Edge States: Topological Insulators, Superconductors

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Work with Manuel Asorey and Juan Manuel Perez Pardo

For earlier work, see especially:

Bal et al., IJMPA A9 3341 (1994)

T. R. Govindarajan and R. Tibrewala, Phys. Rev. D 83 124045 (2011)

"The steady progress of physics requires for its theoretical formulation a mathematics which get continually more advanced. This is only natural and to be expected. (...) the advance in physics is to be associated with continual modification and generalisation of the axioms at the base of mathematics rather than with a logical development of any one mathematical scheme on a fixed foundation."

P. A. M. Dirac in the paper on magnetic monopoles (1931)

"...under the demoralizing influence of quantum field theoretic perturbation theory (infested with divergences), the mathematics required for a theoretical physicist was reduced to a rudimentary knowledge of the Latin and Greek alphabet."

Res Jost

Consider a manifold M with a boundary ∂M.

Nowadays there are discussions about quantum systems on such M with

- low-lying excitations localised near ∂M,
- bulk levels which are gapped.

Often these edge levels are called "topological".

Of these quantum Hall edge states violate P and T.

We discuss a new class of edge states conserving P and T.

They occur at the interface of say superconductor and normal metal.

They occur for integral and half-integral spin.

They occur in all dimensions.

Provide models for topological insulators?

Emergent Edge States in QHE

Lagrangian

$$L = \frac{k}{4\pi} \int_M d^2x \; \epsilon^{ijk} A_i \partial_j A_k \tag{1}$$

No bulk excitations since

$$F_{ij} = \partial_i A_j - \partial_j A_i = 0 \quad \text{in} \quad M. \tag{2}$$

This is Gauss law

$$\partial_1 A_2 - \partial_2 A_1 = 0 \quad \text{in} \quad M. \tag{3}$$

Write this as

$$\frac{k}{2\pi} \int_M d^2x \epsilon^{ij} \left(\partial_i \Lambda\right) A_j \approx 0 \quad \textit{for} \quad \Lambda|_{\partial M} = 0. \tag{4}$$

Now can consider

$$Q[\chi] = \frac{k}{2\pi} \int_M d^2 x \epsilon_{ij} \, \partial_i \chi \, A_j, \quad \chi|_{\partial M} \neq 0. \tag{5}$$

Also

$$[\, A_i(x) \,,\, A_j(y) \,]_{x^0=y^0} \;=\; i\frac{2\pi}{k}\epsilon_{ij}\delta^2(x,y), \tag{6}$$

$$\Rightarrow [\, Q[\chi_1] \,,\, Q[\chi_2] \,] \;=\; i\frac{k}{2\pi}\int_M d^x \, \epsilon_{ij} \, \partial_i \chi_1 \, \partial_j \chi_2$$

$$= \; i\int_{\partial M} d\theta \, \left(\chi_1 \partial \chi_2\right)(R,\theta). \tag{7}$$

$Q[\chi]$ describes edge states.

This is the algebra of Massless Scalar Fields on ∂M:

$$[\ \varphi(\theta)\ ,\ \dot{\varphi}(\theta')\ ]_{t=t'}\ =\ i\delta(\theta-\theta') \tag{8}$$

$$\Longrightarrow \qquad [\ \partial_\theta\varphi(\theta)\ ,\ \dot{\varphi}(\theta')\ ]\ =\ i\partial_\theta\delta(\theta-\theta'). \tag{9}$$

Let

$$\varphi_+ = \frac{1}{\sqrt{2}}\left(\partial_t + \partial_\theta\right)\varphi(\theta), \tag{10}$$

then

$$[\ \varphi_+(\theta)\ ,\ \varphi_+(\theta')\ ]\ =\ i\partial_\theta\delta(\theta-\theta'). \tag{11}$$

If

$$q[\chi_1] = \int d\theta\ \chi_1(\theta)\varphi_+(\theta), \tag{12}$$

$$[\ q[\chi_1]\ ,\ q[\chi_2]\ ] = i\int d\theta'\ \left(\chi_1\partial_\theta\chi_2\right)(\theta') \tag{13}$$

Bal, Bimonte, Gupta, Stern (1992)

The above fields describe quantum Hall edge states.

We now shift to edge states conserving P and T.

The Laplacian ∇^2 and Robin Boundary Conditions

In QFT, to quantise massless tensor fields φ, we solve

$$-\nabla^2 u_n = \lambda_n u_n, \tag{14}$$

and expand

$$\varphi = \sum a_n u_n. \tag{15}$$

We need

$$\lambda_n \geq 0. \tag{16}$$

That is ok on $\mathbb{R}^d$.

But suppose manifold M has boundary ∂M. Then

$$-\nabla^2 \ngeq 0 \tag{17}$$

for Robin boundary conditions

$$\left(\vec{n} \cdot \vec{\nabla}\right) u = \overline{m} u, \quad \overline{m} > 0. \tag{18}$$

Known already to Lieb and Liniger (1993).

Later found by Bal, Chandar, Ercolessi, Govindarajan and Shankar (1994).

Used for black hole physics by Govindarajan and Tibrewala (2011).

Example in $1d$: Laplacian on $M = [0, \infty)$

Consider

$$\int_M dx\, \overline{u}\left(-\partial_x^2\right)\ u(x) := \left(u, -\partial_x^2 u\right) = (\partial_x u, \partial_x u) + \overline{u}(0)\partial_x u(0).$$

Here $\lim_{x\to\infty} u(x) = 0$.

Now, fix Robin boundary conditions:

$$\partial_x u(0) = -\vec{n}\cdot\vec{\nabla} u(0) = -\overline{m} u(0).$$

Therefore,

$$\left(u, -\partial_x^2 u\right) = \underbrace{(\partial_x u, \partial_x u)}_{>0} \underbrace{-\overline{m}\ \overline{u}(0) u(0)}_{<0}$$

One concludes that $\exists\ u$ with $\vec{n}\cdot\vec{\nabla} u = -\overline{m} u$ localised near $x = 0$ such that

$$\left(u, -\partial_x^2 u\right) < 0.$$

Argument valid in all dimensions d if M has compact boundary ∂M

Set

$$\int_M dV_M \,\overline{u}(x)\left(-\nabla^2\right)u(x) := \left(u, -\nabla^2 u\right) \tag{19}$$

Then

$$\begin{aligned}\left(u, -\nabla^2 u\right) &= \left(\vec{\nabla} u, \vec{\nabla} u\right) - \int_{\partial M} dV_{\partial_M}\, \overline{u}\ \vec{n}\cdot\vec{\nabla} u \\ &= \underbrace{\left(\vec{\nabla} u, \vec{\nabla} u\right)}_{>0} \underbrace{-\overline{m}\int_{\partial M} dV_{\partial_M}\, \overline{u}\, u}_{<0}\end{aligned} \tag{20}$$

In fact, Asorey, Ibort and Marmo:

$\exists$ sequence $u^{(n)}$, with $\left(u^{(n)}, u^{(n)}\right) = 1$,

1. which fulfil $\vec{n} \cdot \vec{\nabla} u^{(n)}|_{\partial M} \;=\; n\overline{m}\; u^{(n)}$,

2. are edge localised with width

$$\sim \frac{1}{n\overline{m}} \to 0, \quad \text{as} \quad n \to \infty, \tag{21}$$

3. $\left(u^{(n)}, -\nabla^2 u^{(n)}\right) \to -\infty, \quad \text{as} \quad n \to \infty!$

Thus,

A) $\left(u^{(n)}, u^{(n)}\right) = 1$ implies that $u^{(n)}$ is in Hilbert space for all n.

B) By 3), $\left(u^{(n)}, -\nabla^2 u^{(n)}\right) \approx$ non-relativistic energy $\longrightarrow -\infty$.

C) By 2), they get narrower and narrower as $n \longrightarrow \infty$

Item c) means $\left|\left(\psi, u^{(n)}\right)\right| \longrightarrow 0$ for any fixed ψ in Hilbert space. We say $u^{(n)} \longrightarrow 0$ weakly.

As $n \to \infty$, we approach Dirichlet boundary condition.

So, there exist a sequence of progressively localised edge states $u^{(n)}$ with energies $\to -\infty$ as $\overline{m} \to \infty$ and weakly converging to 0 !

Can we understand them?

Boundaries ∂M of Superconductors

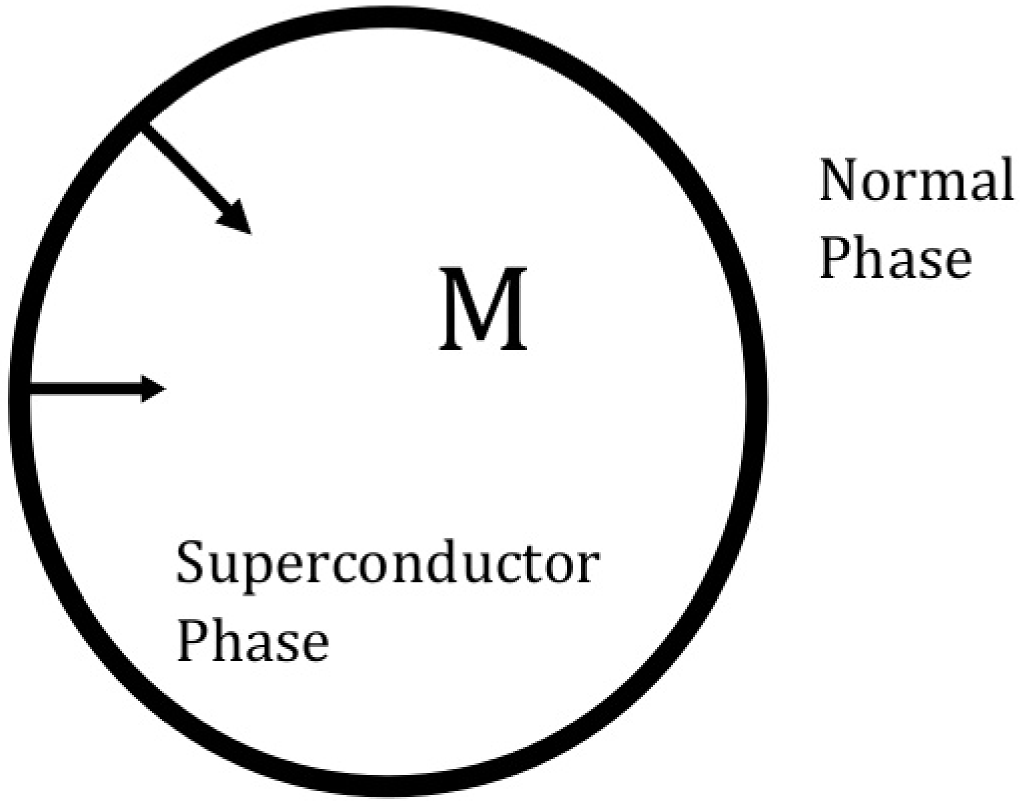

Figure: Electromagnetic field is massless in $\mathbb{R}^N \backslash M$, but it has London mass m in M, the Meissner effect: $A_i \sim e^{mr} a_i$.

Meissner effect

For static solution

$$A_i \sim e^{m(r-R)}, \tag{22}$$

as r decreases from R, i.e., ∂M.

This implies $\dot{A}_i = mA_i$: Robin boundary condition.

Also

$$\left(-\nabla^2 + m^2\right) A_i = 0. \tag{23}$$

So Laplacian in superconductor has Robin boundary condition $\dot{A}_i = mA_i$ on ∂M.

Hence it has edge states!

Photon massive in M and its mass lifts the edge levels from negative to positive values. Numerically

$$m^2 \geq \overline{m}^2. \tag{24}$$

On Edge Levels

Let us consider massive scalar fields φ with Robin boundary condition:

$$n \cdot \nabla \varphi = \overline{m} \varphi, \quad \overline{m} > 0 \quad \text{on} \quad \partial M. \tag{25}$$

The field φ can be a "pseudo-Goldstone" boson.

Scalar fields are easier to discuss.

In spherical coordinates,

$$-\nabla^2 = -\frac{\partial^2}{\partial r^2} - \frac{1}{r^2} \nabla_\theta^2, \tag{26}$$

$$-\nabla_\theta^2 \geq 0. \tag{27}$$

Lowest level has

$$-\nabla_\theta^2 \simeq 0 \quad \text{for zero angular momentum.} \tag{28}$$

Angular excitation are spaced by $\frac{1}{r^2}$.

On Bulk Levels

If χ is a bulk level, $\chi|_{\partial M} = 0$, then

$$\begin{aligned}(\chi, (-\nabla^2 + m^2)\,\chi) &= \left(\vec{\nabla}\chi, \vec{\nabla}\chi\right) + m^2\,(\chi, \chi) \\ &\geq m^2 \end{aligned} \tag{29}$$

for normalised χ.

Bulk is gapped.

On Parity P and Time-Reversal T

Robin condition preserve P and T

- P is ok: $\vec{n} \cdot \vec{\nabla}$ is P or orientation-reversal invariant.
- T is ok: $\overline{m}$ is real.

Edge states must observable.

May affect Casimir energies?

An Example

$M = [0, L]$ with Dirichlet at 0 and Robin at L.

Figure: This example is prototype of what happens in all dimensions.

There is one level ψ_0 localised near $x = L$ with energy $\approx -\overline{m}^2$ for $-\partial_x^2$:

$$-\partial_x^2 \psi(x) \simeq -\overline{m}^2 \psi(x). \tag{30}$$

For let

$$\psi(x) = \alpha e^{-Kx} + \beta e^{Kx} \tag{31}$$

$$-\partial_x^2 \psi(x) = -K^2 \psi(x). \tag{32}$$

- $x = 0$:

$$\begin{aligned} \psi(0) = 0 \;&\Rightarrow\; \alpha + \beta = 0 \\ &\Rightarrow\; \psi(x) = e^{-Kx} - e^{Kx}. \end{aligned} \tag{33}$$

- $x = L$:

$$\begin{aligned} \partial_x \psi(L) &= \overline{m}\psi(L) \\ \Rightarrow\; \overline{m} \;&=\; K \frac{e^{KL} + e^{-KL}}{e^{KL} - e^{-KL}} \equiv f(K). \end{aligned} \tag{34}$$

Can assume $K > 0$ since $K < 0$ only makes $\psi \to -\psi$.

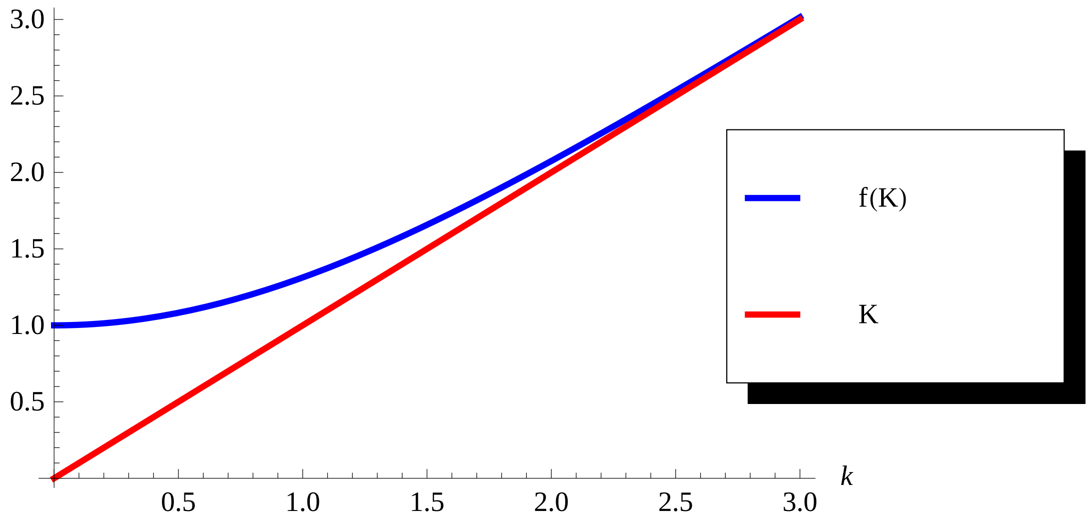

FIGURE: Plot of $f(K) = K \frac{e^{KL}+e^{-KL}}{e^{KL}-e^{-KL}}$ versus K

For large $\overline{m}$, near Dirichlet point, $f(K) \simeq K$,

$$K \simeq \overline{m} \qquad \text{and} \tag{35}$$

$$\psi(x) \approx e^{-\overline{m}x} - e^{\overline{m}x}. \tag{36}$$

$e^{+\overline{m}x}$ dominates near $x = L$.

No edge states near $x = 0$ where $|\psi|$ is small.

We see

$$-\partial_x^2 \psi \simeq -\overline{m}^2 \psi. \tag{37}$$

So for mass $m \simeq \overline{m}$,

$$-\partial_x^2 + m^2 \tag{38}$$

has a low-lying level localised at $x = L$.

Bulk is gapped by m^2

The Edge States for Dirac Operator

Topological insulators: described in literature by Dirac operator with edge states with small gap and gapped bulk.

P, T preserved for space dimension d odd.

There is "spin-momentum locking".

Such Dirac operators emerge from previous analysis.

The d-dimensional Dirac operator D is the Dirac-Hamiltonian of $(d+1)$ dimensions:

$$D = -i\gamma^i \nabla_i + m\gamma^{d+1} \tag{39}$$

$$\gamma^{d+1} = \beta \quad \text{of} \quad d = 3,$$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\delta^{\mu\nu} \tag{40}$$

It implies $D^2 = -\nabla^2 + m^2 = -\text{Laplacian} + m^2$.

QUESTION: If D^2 has edge states, does D have them too?

For that we must understand boundary conditions for D

Boundary Conditions for D

We have Hilbert space

$$\mathcal{H}(M) \text{ on } M, \text{ with } (\psi, \chi) = \int_M dV_M \, \psi^\dagger \chi.$$

$$\mathcal{H}(\partial M) \text{ on } \partial M, \text{ with } \langle \psi, \chi \rangle = \int_{\partial M} dV_{M-1} \, \psi^\dagger \chi.$$

Let K be any self-adjoint operator on ∂M with no zero eigenvalue and anti-commuting with $\gamma \cdot n$:

$$\gamma \cdot n \, K = -K \, \gamma \cdot n. \tag{41}$$

We can write

$$\mathcal{H}(\partial M) = \underbrace{\mathcal{H}^{(-)}(\partial M)}_{K<0} \oplus \underbrace{\mathcal{H}^{(+)}(\partial M)}_{K>0},$$

$$\left(\psi^{(+)}, \psi^{(-)}\right) = 0 \qquad \text{if} \qquad \psi^{(\pm)} \in \mathcal{H}^{(\pm)}(\partial M).$$

Boundary condition – or domain – of D:

$$\psi|_{\partial M} \in \mathcal{H}^{(-)}(\partial M). \tag{42}$$

So

$$(\chi, D\psi) - (D\chi, \psi) = -i\langle \chi, \gamma \cdot n \; \psi \rangle. \tag{43}$$

But if

$$K\psi_j = -|\lambda_j|\psi_j, \;\; |\lambda_j| > 0,$$

$$K \; \gamma \cdot n \; \psi_j = +|\lambda_j| \; \gamma \cdot n \; \psi_j. \tag{44}$$

$\Longrightarrow$

$$\gamma \cdot n \; \mathcal{H}^{(-)}(\partial M) = \mathcal{H}^{(+)}(\partial M)$$

$$\langle \chi, \gamma \cdot n \; \psi \rangle = 0. \tag{45}$$

So D is symmetric. Self-adjointness also follows easily.

Atiyah-Patodi-Singer (APS) boundary condition:

$$K = i\gamma \cdot n \; A(m). \tag{46}$$

where

$$D = -i\gamma \cdot n \; D_r + A(m), \tag{47}$$

$$A(m) = -i\vec{\gamma}_\theta \cdot \nabla_\theta + m\gamma^{d+1}, \tag{48}$$

with $-i\vec{\gamma}_\theta \cdot \nabla_\theta$ the tangential part of D.

Below we choose

$$K = i\gamma \cdot n \; A(\mu) \tag{49}$$

to get low-lying edge states.

Here $m, \; \mu$ can be different.

$m, \; \mu$ here are like $m, \; \bar{m}$ for $-\nabla^2$.

We can tune m and μ for optimal results.

Example: $M = (-\infty, 0]$, $\partial M = \{0\}$

$$D = -i\sigma^1\partial_1 + \sigma^2 m, \tag{50}$$

$$D^2 = -\partial_1^2 + m^2, \tag{51}$$

and

$$A(m) = \sigma^2 m. \tag{52}$$

So

$$K = i\sigma^1 A(m) = -m\sigma^3 \tag{53}$$

$$\sigma^1 K = -K\sigma^1. \tag{54}$$

In addition, for edge state at $x = 0$, we look for ψ fulfilling

$$\dot{\psi} = \overline{m}\psi. \tag{55}$$

This is suggested by Robin boundary conditions.

Now we want $(D\psi, D\psi)$ to be small for edge state:

$$(D\psi, D\psi) = i\psi^\dagger(0)\ \sigma^1\ D\psi(0) + \left(\psi, D^2\psi\right), \tag{56}$$

where the second term is governed by a low energy edge state by (55).

Now,

$$\begin{aligned} i\sigma^1\left(-i\sigma^1\partial_1 + m\sigma^2\right)\psi|_{x=0} &= i\sigma^1\left[-i\sigma^1\dot{\psi}(0) + m\sigma^2\psi(0)\right] \\ &= \left(\overline{m} - m\ \sigma^3\right)\psi(0). \end{aligned}$$

gives boundary term in $(D\psi, D\psi)$:

$$\overline{m}\ \psi(0)^\dagger\psi(0) - m\ \psi(0)^\dagger\ \sigma^3\ \psi(0). \tag{57}$$

First term is large. So cancel it with

$$\psi(0) = \begin{pmatrix} \xi \\ 0 \end{pmatrix}, \tag{58}$$

and regularising mass

$$\overline{m} \;=\; m. \tag{59}$$

Note that

$$K\psi(0) = -m\sigma^3\psi(0) = -m\psi(0) < 0, \tag{60}$$

as needed.

Then the edge state is

$$\psi(x) = e^{mx} \begin{pmatrix} \xi \\ 0 \end{pmatrix}. \tag{61}$$

It has zero energy

$$D\psi(x) = 0. \tag{62}$$

State ψ is normalisable and localised near $x = 0$.

D has also eigenstates of form

$$e^{ikx}\ \eta + e^{-ikx}\ \zeta. \tag{63}$$

Its full set of eigenstates are complete.

Can be generalised to higher dimensions.

Remarks

- We have explicitly calculated the low-lying edge states for disk.

- Proved spin-momentum locking or net spin transport.

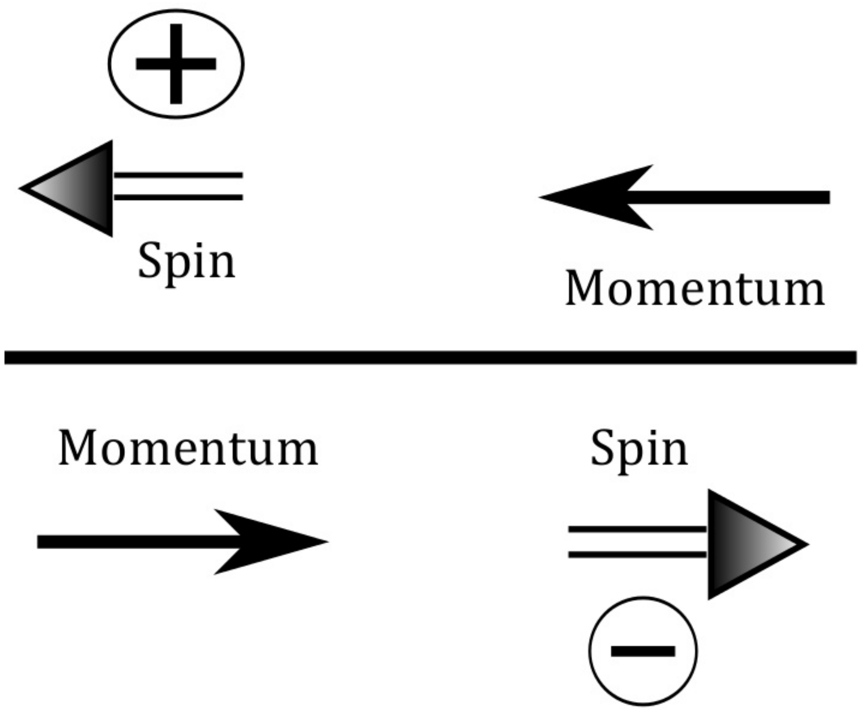

Figure: It shows net ⊕ spin transport to left.

- A Majorana or reality condition on K.
- P and T invariance for Dirac operator if d is odd.
- P and T seems broken by mass term of Dirac operator if d is even.
- Last two points are still being examined.