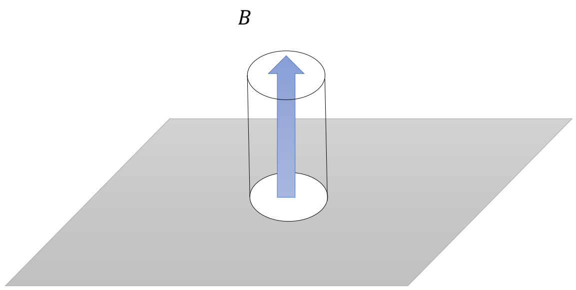
Séptima Escuela de Fsica Matemática. Topological quantum matter: from theory to applications. Excercises

SESSION I

Problem 0 - Aharonov-Bohm effect

Consider a uniform magnetic field inside a solenoid that pierces a 2d plane as shown in the figure:



(a) Compute the vector potential, A, outside of the solenoid by expressing it as the gradient of scalar function $A = \nabla f$. Is f a continuous function on the plane in general? (Hint: gradient in polar coordinates of a function that depends only on the polar angle θ, $f(\theta)$ is: $\nabla f = \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta}$)
(b) Consider a galilean particle moving in the plane in the region outside the solenoid (the particle only "sees" the vector potential, not the magnetic field). Demonstrate that Eq.(2.5) of Jiannis Pachos book is a solution of Eq.(2.4).
(c) What is the relation between the winding of the phase of the wavefunction along a loop encircling the solenoid and the magnetic flux inside it?. What is the condition on the flux through the solenoid for the wavefunction outside to be single-valued and continuous?

Problem 1 - Berry phases

Consider the Hamiltonian:

$$H(h) = \begin{pmatrix} 0 & h_x - ih_y \\ h_x + ih_y & 0 \end{pmatrix} = |h| \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}, \tag{1}$$

we will view the Hamiltonian as a function of the parameter $h \in \mathbb{C}$.
(a) Compute the orthonormalized eigenstantes $|v_s(h)\rangle$, where $s = \pm 1$ and the eigenvalues are $s|h|$, in a gauge such that the upper component of each eigenvector is real.
(b) Compute the Berry connections, $A_{|h|}$ and A_ϕ, from the following definition:

$$\langle v_s(h)|v_s(h + \Delta h)\rangle = 1 + i(A_\phi \Delta\phi + A_{|h|}\Delta|h|) + \mathcal{O}(\Delta h)^2 \tag{2}$$

(c) Consider a closed loop γ traced by the variable h in $\mathbb{C}$ that never crosses the origin. Demonstrate that:

$$\oint_\gamma A_\phi d\phi + A_{|h|}d|h| = \pi w(\gamma), \tag{3}$$

where $w(\gamma) \in \mathbb{Z}$ counts how many times the loop winds around the origin.

(d) (optional) Try to find a different gauge choice for the eigenstates in which the Berry connection vanishes essentially everywhere in the complex plane. What's going on? (Notice that in such gauge the eigenvectors are continous functions of $h \in \mathbb{C}$ almost everywhere except along a ray where there is a branch cut).

Problem 2 - Hall conductivity of massive Dirac fermions

Consider the Hamiltonian describing two-dimensional massive Dirac fermions:

$$H = \begin{pmatrix} m & p_x - ip_y \\ p_x + ip_y & -m \end{pmatrix} \tag{4}$$

(a) Compute the energy dispersion $\epsilon_s(p)$ and the eigenstates $|u_p^s\rangle$, where p is momentum and $s = \pm 1$ denotes the conduction/valence band (positive/negative energy).

(b) compute the Berry connection $A_p^s = -i\langle u_p^s|\nabla_p|u_p^s\rangle$, and the Berrry curvature $\Omega^s(p) = \nabla_p \times A_p^s$, where $|u_p^s\rangle$ is the eigenstate of momentum p and $s = \pm 1$ denotes the conduction/valence band (positive/negative energy).

(c) Compute the Hall conductivity, which is the average of the Berry curvature over occupied states, as a function of chemical potential $\mu \in \mathbb{R}$, as follows:

$$\sigma_{xy}(\mu) = \sum_s \frac{e^2}{\hbar} \int \frac{d^2p}{(2\pi)^2}\, n_F^s(p)\, \Omega^s(p) \tag{5}$$

where $n_F^s(p) = \theta(\mu - \epsilon_s(p))$. $\int_0^y dx \frac{x}{(1+x^2)^{3/2}} = 1 - \frac{1}{\sqrt{1+y^2}}$.

(d) Show that the conductivity is a constant $\sigma_{xy}(\mu) = \mathrm{sign(m)}\frac{e^2}{2h}$ when the chemical potential is in the gap: $|\mu| < |m|$. Why is it half-integer-quantized instead of integer-quantized?