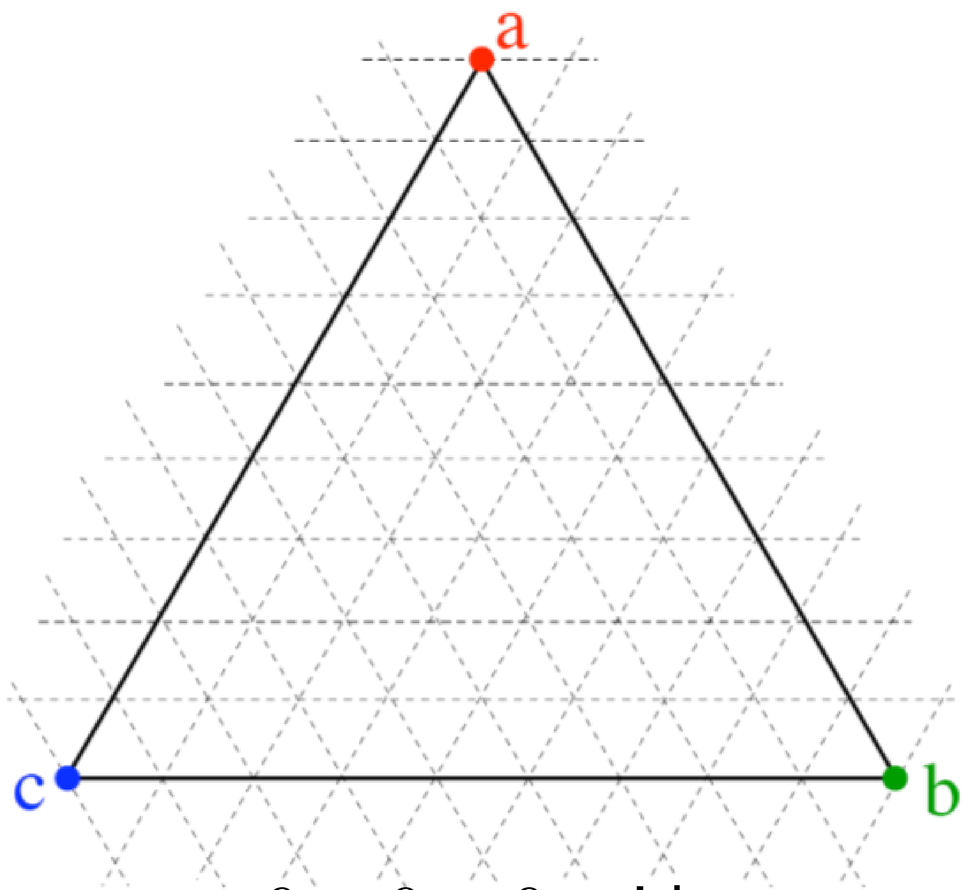
Face-counting drawing procedure

[Schnyder'90]



9 inner faces

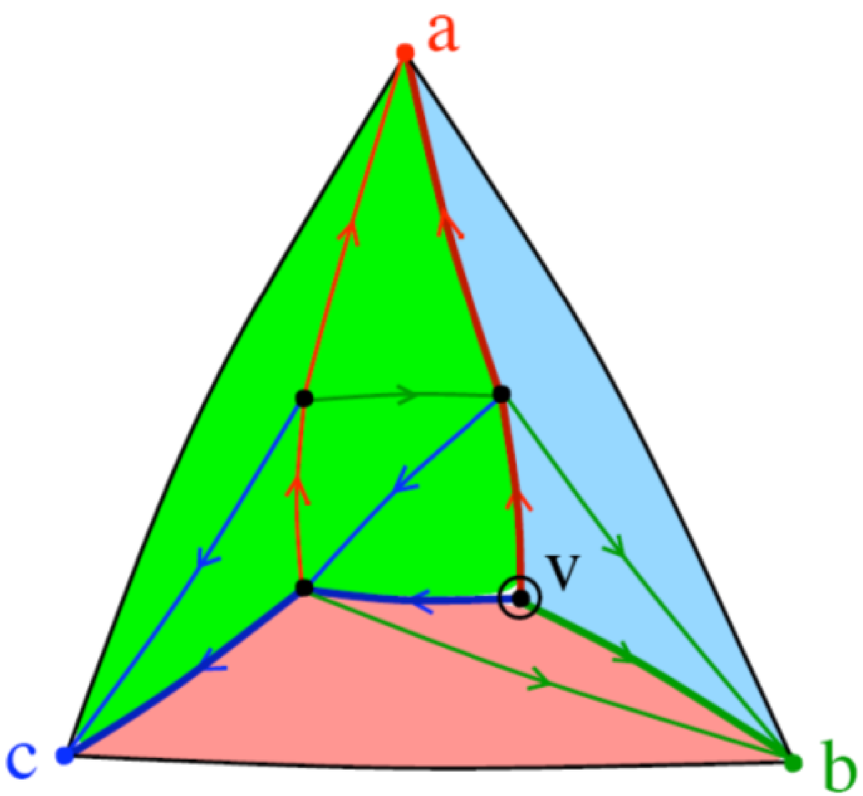
for v: red area: 2 faces
green area: 5 faces
blue area: 2 faces

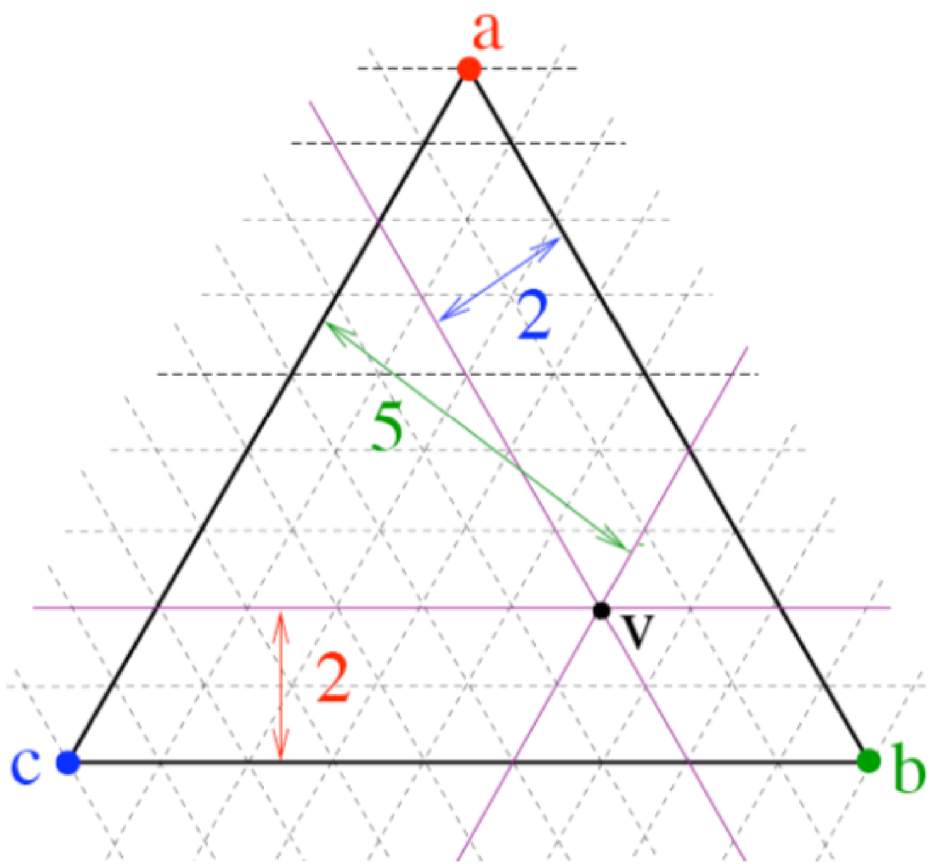


$9 \times 9 \times 9$ grid

Face-counting drawing procedure

[Schnyder'90]



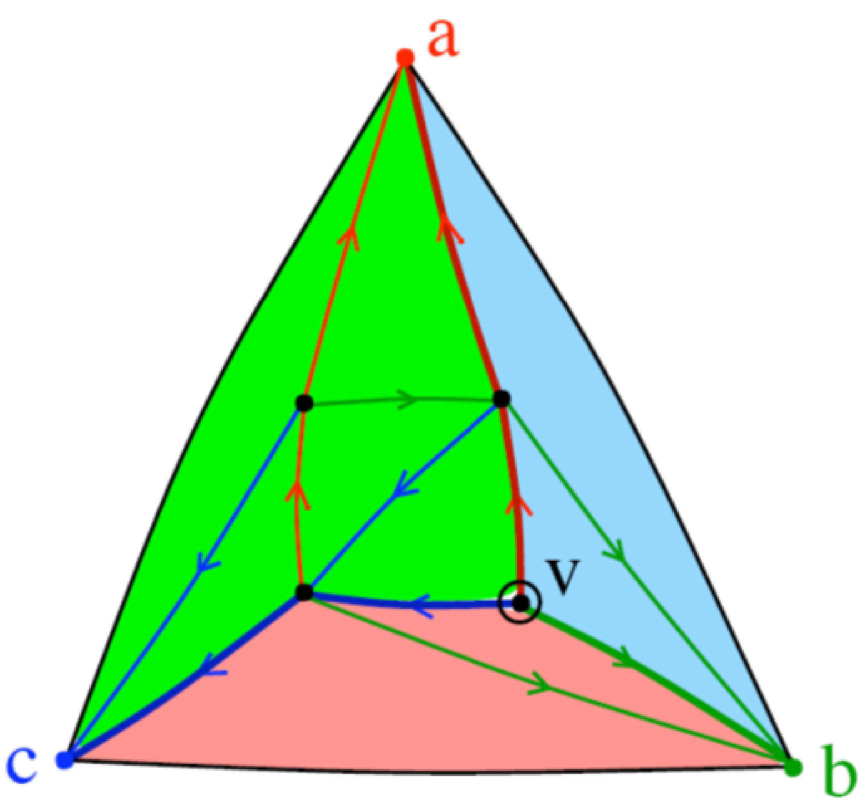


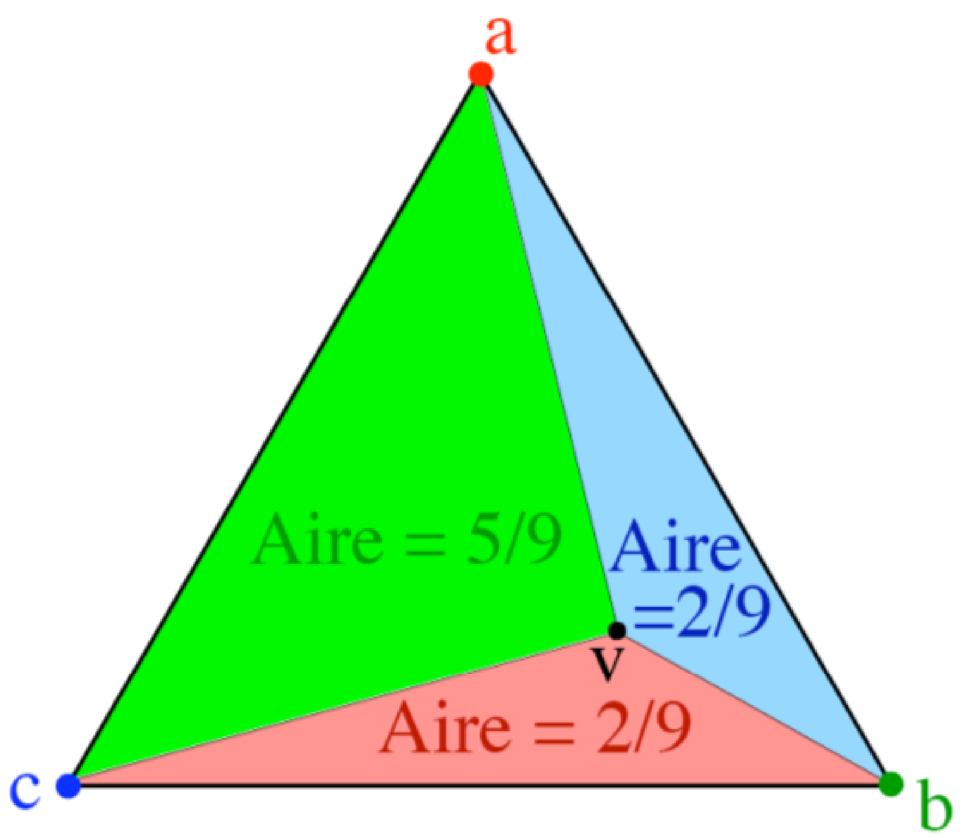
for v: red area: 2 faces
green area: 5 faces
blue area: 2 faces

draw v at the barycenter of $\{a, b, c\}$
with weights $\frac{2}{9}$, $\frac{5}{9}$, $\frac{2}{9}$

Face-counting drawing procedure

[Schnyder'90]



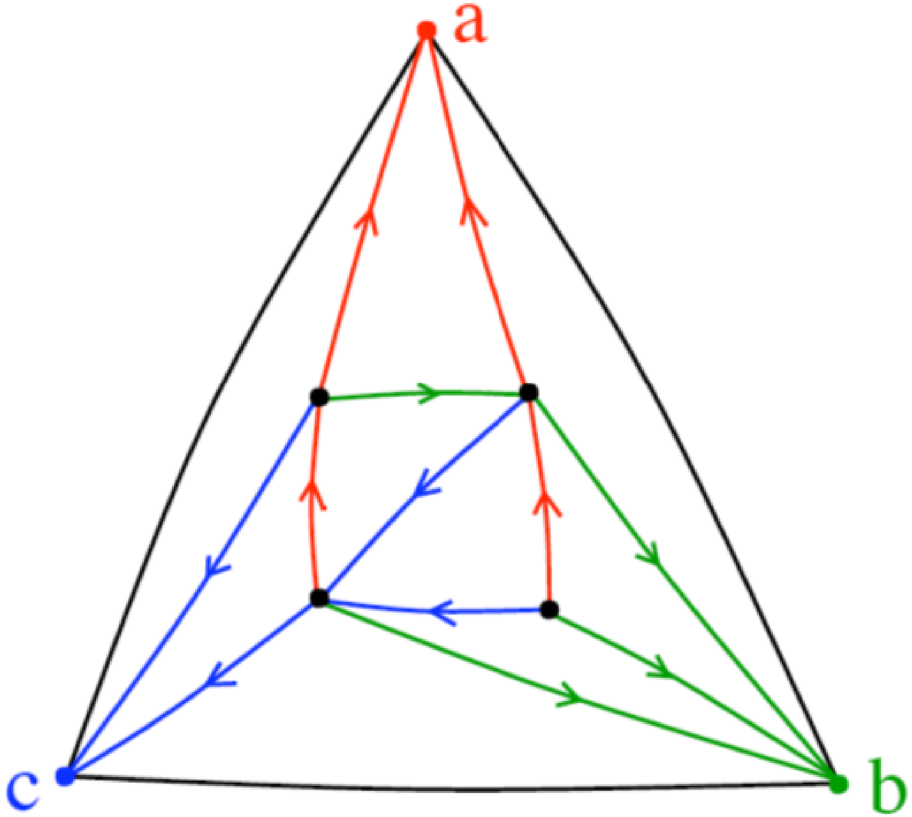


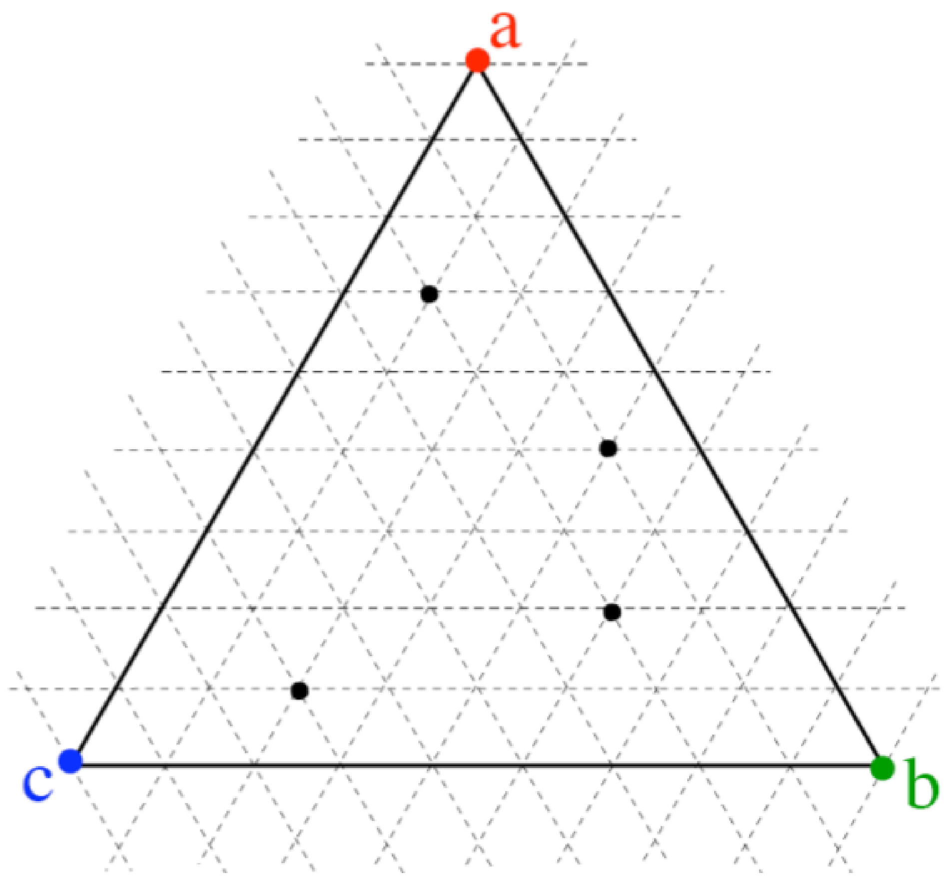
for v: red area: 2 faces
green area: 5 faces
blue area: 2 faces

draw v at the barycenter of $\{a, b, c\}$
with weights $\frac{2}{9}$, $\frac{5}{9}$, $\frac{2}{9}$

Face-counting drawing procedure

[Schnyder'90]

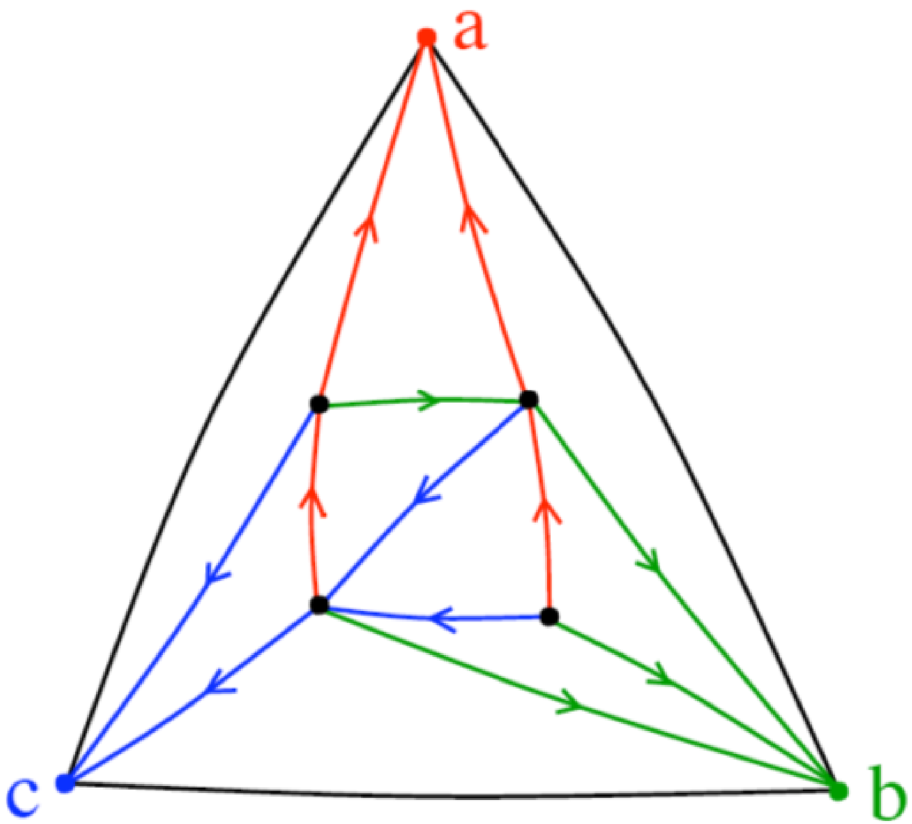


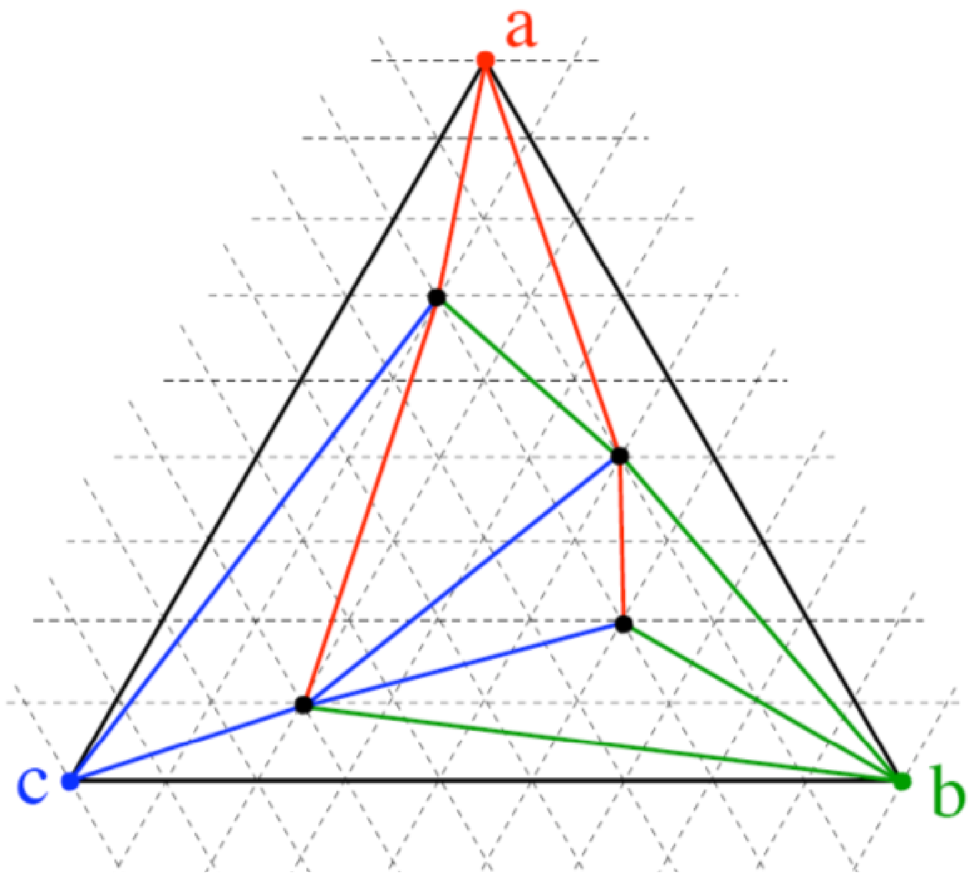


draw the other vertices
according to the same rule

Face-counting drawing procedure

[Schnyder'90]

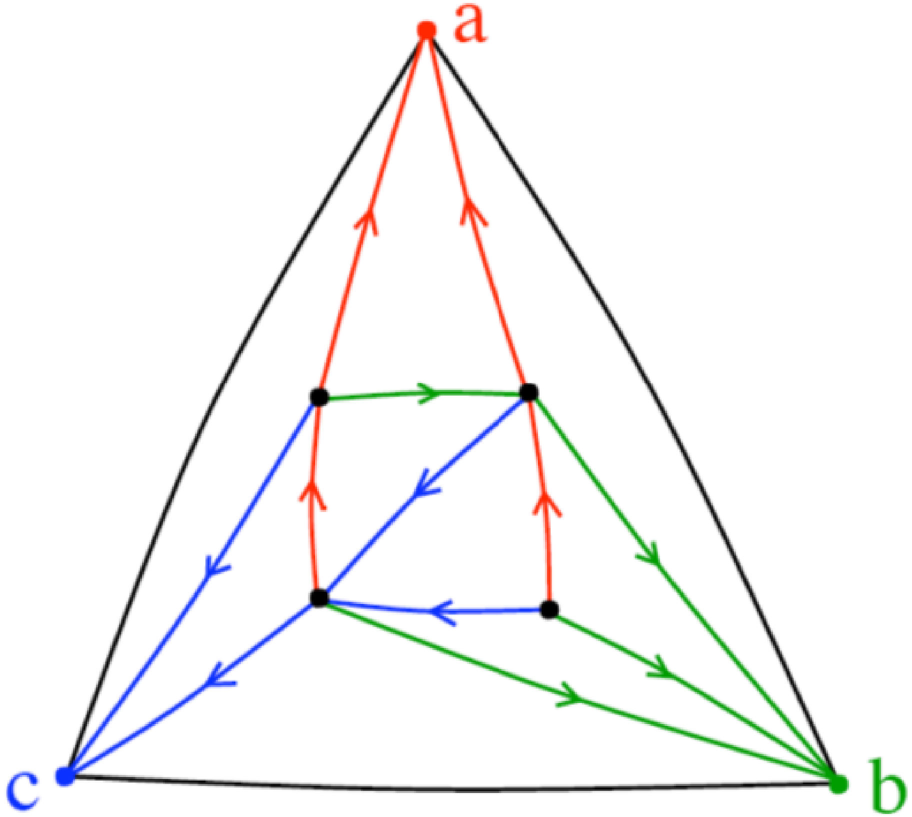


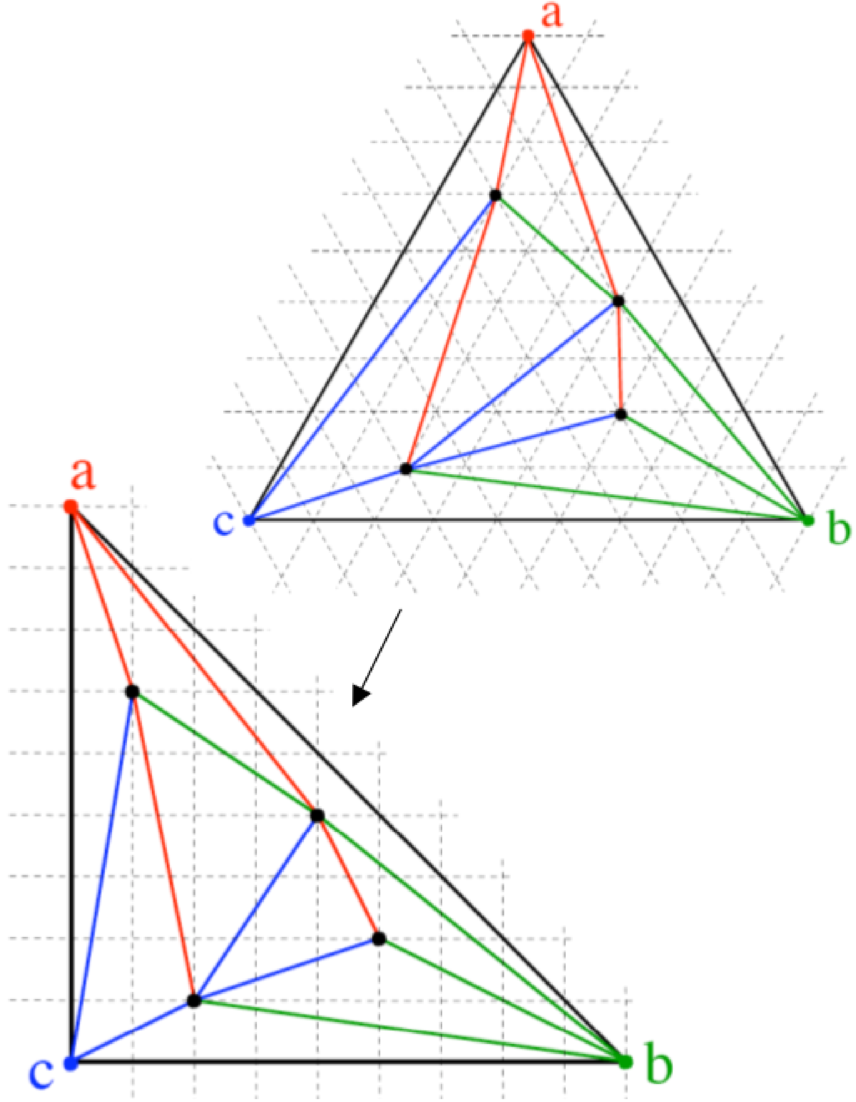


draw the edges as segments

Face-counting drawing procedure

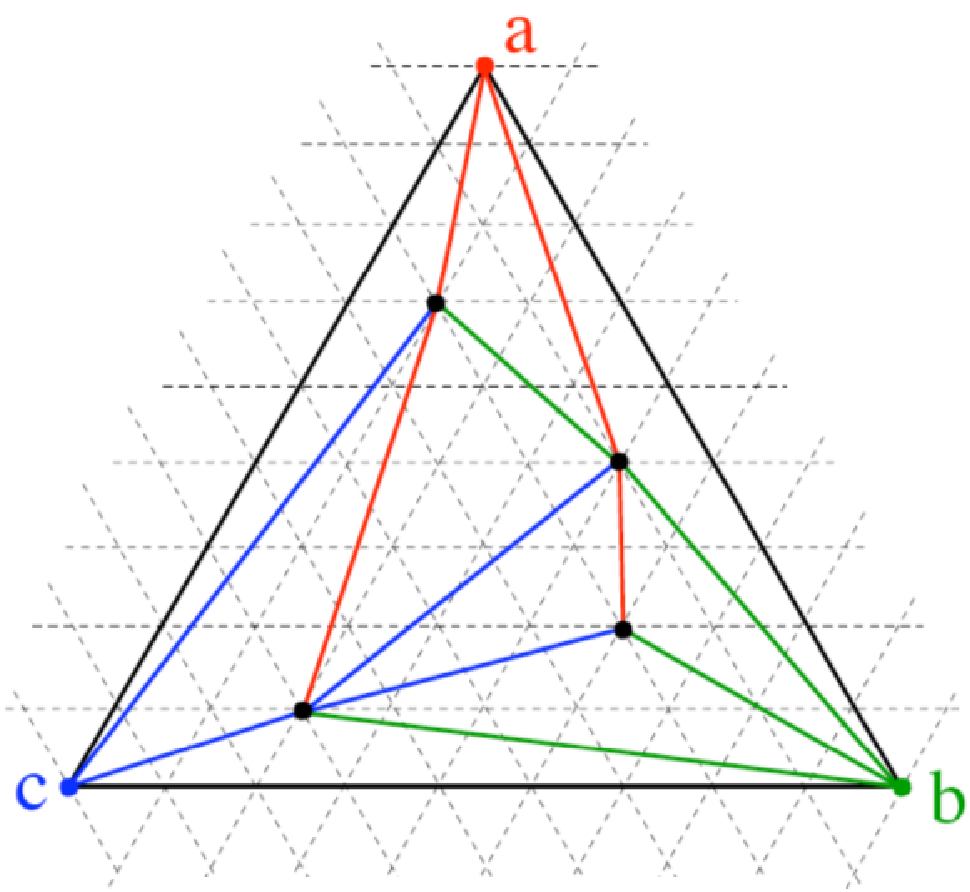
[Schnyder'90]



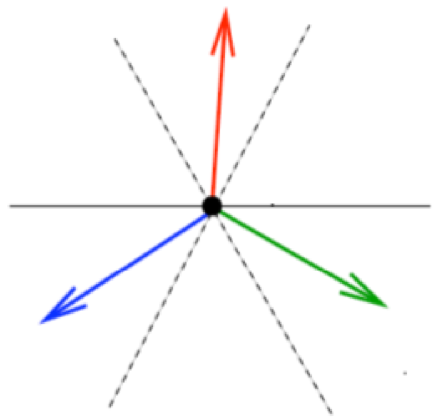


For any triangulation T with n vertices, this procedure gives a planar straight-line drawing on the regular $(2n-5) \times (2n-5)$ grid

Proof of planarity



at each inner vertex:

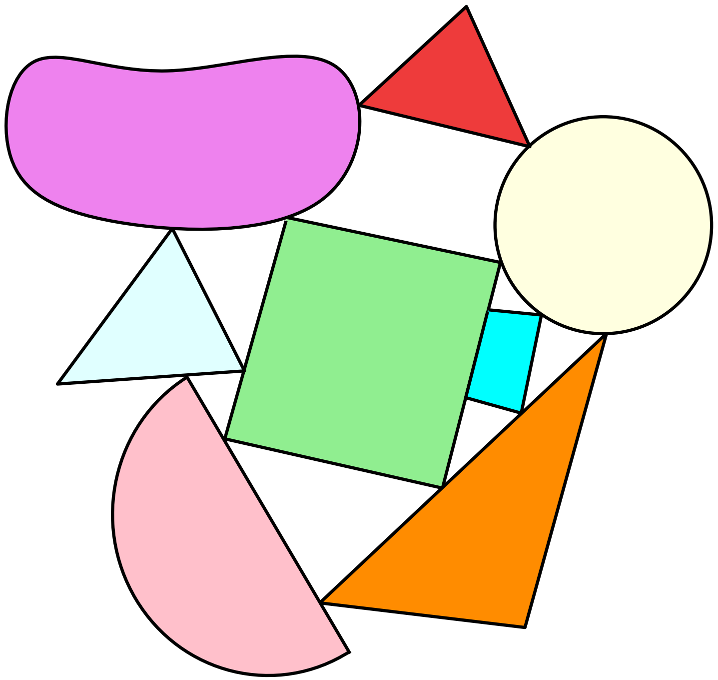


(hence inside the convex hull of neighbours)

Contact representations of planar graphs

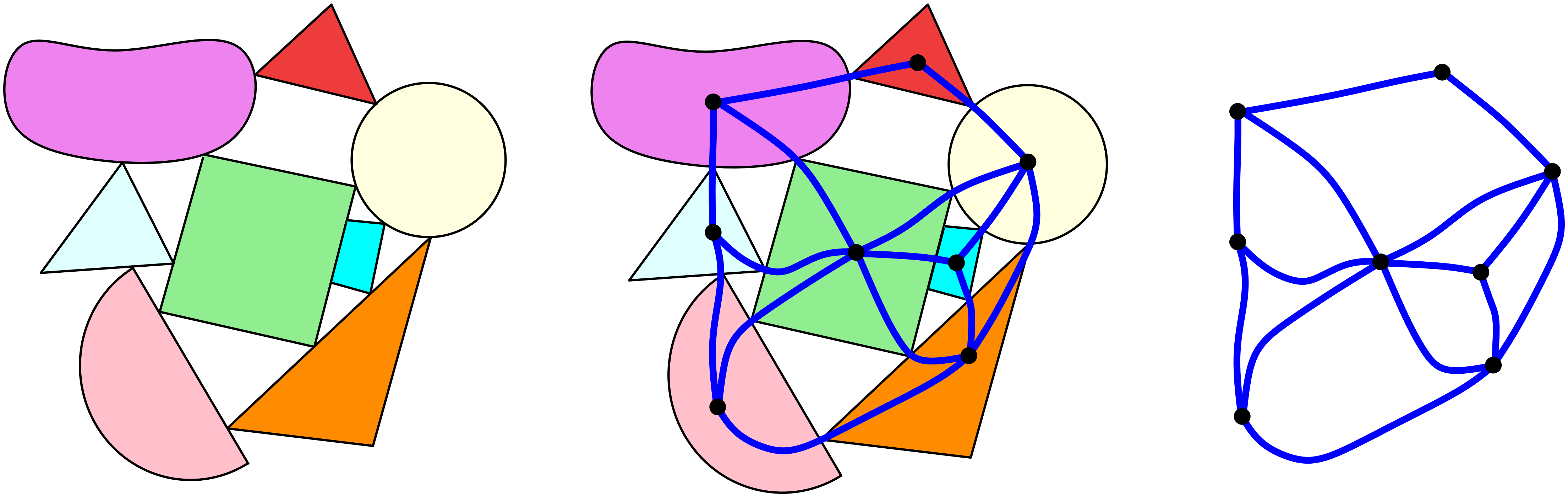
General formulation

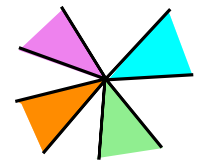
Contact configuration = set of “shapes” that can not overlap
but can have contacts



General formulation

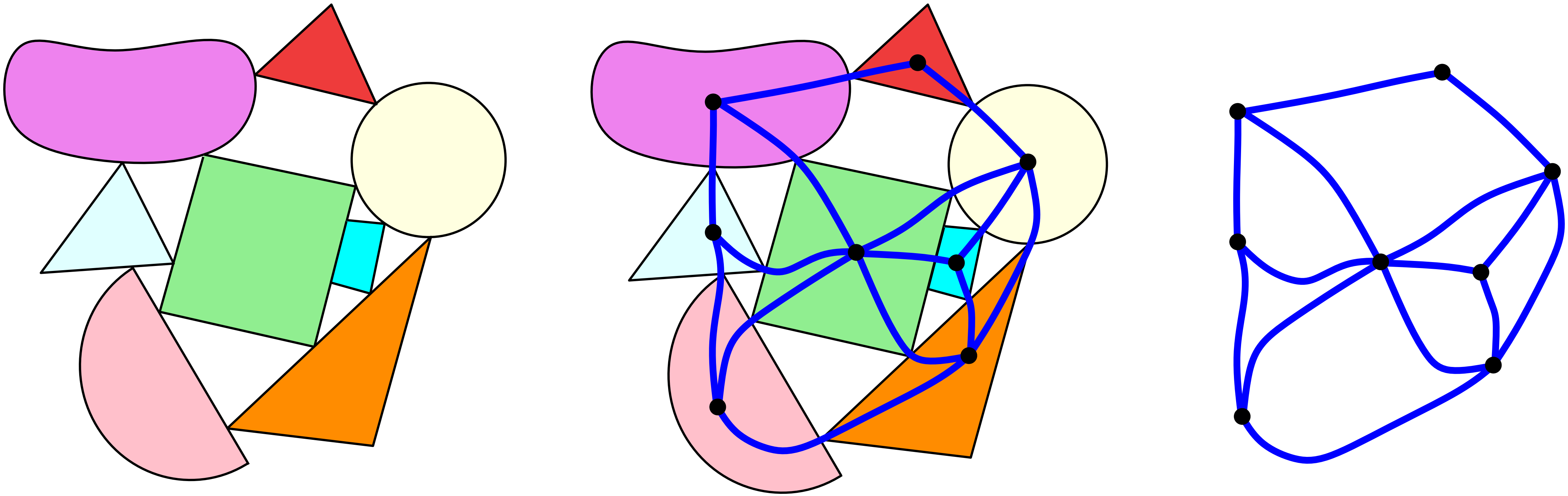
Contact configuration = set of "shapes" that can not overlap
but can have contacts

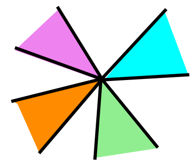


yields a planar map (when no )

General formulation

Contact configuration = set of “shapes” that can not overlap
but can have contacts



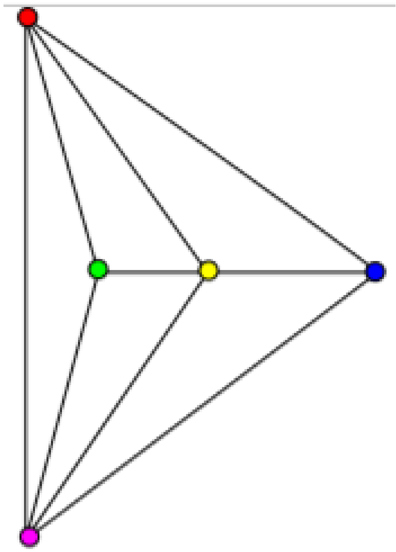
yields a planar map (when no )

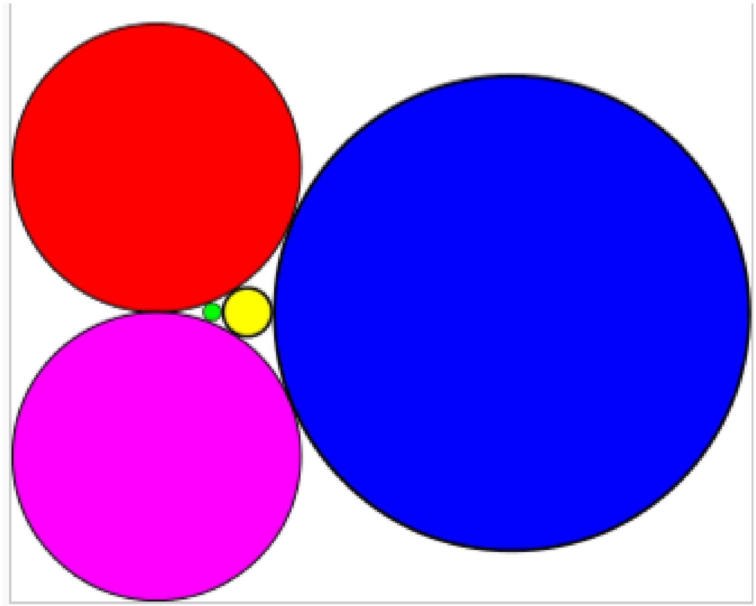
Problem: given a set of allowed shapes, which planar maps can be realized as a contact configuration? Is such a representation unique?

Circle packing

[Koebe'36, Andreev'70, Thurston'85]: every planar triangulation admits a contact representation by disks

The representation is unique if the 3 outer disks have prescribed radius

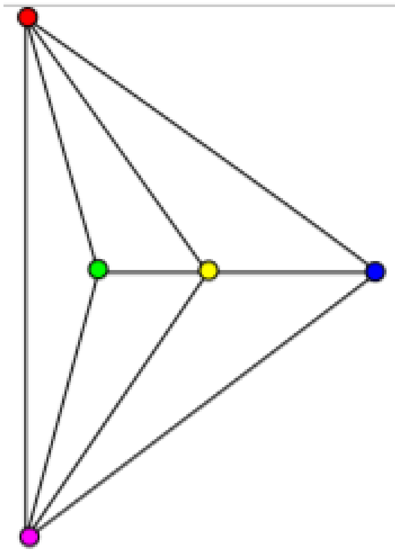


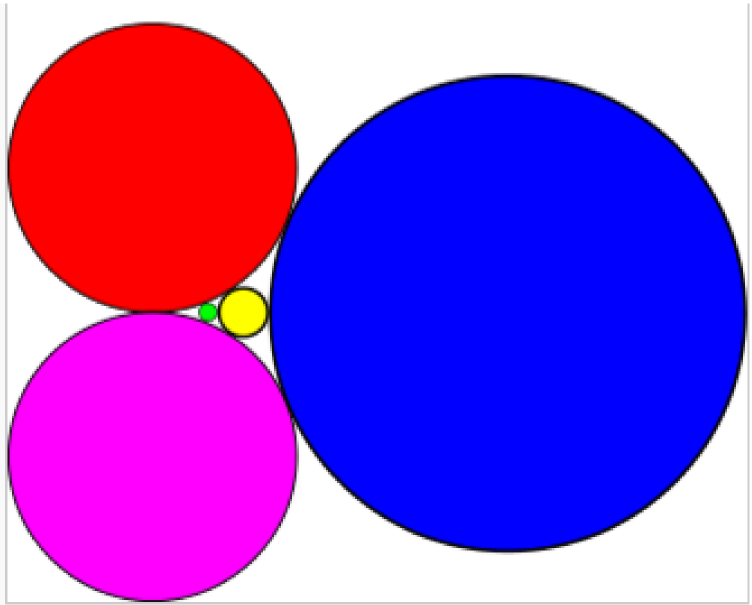


Circle packing

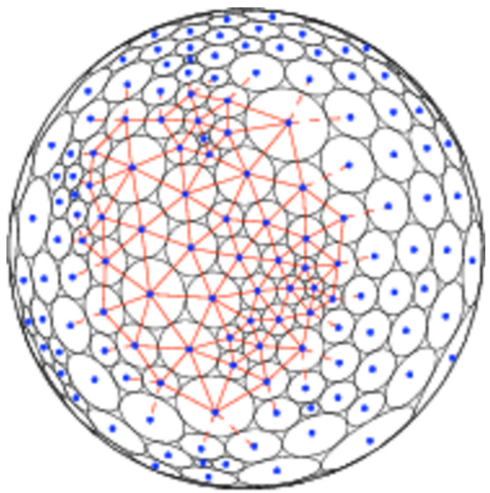
[Koebe'36, Andreev'70, Thurston'85]: every planar triangulation admits a contact representation by disks

The representation is unique if the 3 outer disks have prescribed radius





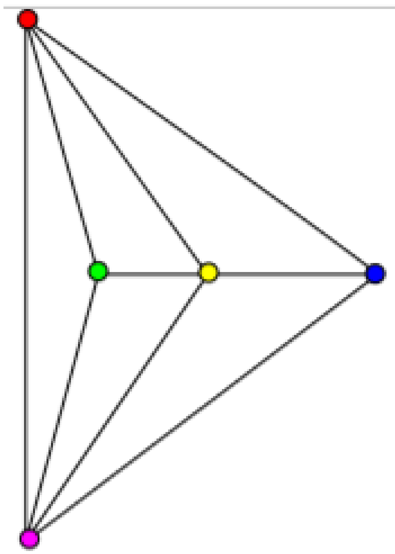
Exercise: the stereographic projection maps circles to circles (considering lines as circle of radius $+\infty$).

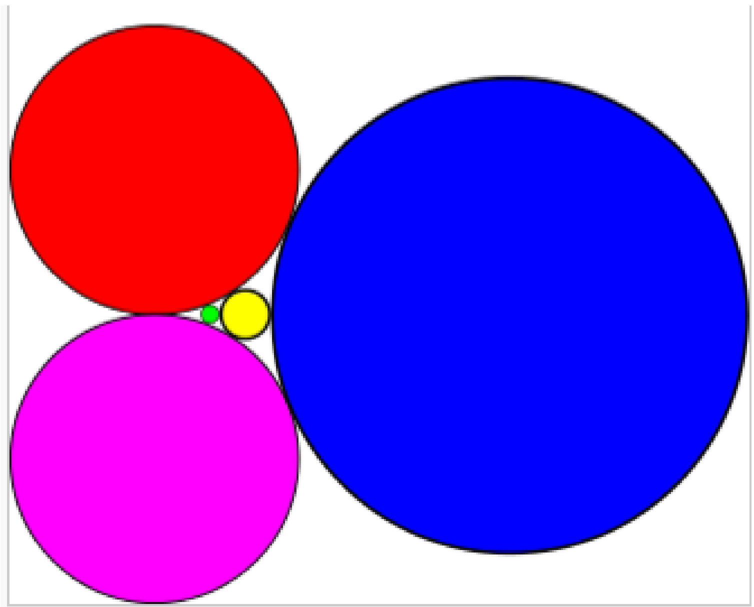


Circle packing

[Koebe'36, Andreev'70, Thurston'85]: every planar triangulation admits a contact representation by disks

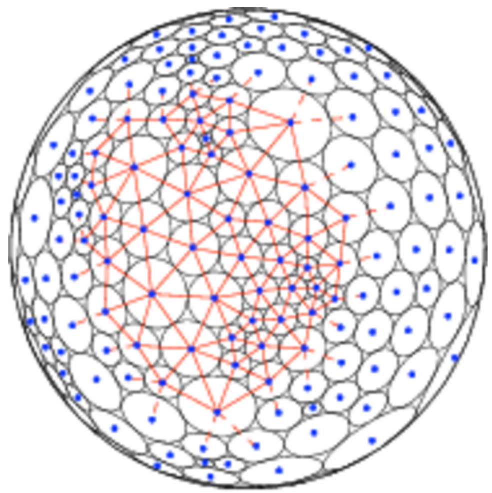
The representation is unique if the 3 outer disks have prescribed radius





Exercise: the stereographic projection maps circles to circles (considering lines as circle of radius $+\infty$).

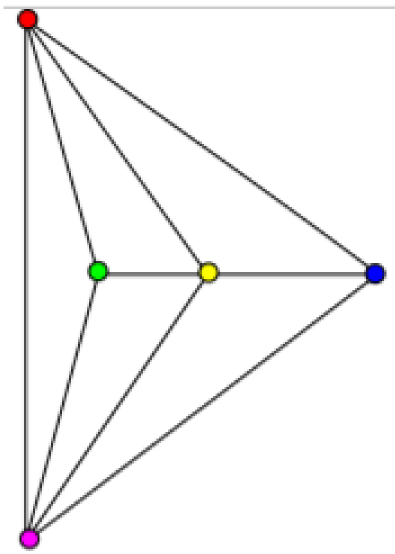
Hence one can lift to a circle packing on the sphere

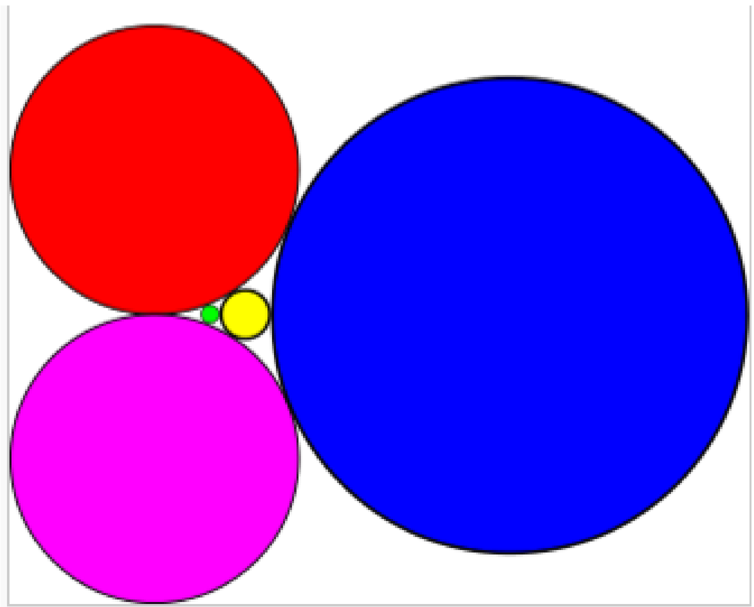


Circle packing

[Koebe'36, Andreev'70, Thurston'85]: every planar triangulation admits a contact representation by disks

The representation is unique if the 3 outer disks have prescribed radius

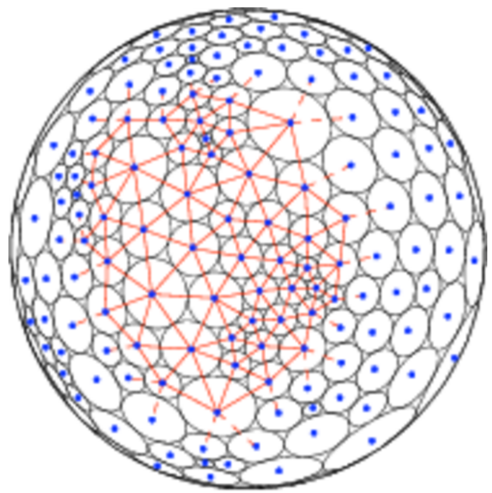




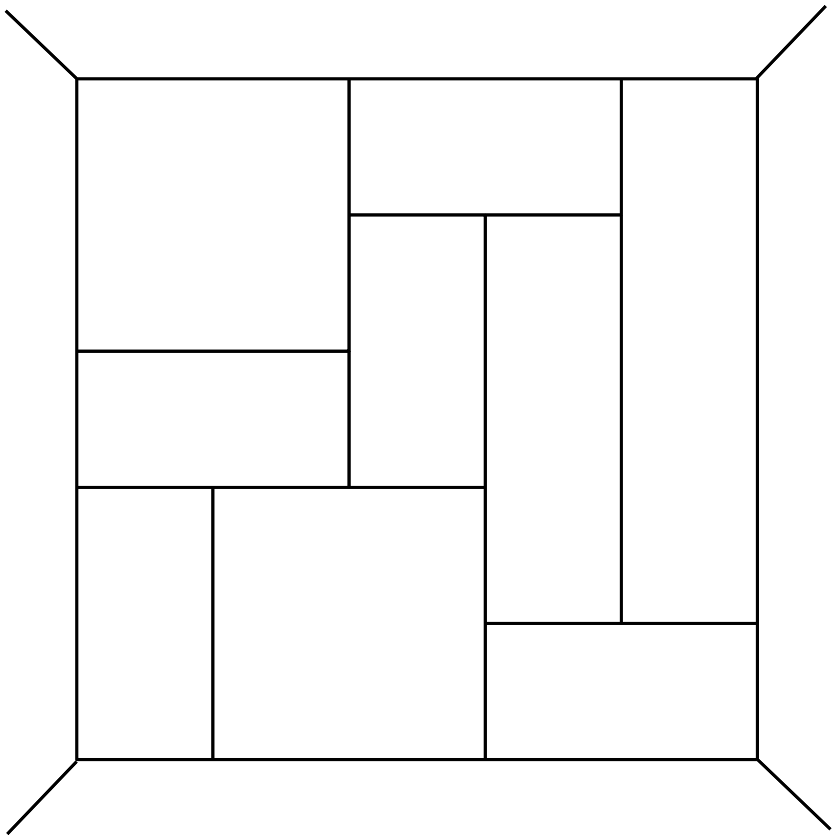
Exercise: the stereographic projection maps circles to circles (considering lines as circle of radius $+\infty$).

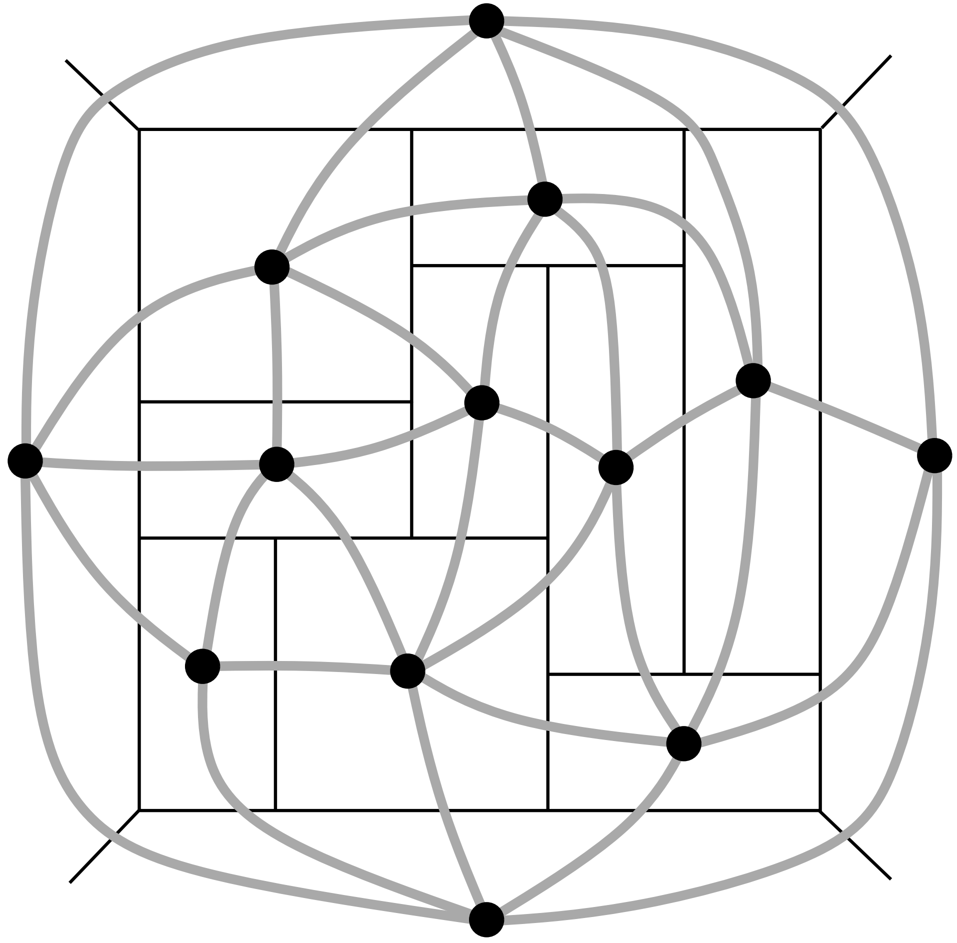
Hence one can lift to a circle packing on the sphere

There is a unique representation where the centre of the sphere is the barycenter of the contact points

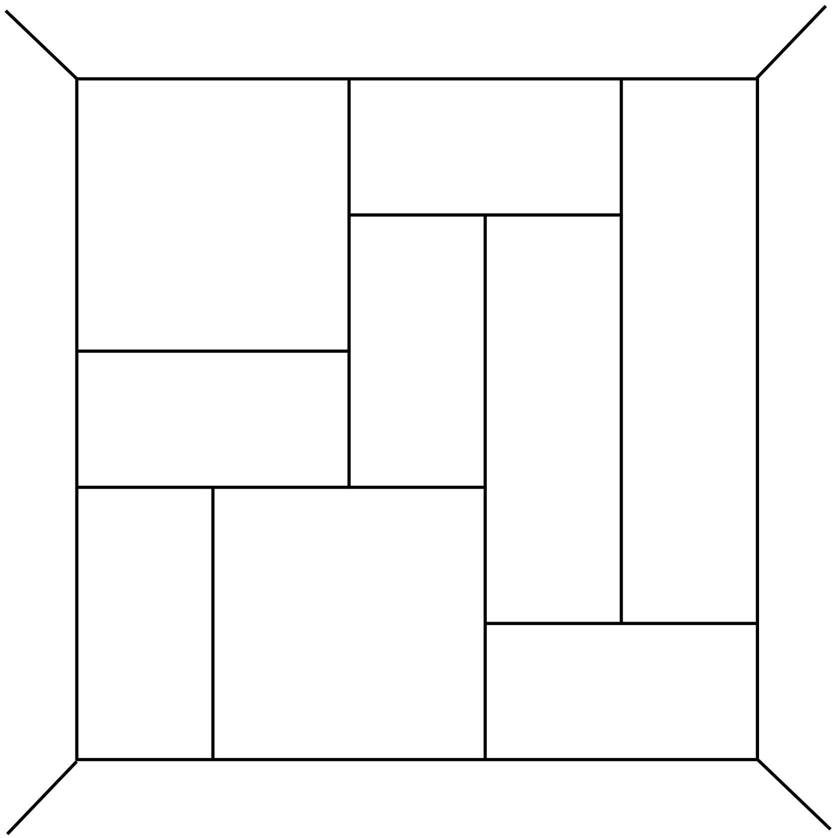
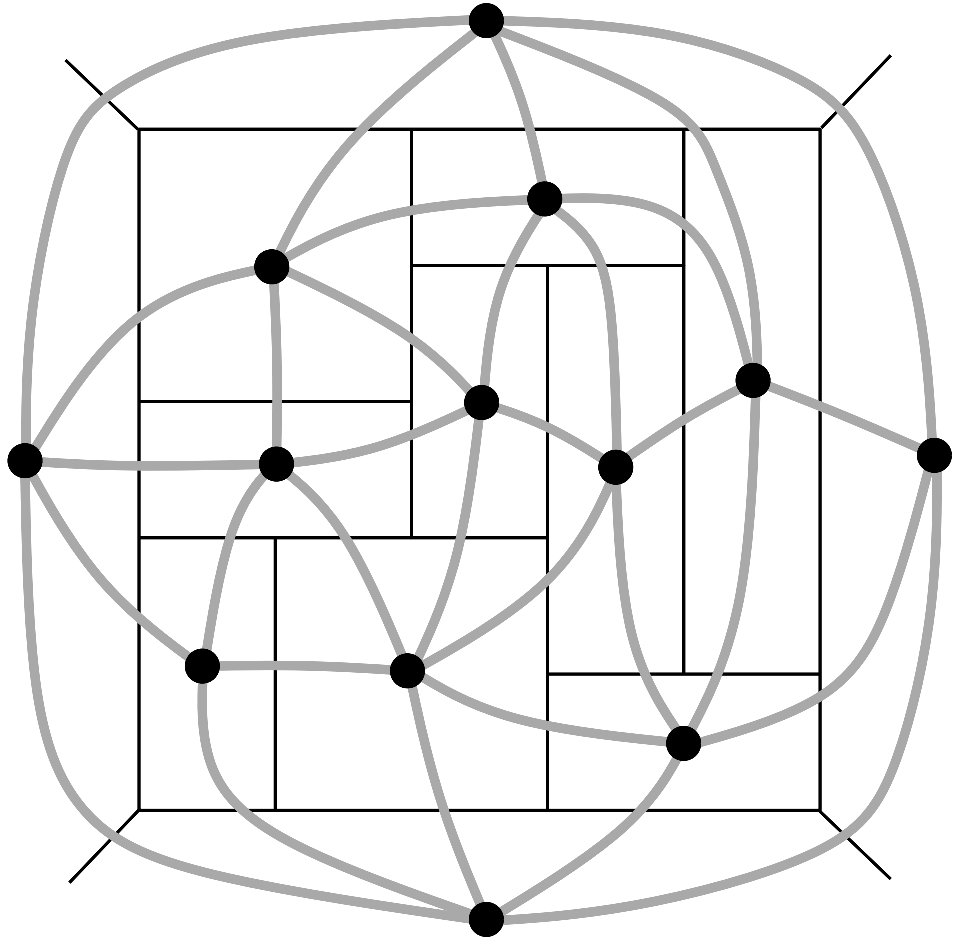


Axis-aligned rectangles in a box



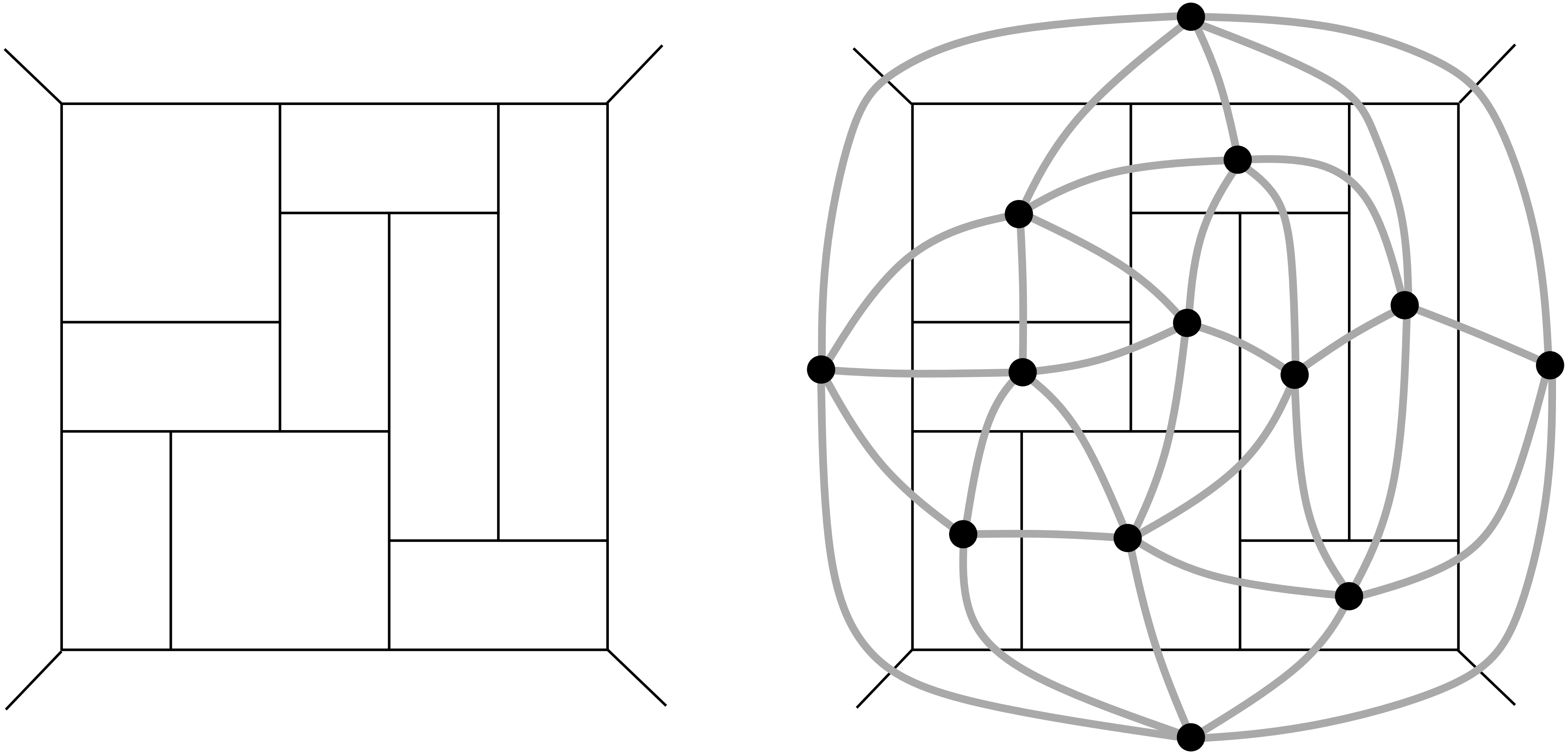


Axis-aligned rectangles in a box

- The rectangles form a tiling. The contact-map is the dual map
- This map is a triangulation of the 4-gon, where every 3-cycle is facial

Axis-aligned rectangles in a box

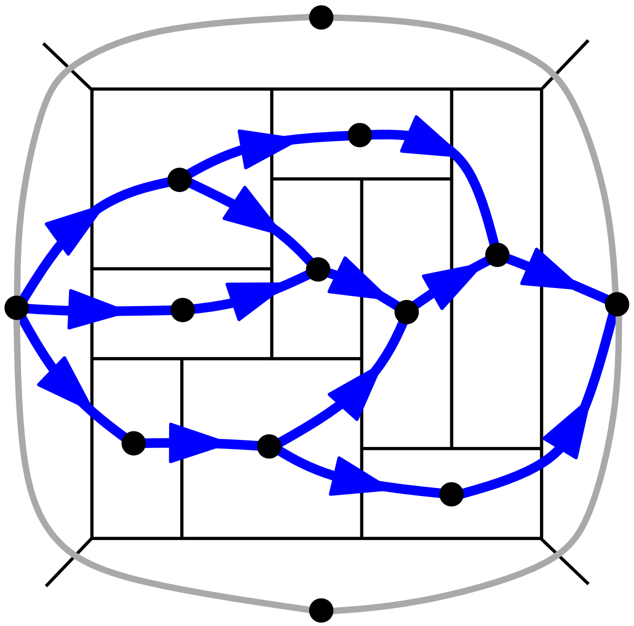


- The rectangles form a tiling. The contact-map is the dual map
- This map is a triangulation of the 4-gon, where every 3-cycle is facial

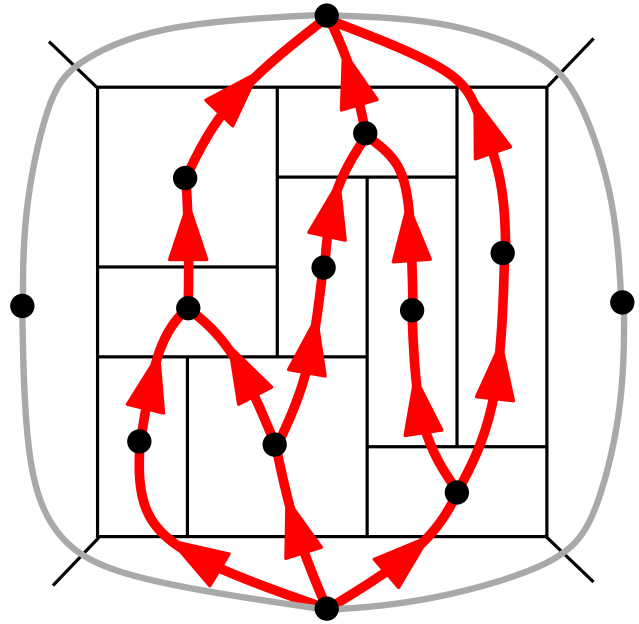
Is it possible to obtain a representation for any such triangulation?

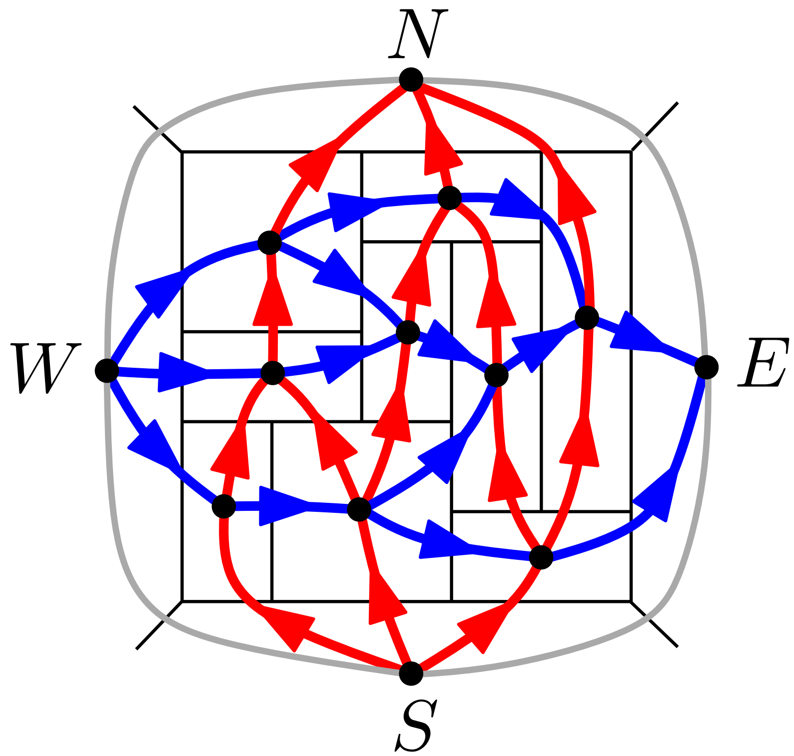
Two partial dual Hasse diagrams

dual for vertical edges



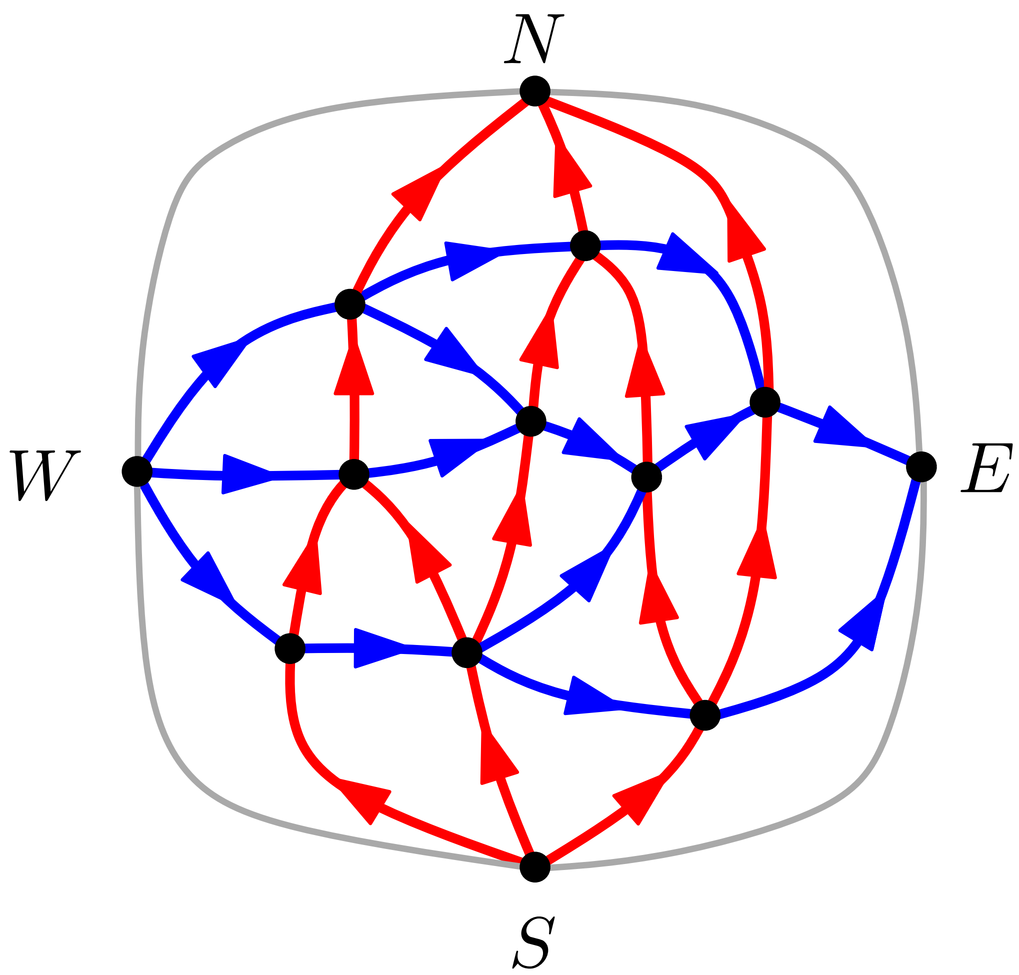
dual for horizontal edges



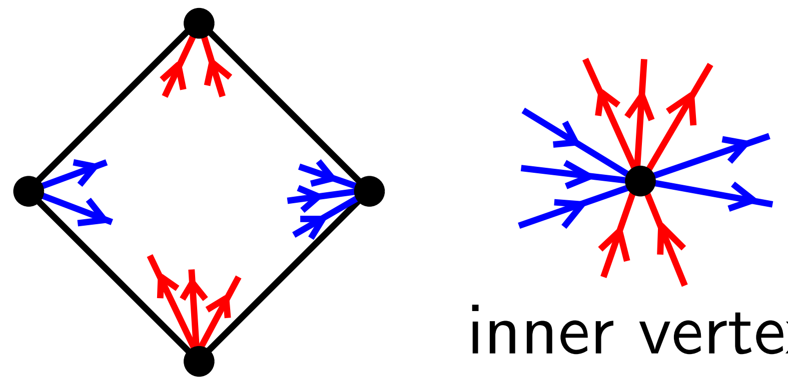


Transversal structures

For T a triangulation of the 4-gon, a transversal structure is a partition of the inner edges into 2 transversal Hasse diagrams



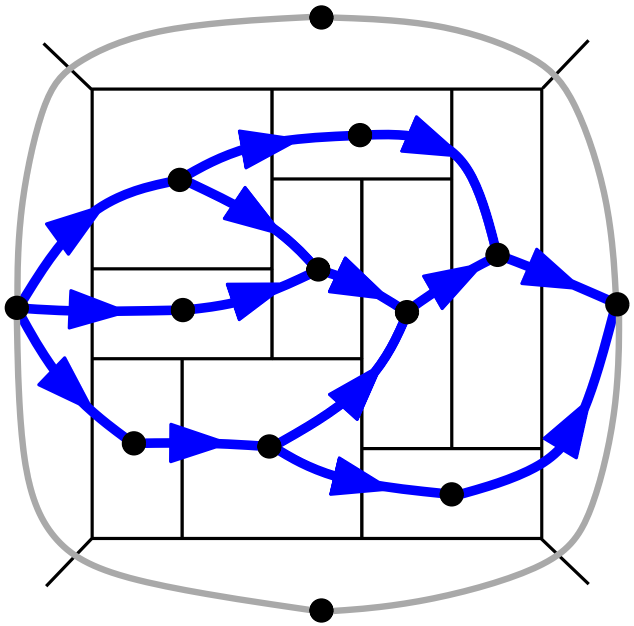
characterized by local conditions:



T admits a transversal structure iff every 3-cycle is facial

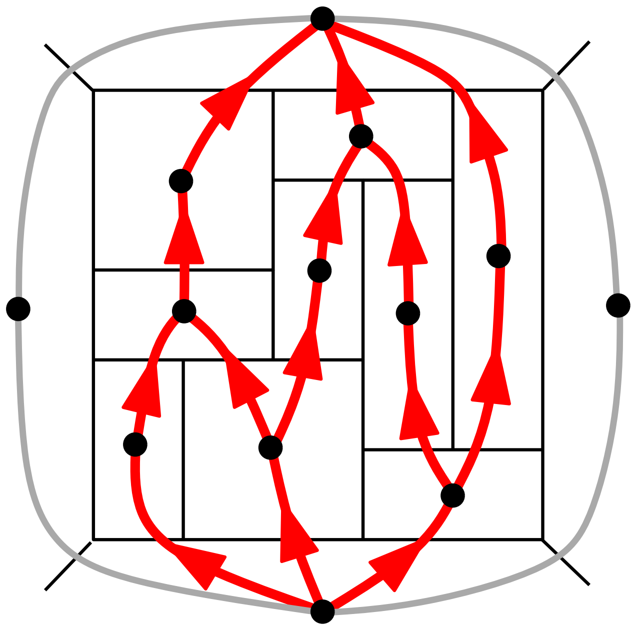
Face-labelling of the two Hasse diagrams

dual for vertical edges



a horizontal segment in each face

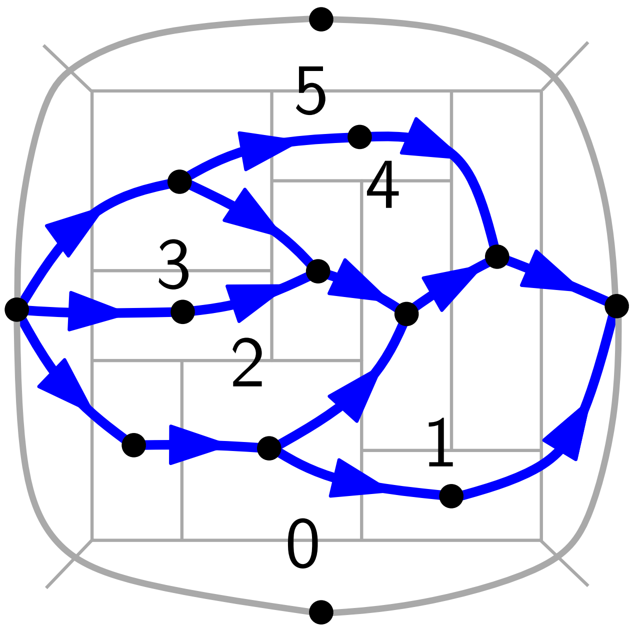
dual for horizontal edges



a vertical segment in each face

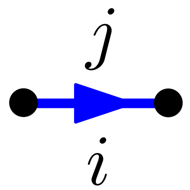
Face-labelling of the two Hasse diagrams

dual for vertical edges



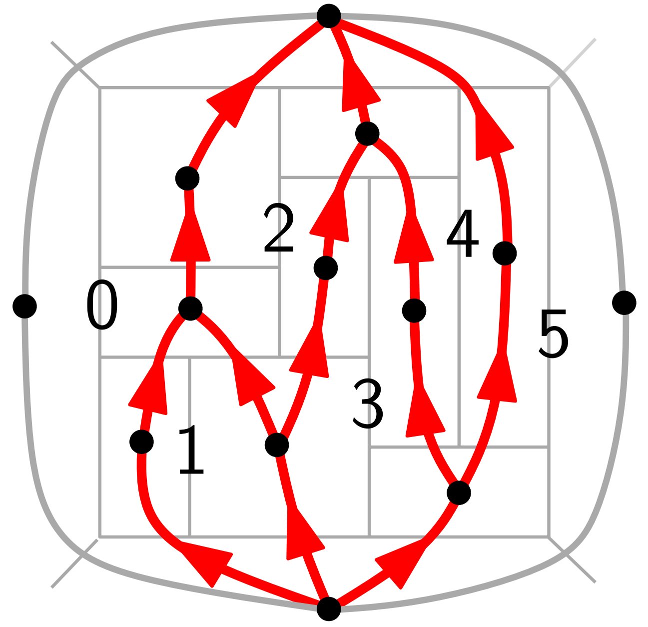
a horizontal segment in each face

label the face by the y-coordinate of segment



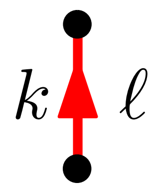
$j > i$

dual for horizontal edges



a vertical segment in each face

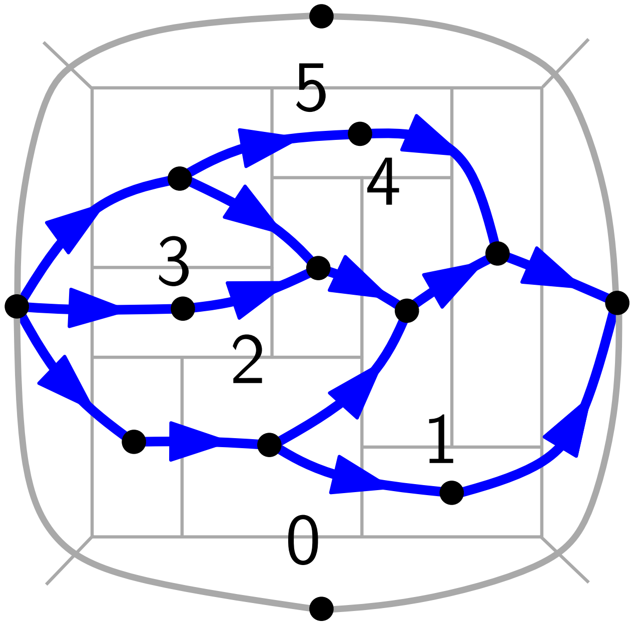
label the face by the x-coordinate of segment



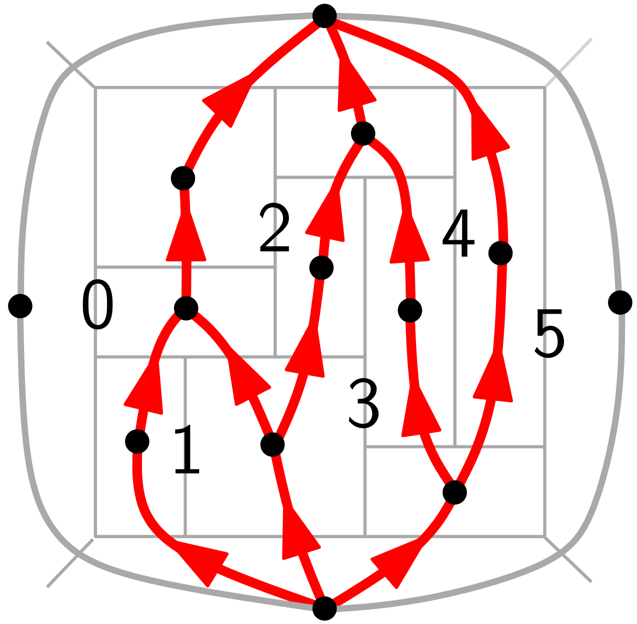
$\ell > k$

Face-labelling of the two Hasse diagrams

dual for vertical edges

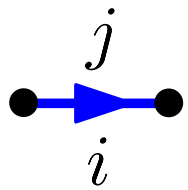


dual for horizontal edges



a horizontal segment in each face

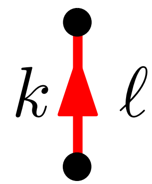
label the face by the y-coordinate of segment



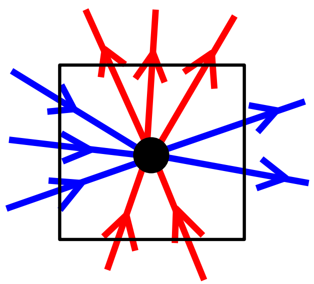
$j > i$

a vertical segment in each face

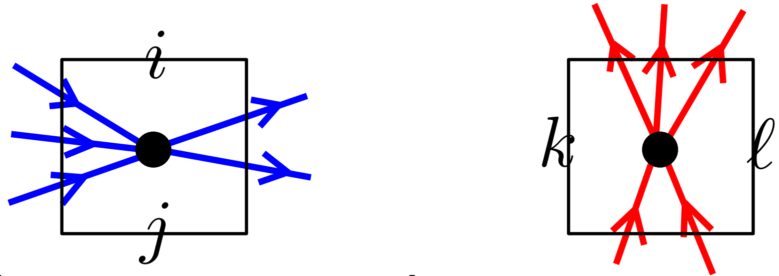
label the face by the x-coordinate of segment



$\ell > k$



vertex $v \leftrightarrow$ rectangle $R(v)$

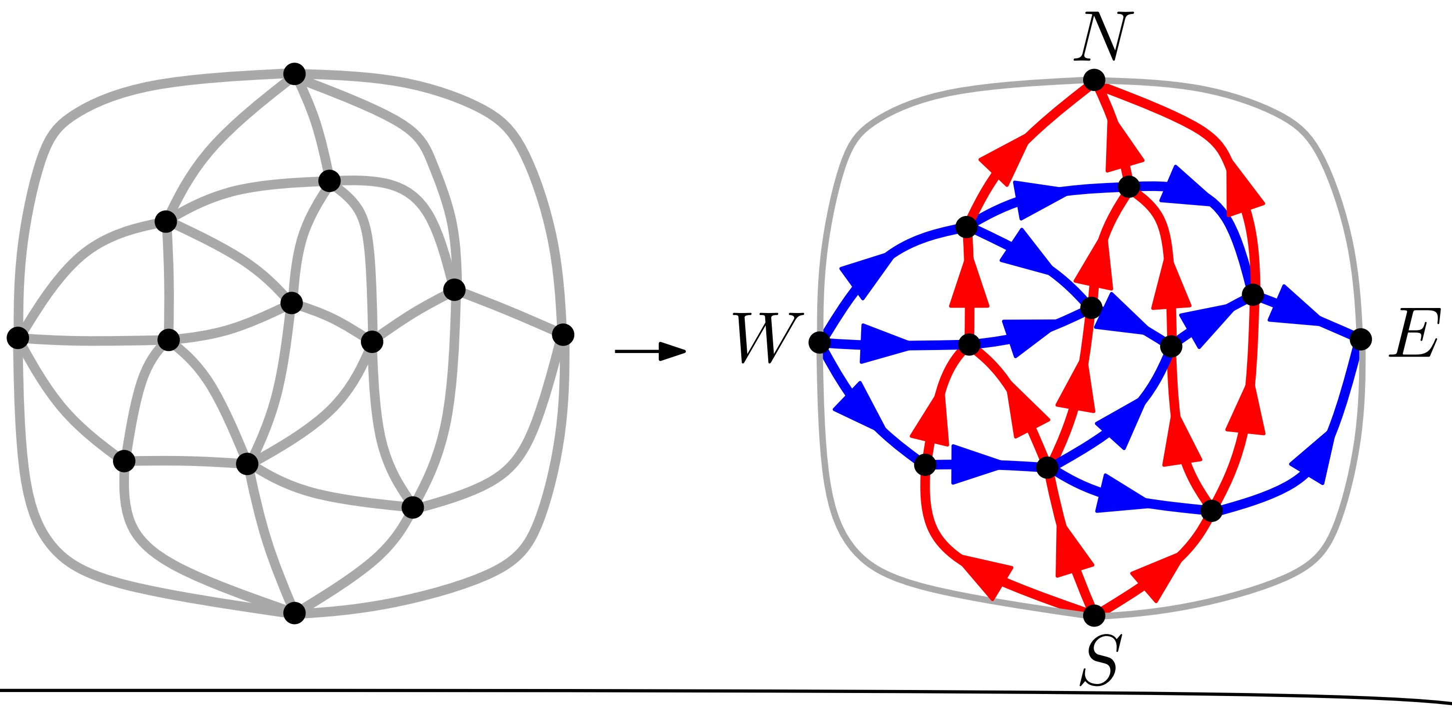


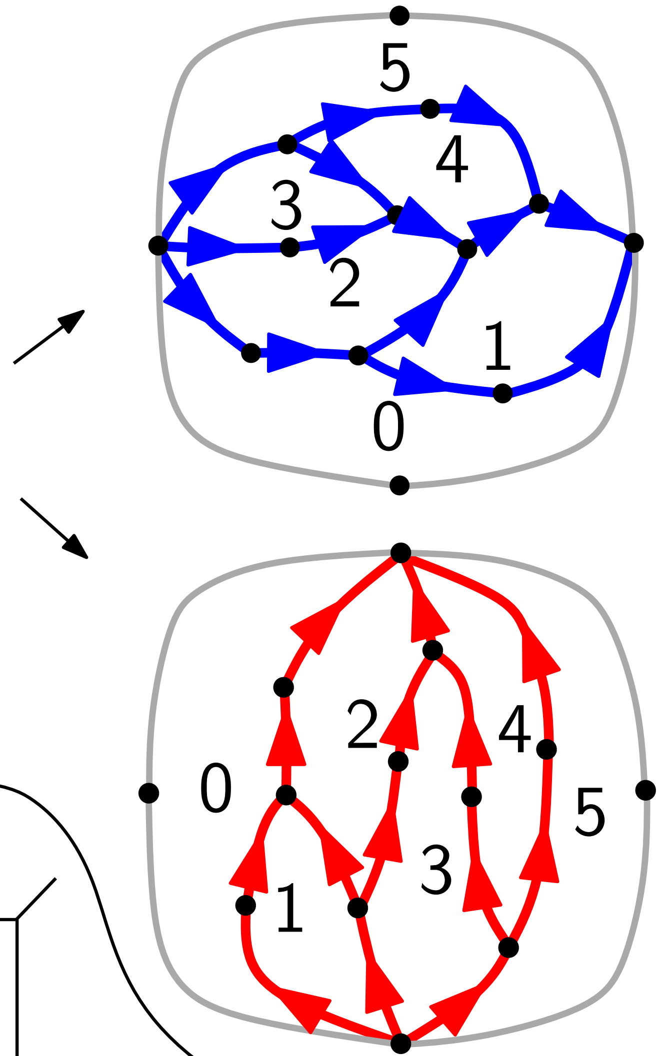
bounding x, y-coordinates given by labels

Algorithm by reverse-engineering

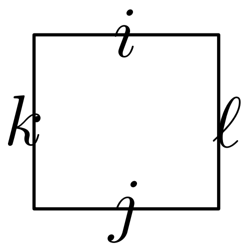
[Kant, He'92]

For T a triangulation of the 4-gon without separating 3-cycle

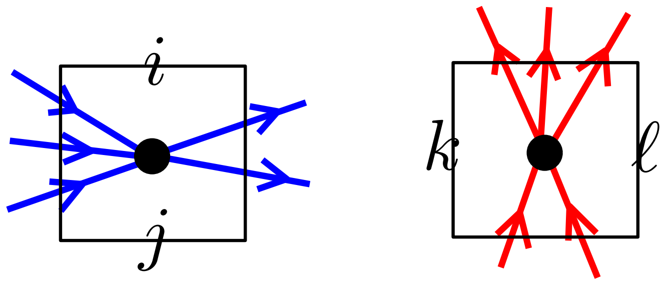


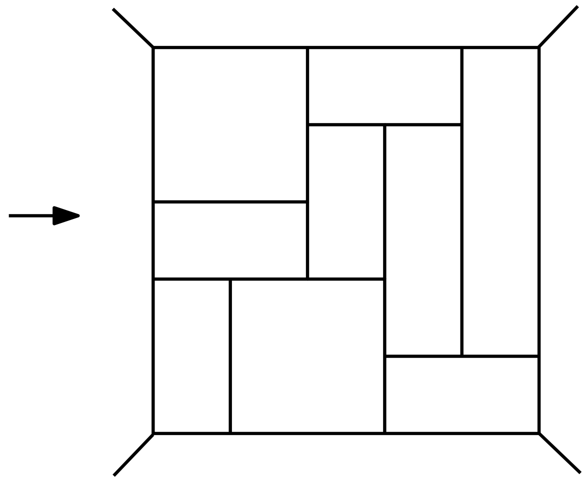


each vertex → box



where





Square tilings

[Schramm'93]

There is a unique tiling where every box is a square

(needs no separating 4-cycle to be sure there is no degeneracy)