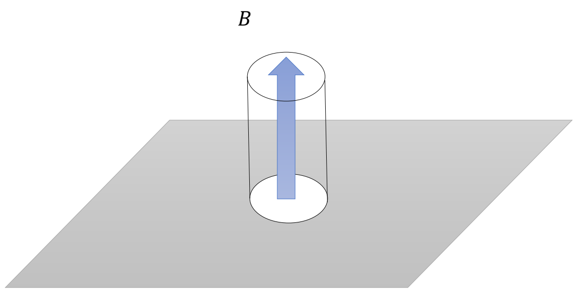
Séptima Escuela de Fsica Matemática. Topological quantum matter: from theory to applications. Excercises

SESSION 0

Problem 0 - Aharonov-Bohm effect

Consider a uniform magnetic field inside a solenoid that pierces a 2d plane as shown in the figure:



(a) Compute the vector potential, A, outside of the solenoid by expressing it as the gradient of scalar function $A = \nabla f$. Is f a continuous function on the plane in general? (Hint: gradient in polar coordinates of a function that depends only on the polar angle θ, $f(\theta)$ is: $\nabla f = \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta}$)
(b) Consider a galilean particle moving in the plane in the region outside the solenoid (the particle only "sees" the vector potential, not the magnetic field). Demonstrate that Eq.(2.5) of Jiannis Pachos book is a solution of Eq.(2.4).
(c) What is the relation between the winding of the phase of the wavefunction along a loop encircling the solenoid and the magnetic flux inside it?. What is the condition on the flux through the solenoid for the wavefunction outside to be single-valued and continuous?

Problem 1 - Jones Polynomials

Work out the details of Section 8.4 "Example I: Kauffman bracket of simple links" from Jiannis Pachos book (page 172).

Problem 2 - Quantum Hall Effect

Reference: "Introduction to the Physics of the Quantum Hall Regime", arXiv:cond-mat/9410047 v1, A. H. MacDonald.
(a) Show the single-particle wavefunctions of electrons in the lowest Landau level in the symmetric Gauge to be given by Eq.(26) in Allan's arxiv notes.
(b) Compute the mean square radius of these wavefunctions, i.e. derive Eq. (29) from these notes in the case of $n = 0$.

Problem 3 - Kitaev's wire

Reference: "Unpaired Majorana fermions in quantum wires", arXiv:cond-mat/0010440, A. Kitaev.
Consider spinless electrons hopping in a 1D chain of N-sites in the presence of a p-wave superconducting pairing term:

$$H = \sum_{j=1}^{N} -t(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu c_j^\dagger c_j + \Delta c_j c_{j+1} + \Delta^* c_{j+1}^\dagger c_j^\dagger \tag{1}$$

where $t > 0$ is the hopping amplitude, $\mu \in \mathbb{R}$ is the chemical potential, and $\Delta \in \mathbb{C}$ is the superconducting order parameter.

(a) Eliminate the phase of Δ, $\Delta = |\Delta| e^{i\phi}$, by the gauge change: $c_j \to e^{i\phi/2} c_j$. Assume periodic boundary conditions, namely $c_{N+1} \equiv c_1$, and rewrite the Hamiltonian in momentum space by using: $c_j = \frac{1}{\sqrt{L}} \sum_k e^{ikj} c_k$. Choose the Brilloiun zone to be $k \in (-\pi, \pi]$. What are the discrete allowed values for k?

(b) Re-write this Hamiltonian as:

$$H = \sum_k \Psi_k^\dagger \begin{pmatrix} 2t\cos k + \mu & 2i|\Delta|\sin k \\ -2i|\Delta|\sin k & -(2t\cos k + \mu) \end{pmatrix} \Psi_k \tag{2}$$

where $\Psi_k = \begin{pmatrix} c_{-k}^\dagger \\ c_k \end{pmatrix}$. Find the energy dispersion of the quasi-particles.

(c) For a fixed $t > 0$ find the regions in the real axis of $\mu \in \mathbb{R}$ for which the spectrum is gapped and the points at which it is gapples. What is the parity of each of these regions for chains of a total even number of sites and for chains with odd number of sites (hint: assume the chemical potential is always in a gap and compute the parity from the problem without superconductivity $\Delta = 0$. Superconductivity destroys particle conservation but not parity.)

(d) For antiperiodic boundary conditions, $c_{N+1} \equiv -c_1$, what is the fermion parity of each of the regions found in (c)? The topological phase is that for which the fermion parity differs for periodic and anti-periodic boundary conditions.