The goal of this exercise is to explore some properties of Gaussian matrices in order to derive deeper and fundamental results in Random Matrix Theory.

Wick's theorem

We begin by recalling the joint probability density for a $N \times N$ Gaussian hermitian matrix $\mathbf{X} = \{x_{ij}\}$, $i, j = 1, \ldots, N$ of variance 1 and mean 0:

$$P(x_{11}, x_{12}, \ldots, x_{NN}) = C_N \mathrm{e}^{-\frac{N}{2}\mathrm{Tr}\mathbf{X}^2} \tag{1}$$

1. Write the probability density function in terms of the entries of the matrix $\{x_{ij}\}$.

2. Calculate $\langle x_{ij} \rangle$, $\left\langle x_{ij}^2 \right\rangle$ and $\langle x_{ij} x_{lp} \rangle$.

3. Calculate $\left\langle \mathrm{e}^{t \sum_{i,j} k_{ij} x_{ij}} \right\rangle$.

This last item can be shown to have a more general form:

$$\left\langle \mathrm{e}^{t \sum_{ij} k_{ij} x_{ij}} \right\rangle = \mathrm{e}^{\frac{t^2}{2} \sum_{i,j,l,p} k_{ij} k_{lp} \langle x_{ij} x_{lp} \rangle}. \tag{2}$$

5. Expanding both sides in powers of t, show that all n-point correlation functions of the Gaussian matrix can be expressed as a sum of 2-point correlation functions. We name this result Wick's theorem and it is quite more general than this small exercise shows.

The Green's function

We turn our attention to the eigenvalues $\{\lambda_i\}$ of a hermitian Gaussian random matrix $\boldsymbol{X}$. We define the following matrix

$$\mathbf{G}_N(z) = \left\langle (z\mathbb{1}_N - \mathbf{X})^{-1} \right\rangle, \tag{3}$$

where z is a complex number and the average is taken over the measure of $\mathbf{X}$.

Using $\mathbf{G}_N(z)$, we define the so called Green's function or resolvent,

$$G_N(z) = \frac{1}{N}\mathrm{Tr}\mathbf{G}_N(z) \tag{4}$$

6. Write $G_N(z)$ as a function of z and $\{\lambda_i\}$.

7. What is the domain of definition of $G_N(z)$? Knowing that the eigenvalues of a Gaussian random matrix, when correctly normalized, fall within a compact support, what will intuitivly happen to $G_N(z)$ when $N \to \infty$?

To match this intuitive notion, we define the average density of eigenvalues

$$\rho(\lambda) = \left\langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \right\rangle, \tag{5}$$

which allows us to write the resolvent, in the large N limit, as

$$G_N(z) = \int \frac{\rho(x)}{z - x} \mathrm{d}x. \tag{6}$$

8. Use the Sokhotski–Plemelj identity

$$\lim_{\epsilon \to 0^+} \frac{1}{x \pm i\epsilon} = \Pr \frac{1}{x} \mp i\pi\delta(x), \tag{7}$$

where Pr stands for Cauchy principal part to obtain a way of, having $G_N(z)$, finding $\rho(x)$

The goal of this exercise is to derive Wigner's semicircle law using the resolvent technique. We recall the probability density function $\mathcal{P}_\beta(\boldsymbol{\lambda})$ for the eigenvalues of the Gaussian ensemble:

$$\mathcal{P}_\beta(\boldsymbol{\lambda}) = \frac{1}{Z_{N,\beta}} \mathrm{e}^{-\frac{\beta N}{2}\sum_{j=1}^N \lambda_j^2} \prod_{i>j} |\lambda_i - \lambda_j|^\beta \tag{8}$$

1. Write the probability density function (8) as a Boltzmann weight, i.e., a function of the form:

$$\mathcal{P}_\beta(\boldsymbol{\lambda}) = \frac{\mathrm{e}^{-\beta E[\boldsymbol{\lambda}]}}{Z_{N,\beta}} \tag{9}$$

   and identify the function $E[\boldsymbol{\lambda}]$.

2. We introduce the density $\rho(\lambda)$, defined as:

$$\rho(\lambda) = \frac{1}{N}\sum_{i=1}^N \delta(\lambda - \lambda_i). \tag{10}$$

   Note that with this function we may exchange sums for integrals: $\sum f(\lambda_i) \to \int \rho(x) f(x) \mathrm{d}x$. Using this function, write $E[\boldsymbol{\lambda}] \to N^2 S[\rho]$ in terms of integrals of $\rho(x)$, recalling the fact that we are in the large-N limit.

3. Apply the saddle-point method on $\mathcal{P}_\beta(\boldsymbol{\lambda})$ and write it as a single exponential (instead of the integral of an exponential) whose exponent is the function obtained previously applied to a certain $\rho^\star(\lambda)$. What is the interpretation of $\rho^\star$?

4. Show that by differetiating functionally $S[\rho]$ with respect to ρ we obtain the integral equation for $\rho^\star$.

$$x = \mathrm{Pr}\int_{-\infty}^{\infty} \frac{\rho^\star(y)}{x-y}\mathrm{d}y, \tag{11}$$

   where Pr stands for Cauchy principal value and $x \in \mathrm{supp}(\rho^\star)$.

5. We define the resolvent $G(z)$:

$$G(z) = \int \frac{\rho(y)}{z-y}\mathrm{d}y, \tag{12}$$

   which is an analytic function everywhere outside of the support of $\rho(x)$. Multiplying (11) by $\frac{\rho(x)}{z-x}$ and integrating it over x yields

$$\int x\frac{\rho(x)}{z-x}\mathrm{d}x = \iint \frac{\rho(x)}{z-x}\frac{\rho^\star(y)}{x-y}\mathrm{d}y\mathrm{d}x. \tag{13}$$

   (a) Using the identity

$$\frac{1}{(z-x)(x-y)} = \left(\frac{1}{z-x} + \frac{1}{x-y}\right)\frac{1}{z-y} \tag{14}$$

   show that the right hand side (RHS) of equation (13) can be written as $G(z)^2/2$.

   (b) Using the quite evident identity $x = x + z - z$, show that the LHS of equation (13) can be written as $-1 + zG(z)$.

   (c) Solve the equation and obtain $G(z)$.

6. Using the method derived on the last exercise class, find $\rho(x)$.

The goal of this exercise is to derive Wigner's semicircle law using the diagrammatic approach, a powerful tool to analyse random matrices without having the full eigenvalue distribution. Let $\mathbf{A}$ be a Gaussian random matrix. We begin by recalling the first exercise class, the definition of the matrix $\mathbf{G}_N(z)$

$$\mathbf{G}_N(z) = \left\langle (z\mathbb{1}_N - \mathbf{A})^{-1} \right\rangle, \tag{15}$$

where z is a complex number and the average is taken over the measure of $\mathbf{A}$.

Using $\mathbf{G}_N(z)$, we define the so called Green's function or resolvent. As shown previously, in the large N limit we may write

$$G_N(z) = \frac{1}{N}\mathrm{Tr}\mathbf{G}_N(z) \to \int \frac{\rho(x)}{z-x}\mathrm{d}x, \tag{16}$$

and we need only to determine $G_N(z)$ to obtain the average density by the identity $-\frac{1}{\pi}\lim_{\epsilon->0} \mathrm{Im}\; G_N(x+\mathrm{i}\epsilon) = \rho(x)$.

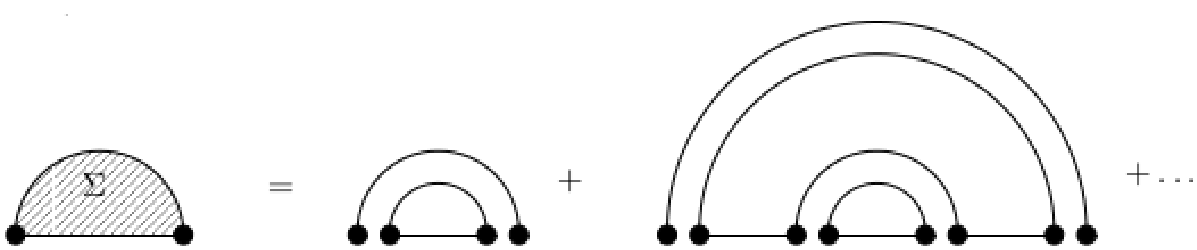
1. Expand equation (15) in powers of $\mathbf{A}$.

We introduce a diagrammatic notation. The element Z^{-1}_{ab} will be noted as a horizontal line between points a and b, while the two-point correlation element $\langle A_{ab}A_{cd}\rangle$ will be noted as a rainbow-like diagram.

3. Using this notation, write the two first non-zero terms of the expansion of equation (15) in powers of $\mathbf{A}$.

4. Use Wick's theorem to express the fourth order term as a sum of products of 2-point correlation functions. Write the result in diagrammatic notation.

It has been shown rigorously by 't Hooft that non-planar diagrams (i.e. diagrams with crossing lines) in the large-N limit are sub-dominant, hence negligeable. We define the self-energy $\mathbf{\Sigma}$ diagrammatically as the um of all orders of "rainbows":



5. Write the remaining diagrams of the expansion of $\mathbf{G}_N(z)$ in terms of $\mathbf{\Sigma}$, show that $\mathbf{G}_N(z) = (z\mathbb{1}_N - \mathbf{\Sigma})^{-1}$. What is the different between this expression and the original expression for $\mathbf{G}_N(z)$?

6. What happens to $\mathbf{G}_N(z)$ when you add a rainbow-like diagram around it? This would be the equivalent of calculating $\sum_{a,b}^{N}[\mathbf{G}_N(z)]_{a,b}\,\langle A_{ia}A_{bj}\rangle$.

The equations

$$\mathbf{G}_N(z) = (z\mathbb{1}_N - \mathbf{\Sigma})^{-1} \qquad\qquad [\mathbf{\Sigma}]_{ij} = \sum_{a,b}^{N}[\mathbf{G}_N(z)]_{a,b}\,\langle A_{ia}A_{bj}\rangle$$

form the Dyson-Schwinger equation, a fundamental result in quantum field theory.

7. Take the trace of both equations to obtain the trace of $\mathbf{G}_N(z)$, the resolvent.

8. Deduce the average density for the eigenvalues of a Gaussian random matrix. Notice that at no point we used the eigenvalue distribution.