3. Spectral statistics.

Random Matrix Theory in a nut shell.

I. The Gaussian Orthogonal Ensemble (GOE)

The set of $N \times N$ **symmetric real** normally distributed random matrices H

$$P_{GOE}(H)\mathrm{d}H = C_N \exp(-\mathrm{tr}H^2) \prod_{i \geq j} \mathrm{d}H_{i,j}$$

II. The Gaussian Unitary Ensemble (GUE)

The set of $N \times N$ **Hermitian complex** normally distributed random matrices H

$$P_{GUE}(H)\mathrm{d}H = C_N \exp(-\mathrm{tr}HH^{\dagger}) \prod_{i \geq j} \mathrm{d}H_{i,j}$$

Eigenvalues distribution:

The spectral density: $\rho(\lambda) = \frac{1}{N}\sum_{j=1}^{N} \delta(\lambda - \lambda_j)$.

The Wigner Semi-circle law: For $N \to \infty \quad \langle \rho(\lambda) \rangle_{GOE,GUE} \to \frac{1}{2\pi}\sqrt{4N - \lambda^2}$

Example: Nearest neigbour spectral distribution

$$s_n = \frac{\lambda_n - \lambda_{n-1}}{\text{mean spacing}} \quad ; \quad P(s) = \frac{1}{\Delta K} \sum_{k=K}^{K+\Delta K} \delta(s - s_k) \quad ; \quad N > \Delta K \gg 1$$

$$P_{GOE}(s) = \frac{\pi s}{2} \exp(-\frac{\pi s^2}{4}) \quad ; \quad P_{GUE}(s) = \frac{32 s^2}{\pi^2} \exp(-\frac{4 s^2}{\pi})$$

Graphs: the spectrum and the spectral statistics

The discrete Laplacian for d-regular graphs:

$$(L\mathbf{f})_i = -\sum_{j\sim i}(f_j - f_i) \;\Rightarrow\; L = -A + d\, I^{(V)}$$

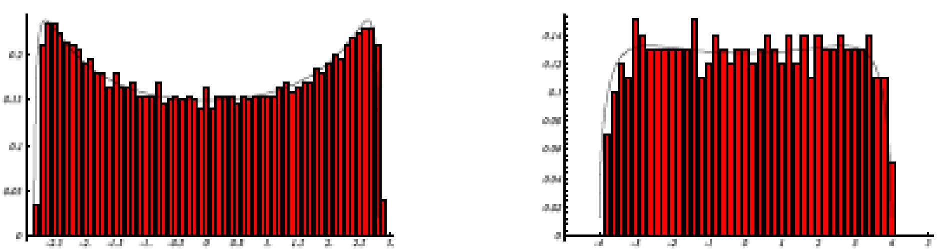
Since L differes from the adjacency matrix A by a constant diagonal matrix, $\Rightarrow$ we study the spectrum of A : $\sigma(\mathcal{G}) = \lambda_0(=d) \geq \lambda_1 \geq, \cdots, \geq \lambda_{V-1}$.

$$\rho(\lambda) = \frac{1}{V-1}\sum_{k=1}^{V-1}\delta(\lambda - \lambda_k)$$

The mean spectral density for d regular graphs, $(V \to \infty\ ,\ d = const)$

Kesten MacKay limit distribution: (Supported in $|\lambda| < 2\sqrt{d-1}$):

$$\rho_{KM}(\lambda) = \lim_{V\to\infty}\frac{1}{V-1}\left\langle\sum_{k=1}^{V-1}\delta(\lambda-\lambda_k)\right\rangle_{\mathcal{G}} = \frac{d}{2\pi}\frac{\sqrt{4(d-1)-\lambda^2}}{d^2-\lambda^2}$$



(a) Cubic graph on 2000 vertices. (b) 5-valent graph on 500 vertices.

Figure 1. Eigenvalue distributions of random graphs *vs* McKay's law

EIGENVALUE SPACINGS FOR REGULAR GRAPHS

DMITRY JAKOBSON, STEPHEN D. MILLER,
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Unfolding the spectrum with the Kesten-McKay density using $\mathcal{N}_{KM}$: The mean spectral counting function.

$$s_j = \mathcal{N}_{KM}(\lambda_j) \quad ; \quad \frac{\mathrm{d}\mathcal{N}_{KM}}{\mathrm{d}\lambda} = \rho_{KM}(\lambda) \ ; \ \mathrm{d}s = \rho_{KM}(\lambda)\mathrm{d}\lambda = \frac{\mathrm{d}\lambda}{\langle \mathrm{d}\lambda \rangle}$$

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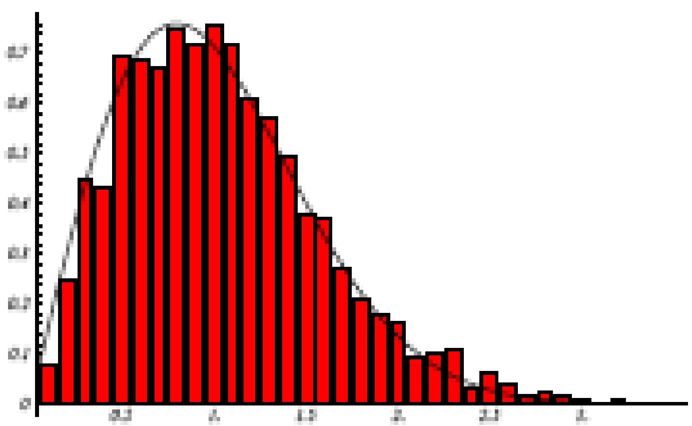
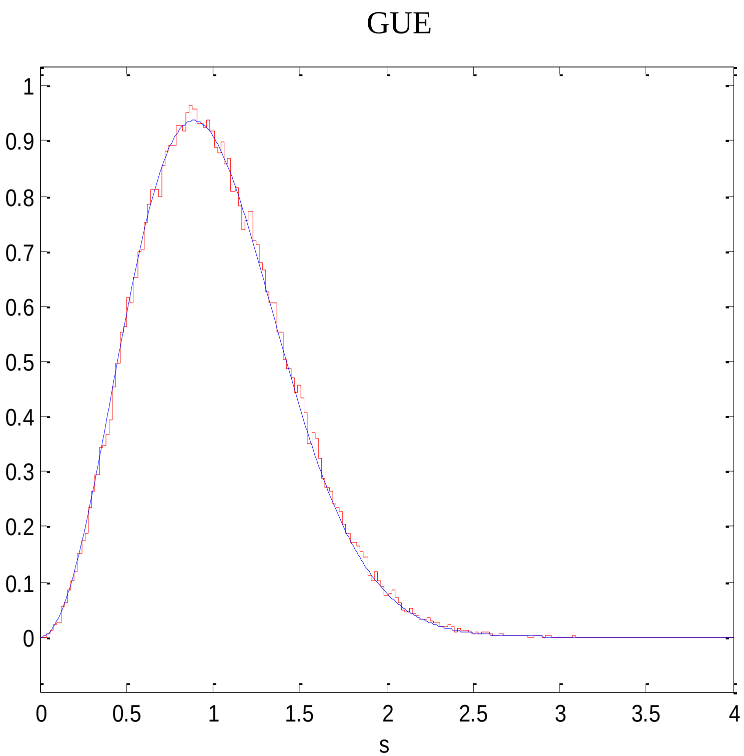


Figure 2. Level spacing distribution of a cubic graph on 2000 vertices *vs* GOE

The "Magnetic" Adjacency Matrix:

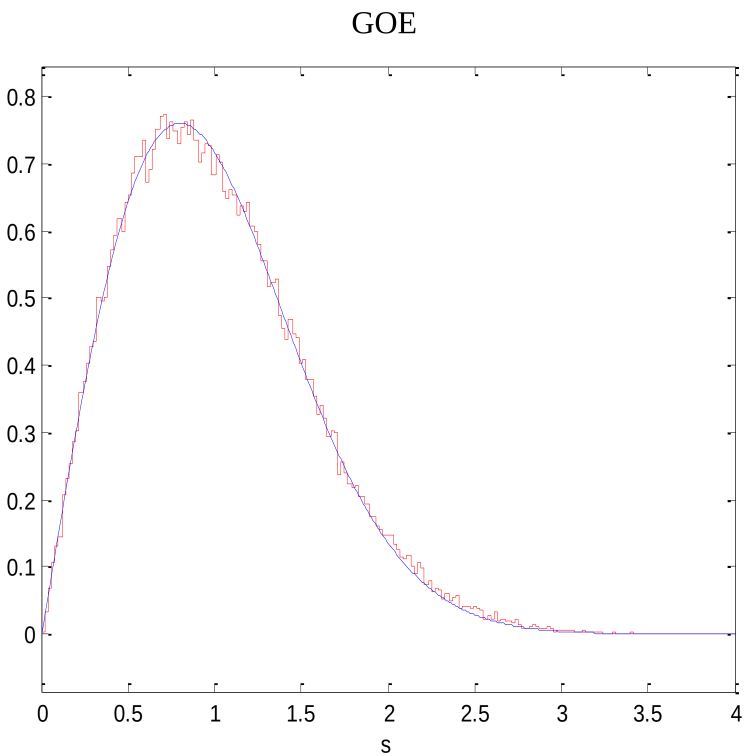
$$A^{(M)}_{i,j} = A_{i,j} e^{i\phi_{i,j}} \quad ; \quad \phi_{j,i} = -\phi_{i,j}$$

$$P_{GUE}(s) = \frac{32 s^2}{\pi^2} \exp(-\frac{4s^2}{\pi})$$



The random $\mathcal{G}$(V,d) ensemble

$$P_{GOE}(s) = \frac{\pi s}{2} \exp(-\frac{\pi s^2}{4})$$



Spectral 2-points correlations:

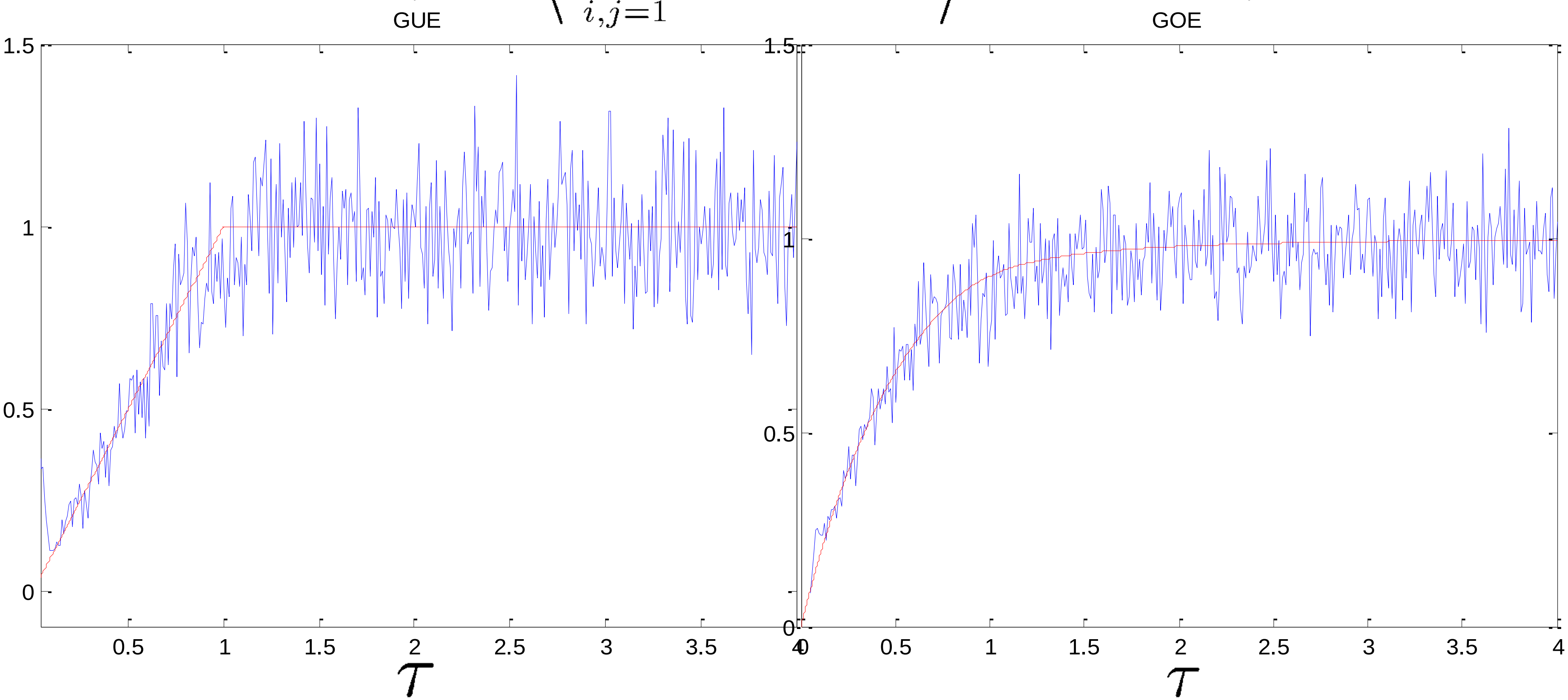
The Circular ensembles (COE,CUE):

$$\theta_j = 2\pi \frac{\mathcal{N}_{MK}(\lambda_j)}{V-1}$$ (mapping the spectrum on the unit circle)

The two points correlation function: $R_2(\eta) = \left\langle \frac{1}{(V-1)} \sum_{i,j}^{V-1} \delta(\eta - (\theta_i - \theta_j) \right\rangle$.

The "spectral form factor": Fourier transform of $R_2(\eta)$

$$K(\tau) = \frac{1}{V-1} \left\langle \sum_{i,j=1}^{V-1} \cos(\theta_i - \theta_j)t \right\rangle , \quad \text{and} \quad \tau = \frac{t}{V-1}$$



Why do random graphs display the canonical spectral statistics?

Counting statistics of cycles *vs* Spectral statistics

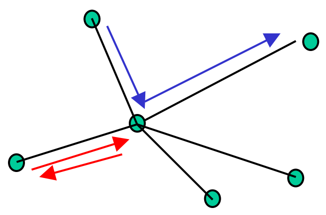
The main tool : Trace formulae connecting

spectral information

and

counts of periodic walks on the graph

The periodic walks to be encountered here are special:
Backscattering along the walk is forbidden.
Notation: non-backscattering walks = n.b. walks



Spectral Statistics

The fluctuating part of the spectral density :

$$\tilde{\rho}(\lambda) = \rho(\lambda) - \frac{d}{2\pi}\frac{\sqrt{4(d-1)-\lambda^2}}{d^2-\lambda^2} = \frac{1}{\pi}\sum_{t=3}^{\infty}\frac{y_t}{\sqrt{4(d-1)-\lambda^2}}\, T_t\left(\frac{\lambda}{2\sqrt{d-1}}\right)$$

Using the Orthogonality of the Chebyshev Polynomials:

$$y_t = 2\int_{-2\sqrt{d-1}}^{2\sqrt{d-1}} \mathrm{d}\lambda\ \tilde{\rho}(\lambda)\ T_t\left(\frac{\lambda}{2\sqrt{d-1}}\right) = 2\int_{-1}^{1}\mathrm{d}u\ \tilde{\rho}(u)\ T_t(u)$$

$$\langle y_t^2\rangle_{\mathcal{G}} = 4\int_{-1}^{1}\int_{-1}^{1} T_t(u)T_t(v)\ \langle\tilde{\rho}(u)\tilde{\rho}(v)\rangle_{\mathcal{G}}\ dudv$$

Map the spectrum to the unit circle: $\phi = \arccos u\ ,\ \ \phi\in[0,\pi]$

$$\heartsuit \qquad \langle y_t^2\rangle_{\mathcal{G}} = 4\int_0^{\pi}\int_0^{\pi}\cos t\phi\cos t\psi\ \boxed{\langle\tilde{\rho}(\phi)\tilde{\rho}(\psi)\rangle_{\mathcal{G}}}\ d\phi d\psi$$

Two-point correlation function. **However:** the spectral variables are not distributed uniformly and to compare with RMT they need **unfolding**

♡ $$\langle y_t^2\rangle_{\mathcal{G}} = 4\int_0^{\pi}\int_0^{\pi} \cos t\phi \cos t\psi \ \langle\tilde\rho(\phi)\tilde\rho(\psi)\rangle_{\mathcal{G}} \ d\phi d\psi$$

Define : $$\widetilde{K}_V(t) \equiv \frac{2}{V-1}\left\langle\left(\sum_{k=1}^{V-1}\cos(t\phi_k)\right)^2\right\rangle_{\mathcal{G}}$$

The (not unfolded) Spectral formfactor

Therefore

$$\langle y_t^2\rangle_{\mathcal{G}} = \frac{2}{V}\widetilde{K}_V(t)$$

Spectral form factor = variance of the number of t-periodic nb - walks

Since $\phi \in [0,\pi]$, the mean-spacing is $\frac{2\pi}{2(V-1)}$.

Define $\tau = \frac{t}{(V-1)}$.

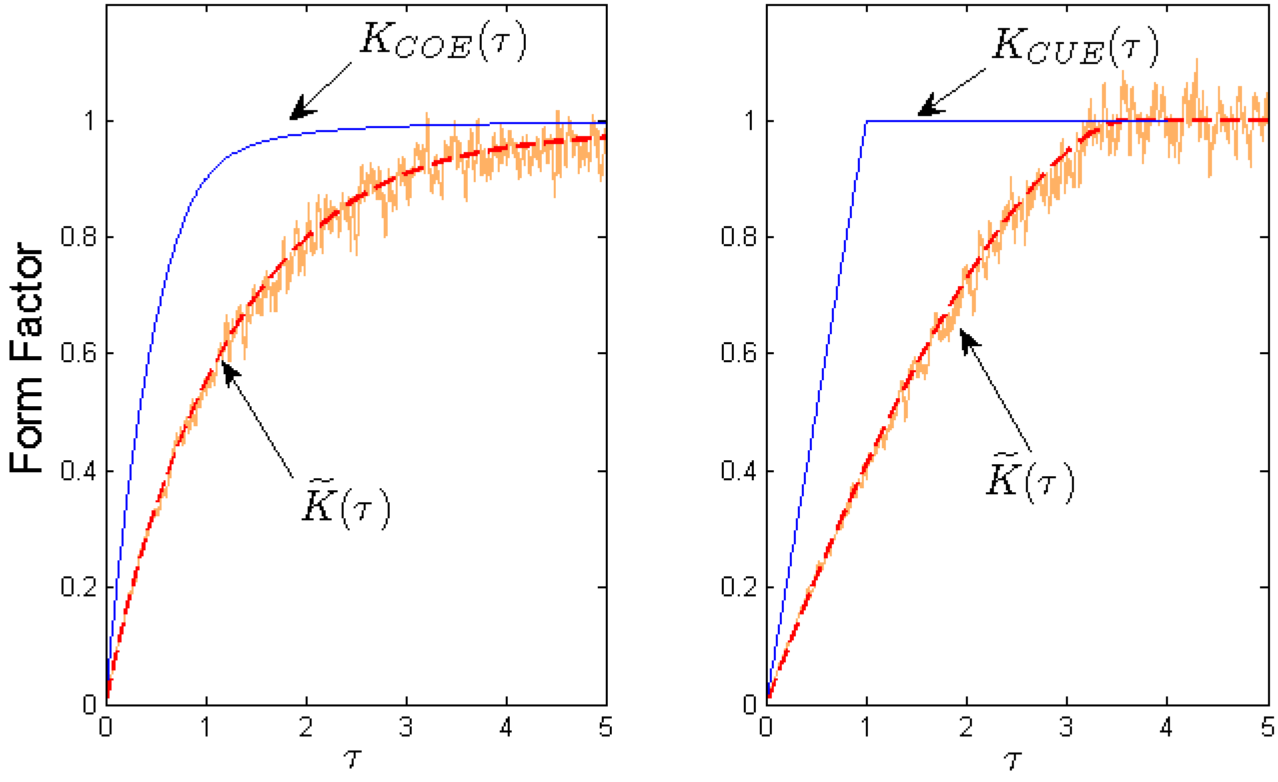
$$C_t = \frac{\mathrm{tr}Y^t}{2t} \approx \left\{ \begin{matrix}\text{\# t-periodic} \\ \text{nb cycles}\end{matrix} \right\}$$

For t < logV/log (d-1) C_t are distributed as a Poissonian variable
Hence: variance/mean =1 (Bollobas, Wormald, McKay)

$$\tilde{K}_V(t) = \frac{t}{V}\left\langle\frac{(C_t - \langle C_t\rangle_{\mathcal{G}})^2}{\langle C_t\rangle_{\mathcal{G}}}\right\rangle_{\mathcal{G}} \xrightarrow[\tau\to 0]{} \tau$$

$$K_V(t) = 2\tilde{K}(\tau) = 2\tau \text{ for } \tau \to 0$$

$$\tilde{K}_V(t) = 2\int_0^{\frac{\pi}{2}} \rho_{KM}(\phi) K\left(\frac{\tau}{2\pi\rho_{KM}(\phi)}\right) \mathrm{d}\phi \quad ; \quad \tau = \frac{t}{V} \ .$$



Conjecture (assuming RMTfor d-regular graphs):

Let C_t denote the number of t-periodic n.b. cycles on a (V,d) graph.
Then: $\frac{\langle (C_t - \langle C_t\rangle)^2\rangle}{\langle C_t\rangle} \to F_{GOE}(\tau)$ in the limit $t, V \to \infty$, $\tau = \frac{t}{V}$ constant.
In particular:
$F_{GOE}(\tau \to 0) \to 1$; $F_{GOE}(\tau \to \infty) \to \frac{1}{\tau}$.

$$\phi_j = \arccos \frac{\mu_j}{2\sqrt{d-1}} \quad ; \quad 0 \le \phi_j \le \pi \ .$$

The Kesten McKay density on the circle is

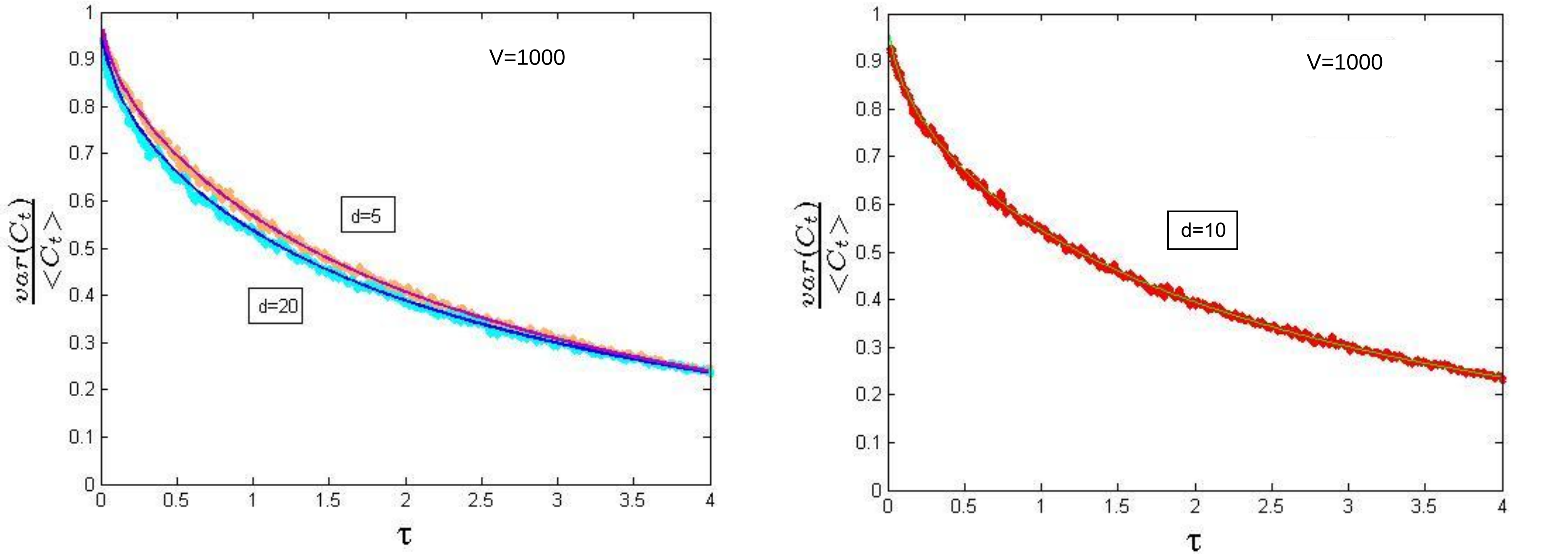
$$\rho_{KM}(\phi) = \frac{2(d-1)}{\pi d} \frac{\sin^2 \phi}{1 - \frac{4(d-1)}{d^2} \cos^2 \phi} \ .$$

$$F_{GOE}(\tau) = \left\langle \frac{(C_t - \langle C_t\rangle)^2}{\langle C_t\rangle} \right\rangle = \frac{2}{\tau} \int_0^{\frac{\pi}{2}} \rho_{KM}(\phi) K_{COE}\left(\frac{\tau}{2\pi\rho_{KM}(\phi)}\right) \mathrm{d}\phi \quad ; \quad \tau = \frac{t}{V} \ .$$

The explicit expressions for the COE is

$$K_{COE}(\tau) = \begin{cases} 2\tau - \tau \log(2\tau+1), & \text{for} \quad \tau < 1 \\ 2 - \tau \log \frac{2\tau+1}{2\tau-1}, & \text{for} \quad \tau > 1 \end{cases} \ .$$

$$\frac{\left\langle \left(C_t - \langle C_t \rangle\right)^2 \right\rangle}{\langle C_t \rangle} = \frac{2}{\tau} \int_0^{\frac{\pi}{2}} \rho_{KM}(\phi) K_{COE}\left(\frac{\tau}{2\pi\rho_{KM}(\phi)}\right) \mathrm{d}\phi \underset{\tau \to 0}{\longrightarrow} 1 + f_2(d)\sqrt{\tau} + \ldots$$



$$\lim_{\tau \to \infty} \tau \frac{var(C_t)}{\langle C_t \rangle} = 1, \quad \text{for } V, t \to \infty; \quad \frac{t}{V} = \tau$$

The magnetic adjacency spectral statistics

The non-backtracking magnetic connectivity:

$$Y^{(M)}_{e',e} = \mathrm{e}^{i\phi_{e'}/2}\ B_{e',e}\ \mathrm{e}^{i\phi_e/2}\ - J_{e',e}$$

$$\mathrm{tr}\left(\left(Y^{(M)}\right)^t\right) = \sum_{L_t} \mathrm{e}^{i\Phi} + \sum_{R_t} \mathrm{e}^{-i\Phi} + |S_t|$$

L_t = the set of t-periodic nb-walks going clockwise
R_t = the set of t-periodic nb-walks going counter-clockwise
S_t = the set of self-tracing t-periodic nb-walks
These are nb-walks which traverse each edge both ways

$$\langle \mathrm{tr}((Y^{(M)})^t)\rangle_{\mathcal{M}} = |S_t| \ \approx\ 0 \text{ for } \ t < \log_{d-1} V \quad ; \quad \text{Denote } \langle\cdot\rangle_{\mathcal{G}M} = \langle\cdot\rangle\ .$$

$$\left\langle \left(\mathrm{tr}\left(\left(Y^{(M)}\right)^t\right) - \left\langle \mathrm{tr}\left(\left(Y^{(M)}\right)^t\right)\right\rangle\right)^2 \right\rangle \approx |L_t| + |R_t| \ \approx\ 2\langle C_t\rangle \cdot t^2$$

$$\approx 4(d-1)^t \left\langle \left(\sum_k \cos(\phi_k)\right)^2 \right\rangle = 8t\langle C_t\rangle \left\langle \left(\sum_k \cos(\phi_k)\right)^2 \right\rangle$$

$$\rightarrow \quad \left\langle \left(\sum_k \cos(\phi_k)\right)^2 \right\rangle \approx \frac{t}{4}$$

By multiplying by $\frac{2}{V-1}$ we get that for short times: $\widetilde{K}(\tau) \approx \tau$.