# Séptima Escuela de Fsica Matemática. Topological quantum matter: from theory to applications. Excercises

## SESSION 3

### Problem 1 - Majorana gymnastics

(a) From a single complex fermion mode $c, c^\dagger$, obtain the "real" (a.k.a. hermitian) majorana fermions modes from: $\gamma_r = c + c^\dagger$, $\gamma_i = \frac{c - c^\dagger}{i}$. What are the commutation relations of these majorana modes? What is their square? Convince yourself that the only non-trivial way to couple them, preserving complex fermion parity, is with a hamiltonian $h = it\gamma_r\gamma_i$.

(b) Consider the most general quadratic Hamiltonian of majorana modes: $H = \frac{i}{4}\sum_{l,m} A_{l,m}\gamma_l\gamma_m$, where $\gamma_m$ are majoranas. What properties are forced upon the matrix $A$ by hermiticity and the majorana commutation relations?

### Problem 2 - Kitaev's wire with Majoranas

Consider our 1D p-wave superconductor again, but this time with open boundary conditions:

$$H = \sum_{j=1}^{N-1} -t(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu c_j^\dagger c_j + |\Delta| c_j c_{j+1} + |\Delta| c_{j+1}^\dagger c_j^\dagger \tag{1}$$

(a) View the majorana modes as two sites at every original fermion site. Transform from the complex fermion to majorana representation $\gamma_{2j-1} = c_j + c_j^\dagger$, $\gamma_{2j} = \frac{c_j - c_j^\dagger}{i}$, and re-express the Hamiltonian above in the Majorana basis.

(b) Consider the following parameters in the trivial phase $|\Delta| = t = 0$, $\mu < 0$. How do the majorana sites pair? are there any un-paired majorana sites?

(c) Consider the following special parameters in the topological phase $|\Delta| = t > 0$, $\mu = 0$. Demonstrate that the Hamiltonian is:

$$H = it\sum_{j=1}^{N-1} \gamma_{2j}\gamma_{2j+1} \tag{2}$$

How do the majorana sites pair? are there any un-paired majorana sites?

(d) Show that there are two degenerate ground states that differ by parity.

### Problem 3 - Majoranas with time reversal symmetry

Consider $n$ Majorana chains running in parallel that couple. Neglect the bulk of the system and focus only on the majorana modes $\gamma_i$, $i = 1, .., n$ on a single side of the chain, let's say the left side. For simplicity assume the majorana modes on the left to transform under time reversal as the real part of the elementary complex fermions that make up the chain, namely $T\gamma_i T^{-1} = \gamma_i$.

(a) Consider the case $n = 2$. What term can you write to couple the Majorana modes? Is this term allowed by time reversal? Show that time reversal acts as particle-hole conjugation (hint: since time reversal is antiunitary it cannot change the norm of states).

(b) Consider the case $n = 4$. How many states does the Hilbert space of 4 majoranas have? find a Hamiltonian that can couple the four majoranas and respects parity and time-reversal symmetry. Whats the degeneracy of the ground and excited states of this Hamiltonian? Show that the square of time-reversal is $-1$ (this means that it acts the same as in spin 1/2).

(c) Can you think of a way to couple $n = 8$ chains so that the ground state has trivial degeneracy (namely 1)?.

**Problem 3 - Toric Code**

Reference: A. Kitaev, “Fault-tolerant quantum computation by anyons”, arXiv:quant-ph/9707021.

(a) Show that all the vertex and plaquette operators in the Toric code model commute with each other.

(b) Convince yourself that the ground state is given by Eq.(5.10) in Jiannis’ book.

(c) Demonstrate that applying a $\sigma_z$ on the ground states makes the two neighboring vertex operators, $A(v)$ to have an eignstate $-1$. Convince yourself that this can be viewed as two quasiparticles that can be separated to arbitrary distances without an energy cost, and hence there is no ”string-tension” to deconfine the particles. These are the $e$ particles. Do the same reasoning for the states created by $\sigma_x$ (these are pairs of $m$ particles).

(d) Consider the composite particle: $\epsilon = e \times m$. Convince yourself that a $2\pi$ rotation of this particle around itself produces a $-1$ phase factor. What does this mean for the mutual statics of the $e$ and $m$ particles.

(e) Convince yourself, using non-contractible loops, that the toric code has four degenerate ground states on the torus.

**Problem 4 - From Hall Conductivity to Chern number**

Starting from the Kubo formula for the Hall conductivity:

$$\sigma_{xy} = \frac{e^2}{i\hbar^2} \sum_{n,n'} \int \frac{d^2k}{(2\pi)^2} \langle n\vec{k}| \frac{\partial H}{\partial k_y} |n'\vec{k}\rangle \langle n'\vec{k}| \frac{\partial H}{\partial k_x} |n\vec{k}\rangle \frac{f_{nk} - f_{n'k}}{(\epsilon_{n'k} - \epsilon_{nk})^2} \tag{3}$$

Show that the Hall conductivity equals the sum of Berry curvare of occupied bands when the chemical potential lies in a bulk gap, which is the Chern number, $C \in \mathbb{Z}$, times the quantum of conductance:

$$\sigma_{xy} = \frac{e^2}{h} \sum_n f_n \int \frac{d^2k}{2\pi i} \nabla_k \times \langle n\vec{k}|\nabla_k|n\vec{k}\rangle = \frac{e^2}{h} C \tag{4}$$

**References on other topics covered**

Kitaev’s wires with time reversal symmetry: Reference: L. Fidkowski and A. Kitaev, “Topological phases of fermions in one dimension”, PRB 83, 075103 (2011).

2D $p + ip$ superconductor: Reference: N. Read and Dmitry Green, “Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum Hall effect”, PRB (2000).

Kitaev’s 16-fold way: Reference: A. Kitaev, “Anyons in an exactly solved model and beyond”, arXiv:cond-mat/0506438.