Classification of Modular Categories

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Séptima escuela de física matemática
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Why Braided Fusion Categories?

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- In **topological quantum computation**, anyons give rise to quantum computational models.

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Short answer (Physics): Anyons

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- $\mathcal{C}$ is non-degenerated: $\det(S_{X,Y}) \neq 0$, where $S_{X,Y} = \mathrm{Tr}_{\mathcal{C}}(\sigma_{X,Y^*} \sigma_{Y^*,X})$.

Modular categories

Summarizing:

Definition

A ***modular category*** (MC) is a non-degenerate braided fusion category over $\mathbb{C}$, with a ribbon structure.

A dictionary of terminologies between anyon theory and UMC theory

Modular categories	***Anyonic system***
simple object	anyon
label	anyon type or anyonic charge
tensor product $a \otimes b$	fusion
fusion rules $a \times b$	fusion rules
triangular space $V_{ab}^{c} := \mathrm{Hom}(a \otimes b, c)$	fusion/splitting space $\vert axb \to c\rangle$
dual	antiparticle
coevaluation /evaluation	creation/annihilation
mapping class group representations	generalized anyon statistics
nonzero vector in $V(Y)$	ground state vector
unitary F-matrices	recoupling rules
twist $\theta_x = e^{2\pi i s_x}$	topological spin
morphism	physical process or operator
colored braided framed trivalent graphs	anyon trajectories
quantum invariants	topological amplitudes

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The category of tilting modules of the quantum groups $U_q(\mathfrak{g})$ (q^2 a lth root of unity) module negligible morphisms. For example:

- $SU(N)_k = \mathcal{C}(\mathfrak{sl}_N, N+k)$,
- $SO(N)_k$,
- $PSU(N)_k \subset SU(N)_k$, for $\gcd(k, N) = 1$.

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Fusion rules

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$$N_{\mathbf{1}a}^b = \delta_{ab} = N_{a\mathbf{1}}^b$$

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$$N_{abc}^u := \sum_e N_{ab}^e N_{ec}^u = \sum_{e'} N_{ae'}^u N_{bc}^{e'}$$

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F-matrices (6j-symbols)

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$$F^d_{abc} : \mathrm{Hom}_{\mathcal{C}}(a \otimes b \otimes c, d) \to \mathrm{Hom}_{\mathcal{C}}(a \otimes b \otimes c, d)$$
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The set of matrices

$$\{F^d_{abc} \in U(N^d_{abc}) | a, b, c, d \in L\}$$

is called the F-matrices and they satisfy the **pentagonal identity (pentagon axiom).**

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- $F^d_{a,b,c} = \omega(a, b, c)\delta_{abc,d}$, so is a function $\omega : G^{\times 3} \to U(1)$
- Pentagon equation is exactly 3-cocycle condition of group cohomology:

$$\omega(a, b, c)\omega(b, c, d)\omega(a, bc, d) = \omega(ab, c, d)\omega(a, b, cd)$$

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As an example the fusion rules:

- $L_k = \{\mathbf{1}, \mathbf{x}\}$
- $x^2 = 1 + kx$ $(N_{xx}^1 = N_{xx}^x = k)$, $k \in \mathbb{Z}^{>0}$

define a fusion category if and only if $k = 1$ (Victor Ostrik).

Examples

Ising theory

- $L = \{\mathbf{1}, \sigma, \psi\}$
- fusion rules: $\sigma^2 = 1 + \psi, \psi^2 = 1, \psi\sigma = \sigma\psi = \sigma$.
- $F^{\sigma}_{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $F^{\sigma}_{\psi\sigma\psi} = F^{\psi}_{\sigma\psi\sigma} = -1$.

Remarks

- The ising fusion rules has two possible realization (Isinig or Mayorama fermion) $F^{\sigma}_{\sigma\sigma\sigma} = \frac{-1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.
- Ising categories are particular cases of a more general famililly called Tambara-Yamagami categories.

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- If $\mathcal{C}(G, \omega)$ has a braid structure then G is abelian
- $R^z_{xy} = c(x, y)\delta_{xy,z}$, so is a function $c : G \times G \to U(1)$
- Hexagonal equation is exacly the abelian 3-cocycle condition

$$\omega(y, z, x)c(x, yz)\omega(x, y, z) = c(x, z)\omega(y, x, z)c(x, y)$$
$$\omega(z, x, y)^{-1}c(xy, z)\omega(x, y, z)^{-1} = c(x, z)\omega(x, z, y)^{-1}c(y, z).$$

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R-matrices for Ising theory

$$R^1_{\psi\psi} = -1, R^\sigma_{\sigma\psi} = i, R^1_{\sigma\sigma} = e^{-\pi i/8}, R^\psi_{\sigma\sigma} = e^{3\pi i/8}$$

The Ising category admist tree (non-equivalent) R-matrices.

More examples: the Drinfeld center

Let $\mathcal{C}$ be a (strict) tensor category and let $X \in \mathcal{C}$.

Definition

A *half braiding* $c_{-,X} : \_ \otimes X \to X \otimes \_$ for X is a natural isomorphism such that $c_{Y\otimes Z,X} = (c_{Y,X} \otimes \mathrm{id}_Z)(\mathrm{id}_Y \otimes c_{Z,X})$, for all $Y, Z \in \mathcal{C}$.

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Theorem (Muger)

The Drinfeld center $\mathcal{Z}(\mathcal{C})$ is ***modular*** *if $\mathcal{C}$ is a spherical fusion category over $\mathbb{C}$.*

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Frobenius-Perron dimensions

Set $\mathcal{C}$ be a fusion category.

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Rank finiteness for braided fusion categories

Theorem

*(Bruillard, Ng, Rowell, Wang) 2013 There are **finitely** many modular categories of a given rank r.*

Theorem (Bruillard, G., Ng, Plavnik, Rowell, Wang) 2015

There are **finitely** many braided fusion categories of a given rank r.

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- $PSU(2)_9$, $SU(2)_4$, $SU(5)_1$, $PSU(3)_4$.

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Let $\mathcal{C}$ be a modular category with $\text{FPdim}(\mathcal{C}) = 4m$, where m is an odd square-free integer. Then $\mathcal{C}$ is equivalent to a (Deligne) product of the following: pointed categories, Ising categories and metaplectic categories.

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Recall that:

- A MC is pointed if all its simple objects are invertible. A **cyclic** $\mathcal{P}_n$ of rank n is a pointed MC with the same fusion rules as $\text{Rep}(\mathbb{Z}_n)$.

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- An Ising MC $\mathcal{I}$ is a Tambara-Yamagami category with $G \simeq \mathbb{Z}_2$.
- A metaplectic m.c. $\mathcal{M}_N$ is any MC with the same fusion rules as the MC $SO(N)_2$, for N odd. The rank of $\mathcal{M}_N$ is $\frac{N+7}{2}$, the dimension is $4N$ and it has two 1-dimensional objects and two simple objects of dimension $\sqrt{N}$, while the remaining simple objects have dimension 2. For example, $\mathcal{TY}(\mathbb{Z}_N, \chi, \nu)^{\mathbb{Z}_2}$, for N odd, is a metaplectic MC.

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- *$\mathcal{C}$ is non-integral with no objects of dimension $\sqrt{2}$ and $\mathcal{C} \cong \mathcal{M}_k \boxtimes \mathcal{P}_{m/k}$, with $\mathcal{M}_k \cong \mathcal{TY}(\mathbb{Z}_k, \chi, \nu)^{\mathbb{Z}_2}$, or*

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- $\mathcal{P}_6$, *a cyclic MC of rank* 6.

Application: Rank 7 case

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The only strictly weakly integral rank 7 modular categories are metaplectic categories.

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If $\mathcal{C}$ is an integral modular category of rank 7, then $\mathcal{C}$ is pointed.

Main theorem: rank 8

Theorem (Bruillard, G., Hughes, Plavnik, Rowell, Sun)

There are no rank 8 strictly weakly integral modular categories.