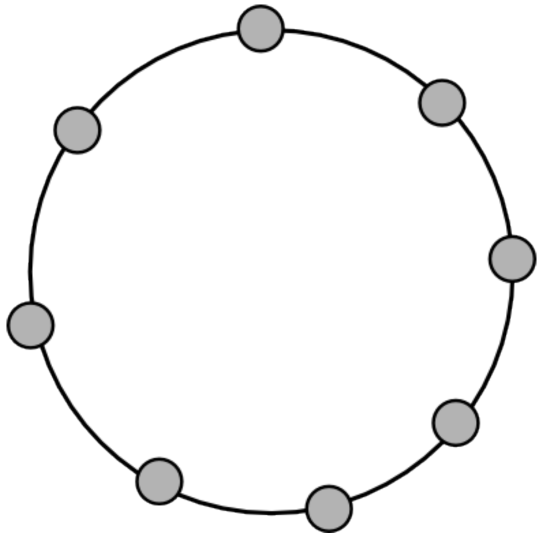
The goal of this exercise is to obtain the trace formula in two simple, yet very enlightening cases. We begin by taking N vertices on a ring



We note f_i the wavefunction associated with vertex i.

1. What are the boundary conditions for this problem?

2. Write the stationary Schrödinger equation using the discrete form of the Laplacian. Show that it can be cast into the form of a recursive equation:

$$f_{i+1} + f_{i-1} = \lambda f_i, \tag{1}$$

   show the relation between λ and the energy.

3. It can be shown that recursive equations such as (1) can be solved by solutions of the form $f_k = r^k$. Determine r.

4. We define $\lambda_k = 2\cos\theta_k$. Using the boundary conditions of the first item, show that θ_k may only take values of the form $\theta_k = \frac{2\pi}{N}k$, $k = 1, \ldots, N$.

5. We define the average density of θ:

$$\rho(\theta) = \frac{1}{N}\sum_{k=1}^{N} \delta\left(\theta - \frac{2\pi}{N}k\right) \tag{2}$$

   (a) Recall the expression of $\delta(x)$ in terms of an infinite sum of exponentials.
   (b) Replace $\delta(x)$ in (2) by this sum of exponentials.
   (c) Get rid of the sum in k using the fact that $\mathrm{e}^{2\pi k/V}$ forms a family of orthogonal functions.
   (d) Rewrite (2) in function of N, θ and a sum over a certain variable.

6. Identify a smooth part and a sum over loops in the expression obtained for $\rho(\theta)$ in the previous item. What kind of formula are we looking at?

7. Repeat this exercise for a finite linear graph.