A Bohmian approach to the non-Markovian non-linear Schrödinger-Langevin equation

Nicolás Morales-Durán†, Andrés F. Vargas* and Pedro Bargueño

Departamento de Física, Universidad de los Andes, Apartado Aéreo *4976*, Bogotá, Distrito Capital, Colombia

† na.morales92@uniandes.edu.co, *af.vargas1540@uniandes.edu.co

1. INTRODUCTION

The theory of open quantum systems is of fundamental importance to properly describe real systems, which do not exist in complete isolation [1, 2]. Among the main approaches usually considered to incorporate environmental effects in the dynamics of the system under study, non-linear Schrödinger equations and system-plus-bath techniques are two of the most representative ones. In this work, a non-Markovian generalized Schrödinger-Langevin equation is derived from the system-plus-bath approach [7]. Specifically Bohmian mechanics is shown to be of great importance in order to obtain a compact expression for the damping potential, which reduces to the well known situations reported in the literature, where Markovian and/or non-linear effects are considered. Finally, an application regarding the generalized uncertainty principle (GUP) as a quantum gravitational principle is presented [8].

2. PRELIMINARIES

Let us consider the Caldeira-Legget model [4] for a one dimensional system. This approach models a massive particle in a heath bath, consisting of an infinite set of harmonic oscillators. From these considerations the following total Hamiltonian arises:

$$H = H_s + H_b + H_{sb}. \tag{1}$$

The first term is the Hamiltonian of an isolated particle in presence of a potential,

$$H_s = \frac{p^2}{m} + V(x). \tag{2}$$

The Hamiltonian of the bath is

$$H_b = \frac{1}{2}\sum_i \left(\frac{p_i^2}{m_i} + m_i\omega_i^2 x_i^2\right), \tag{3}$$

and the coupling between the system and the bath is

$$H_{sb} = \sum_i \left[\frac{d_i^2 f^2(x)}{m_i\omega_i^2} - 2d_i f(x) x_i\right]. \tag{4}$$

The time evolution of the position x of the system is given by the Heisenberg equations of motion. After eliminating the bath degrees of freedom, we can arrive at the generalized Langevin equation (GLE) [2]:

$$\begin{aligned} m\ddot{x}(t) + \frac{\partial V(x)}{\partial x} &+ mf'(x(t))\int_0^t \alpha(t-\tau) f'(x(\tau))\dot{x}(\tau)d\tau \\ &= f'(x(t))\xi(t). \end{aligned} \tag{5}$$

where the memory kernel and the noise term are given, respectively, by

$$\alpha(t) = \frac{1}{m}\sum_i \frac{2d_i^2}{m_i\omega_i^2}\cos(\omega_i t) \tag{6}$$

$$\begin{aligned} \xi(t) = &-\sum_i d_i \left[\left(x_i(0) + \frac{d_i}{m_i\omega_i^2} f(0)\right)\cos(\omega_i t)\right] \\ &- \sum_i d_i \left[\frac{2p_i(0)}{m_i\omega_i} sen(\omega_i t)\right]. \end{aligned} \tag{7}$$

In general we are not going to restrict our study to the Markovian regime, where the kernel is a δ-function in time. We will look for an effective quantum Hamiltonian from which, using the Heisenberg evolution equations, Eq. (5) can be obtained. For that purpose, we start by considering the following Schrödinger equation

$$\imath\hbar\frac{\partial\Psi}{\partial t} = \left[\frac{1}{2m}\left(-\imath\hbar\frac{\partial}{\partial x}\right)^2 + V(x) + V_d + V_r\right]\Psi, \tag{8}$$

where V_r and V_d correspond to random and dissipative effective potentials which try to capture all the information present in the GSLE regarding randomness and dissipation, respectively.

By working in the Heisenberg picture of quantum mechanics we obtain the evolution of the momentum operator and applying the Ehrenfest theorem in Eq. (5) we obtain another equation for expectation values of the forces. After identifying terms in the two previous expressions, the random potential is shown to be given by (the linear and Markovian case is discussed, for example, in [1])

$$V_r(x,t) = -f(x)\xi(t) \tag{9}$$

In particular, if only linear coupling and a Markovian regime is considered, the expression first obtained by Kostin is recovered [3].

$$V_d = \frac{\hbar\alpha}{2\imath}\ln\left(\psi(x,t)/\psi^*(x,t)\right) \tag{10}$$

Now we are up to the challenge of finding the most general dissipative potential such that under the Heisenberg evolution equations it will effectively describe the correct dissipative term appearing in Eq. (5). Following the same procedure as for the random potential, the following integro-differential equation for V_d can be obtained

$$\begin{aligned} &\int \Psi^*\left(-\frac{\partial V_d}{\partial x}\right)\Psi dx = \\ &m\int \Psi^*\left(f'(x(t))\int_0^t \alpha(t-\tau)f'(x(\tau))\dot{x}(\tau)d\tau\right)\Psi dx. \end{aligned} \tag{11}$$

Solving Eq. (11) in terms of V_d represents a tough problem whose solution, under very general assumptions, is far from being trivial [7].

3. A POSIBLE SOLUTION BASED ON THE BOHMIAN APPROACH

Let us tackle the problem of solving Eq. (11) whithin the Bohmian approach. In this formulation the wave-function is expressed in its polar form as $\Psi(x,t) = A(x,t)e^{\frac{\imath}{\hbar}S(x,t)}$. Using the fact that the phase of the wavefunction (the quantum action) is related with the momentum by $\frac{\partial S(x,t)}{\partial x} = p(x,t)$, a non-Markovian effective dissipative potential can be expressed as

$$V_d = \int \frac{df}{dx}\left[\left(\frac{df}{dx}p\right) * \alpha\right] dx \tag{12}$$

Although we have considered the simple one-dimensional case, the generalization to higher dimensions is straightforward, see [7]. Therefore, the non-linear non-Markovian GSLE can be finally written as

$$\imath\hbar\frac{\partial\Psi}{\partial t} = \left[\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) + V_d(\mathbf{r},t) + f(\mathbf{r})\xi(t)\right]\Psi, \tag{13}$$

In addition, we would like to recall the reader that Eq. (13) is free for any ansatz. That is, we have not chosen any particular form for the dissipative force in the Bohmian framework.

4. APPLICATION TO THE GENERALIZED UNCERTAINTY PRINCIPLE

Gravity plays a mayor role in the existence of a fundamental measurable length which modifies the Heisenberg uncertainty principle. This modification known as Generalized Uncertainty Principle (GUP) [8]. To understand this we now define a deformed commutation relation by means of

$$x_i = x_{oi} \quad p_i = p_{oi}(1 - \gamma p_0 + O(p^2)) \tag{14}$$

Where $[x_{oi}, p_{oj}] = ih\delta_{ij}$; $p_o^2 = \Sigma_{j=1}^3 p_{oj}p_{oj}$ and $\gamma = \frac{\gamma_o}{m_p c}$ (γ_o is adimensional).

Consider a one dimensional non-relativistic system that possesses a Hamiltonian of the form: $H = \frac{p^2}{2m} + V(x)$. From the relationships (14) a deformed conmutator can be established. Using the Heisenberg picture of quantum mechanics one can get the evolution equation in configuration space, namely

$$\ddot{x} + \tilde{\gamma}\dot{x} + \frac{V'(x)}{m} = 0 \tag{15}$$

Where $\tilde{\gamma} = -\frac{2\gamma_o m}{m_p c}V'(x)$. This clearly resembles a Schödinger Langevin equation with a position-dependant friction. Using the formalism here developed we can consider (11) with an ohmic dissipation ($\alpha = \gamma\delta(t)$) a non-linear coupling ($f(x)$) and after dropping the noise term is [5]:

$$m\ddot{x} + m\gamma[f'(x)]^2\dot{x} + V'(x) = 0 \tag{16}$$

by comparing (15) and (16) we conclude $\tilde{\gamma} = -m\gamma[f'(x)]^2$ or by recalling the definition of $\tilde{\gamma}$

$$\frac{2\gamma_o V'(x)}{m_p c} = -\gamma[f'(x)]^2 \tag{17}$$

From this one can define a quantum gravity friction as:

$$\gamma_{QG} \equiv \frac{2\gamma_o}{m_p c} \tag{18}$$

By means of (14) and (18) we can describe the system by a classical Hamiltonian given by

$$H = \frac{p^2}{2m} + V(x) - \frac{\gamma_{QG}p^3}{2m} \tag{19}$$

Using the Bohmian approach here presented, under this regime, the dissipative potential takes the form:

$$V_d = -\frac{\gamma_{QG}\tilde{S}}{2} = \gamma_{QG}(pV(x)) \tag{20}$$

Where $\tilde{S}$ is defined as

$$\tilde{S} \equiv \left(\frac{df}{dx}\right)^2 S - 2\int S\frac{df}{dx}\frac{d^2 f}{dx^2}dx$$

And is the coupling-dependant phase of the wave function. Therefore the GUP dissipative Hamiltonian [8] equivalent of equation (19) is

$$H_{GUP} = \frac{p^2}{2m} + V(x)[1 - \gamma_{QG}p] + O(\gamma_o^2) \tag{21}$$

This Hamiltonian is correctly predicted by (13), under this considerations, by introducing the potential given by (20) and dropping the noise term. Namely:

$$H_{GSLE} = \frac{p^2}{2m} + V(x) + \tilde{S} = \frac{p^2}{2m} + V(x) - \gamma_{QG}(pV(x))$$

or regrouping terms

$$H_{GSLE} = \frac{p^2}{2m} + V(x)[1 - \gamma_{QG}p] = H_{GUP} \tag{22}$$

From this, the selected GUP can be seen as a dissipative term in the system. Its remarkable that the generalized Schrödinger Langevin Hamiltonian here presented correctly predicts the Hamiltonian plus dissipative GUP term (H_{GUP}) derived by a different mean.

5. CONCLUSIONS

In this work we have derived a generalized non-Markovian and non-linear Schrödinger–Langevin equation compatible with a Caldeira–Leggett approach. Bohmian techniques have been shown to be of great importance in order to have a compact expression for the non-Markovian damping potential. This potential reduces to the well know situations reported in the literature when simple Markovian and/or non-linear effects are considered as was evidenced in the aplication presented. Although this model does not include temperature, it may be useful to describe simple condensed phase models within the quantum trajectory perspective, which provides an intuitive visualization of the dynamics.

References

[1] M. Razavi, Classical and Quantum Dissipative Systems, Imperial College Press, London (2005).

[2] U. Weiss, Quantum Dissipative systems, in: Series in Modern Condensed Matter Physics, vol. 13, World Scientific (2008).

[3] M. D. Kostin, J. Chem. Phys. **57**, 3589 (1972).

[4] A. O. Caldeira and A. J. Legget, Phys. Rev. Lett **46**, 211 (1981).

[5] P. Bargueño and S. Miret–Artés. Ann. Phys. **346**, 59 (2014).

[6] A. S. Sanz and S. Miret–Artés, A Trajectory Description of Quantum Processes. A Bohmian Perspective, in: Lecture Notes in Physics, vol. 850, Springer, 2012, 831, 2014.

[7] A. F. Vargas N. Morales-Duran and P. Bargueño. Ann. Phys. **356** (2015), pp. 498-504.

[8] P.Bargueño . Physics Letters B 727 . 4-5 (2013), pp. 496-499