Reminder from the previous lecture:

d-regular graphs :

Graphs where all the vertices have the same degree.

Examples:

1. Complete graphs
2. Lattices
3. d-regular infinite trees.

d-regular graphs are an expanding family.

$$E = \frac{Vd}{2} \quad \rightarrow \; V \text{ or } \; d \text{ must be even}$$

$$\beta = V(\tfrac{d}{2} - 1)$$

$\mathcal{G}_{V,d}$. The ensemble of all the d regular graphs with V vertices .

For fixed d and $V \to \infty$: $|\mathcal{G}_{V,d}| \approx \sqrt{2} \mathrm{e}^{\frac{1-d^2}{4}} \left(\frac{d^d V^d}{\mathrm{e}^d (d!)^2} \right)^{\frac{V}{2}}$

$\langle \cdots \rangle_{\mathcal{G}}$: Ensemble average taken with uniform probability distribution.

C_t : Number of t - periodic cycles with no back tracking

$$\langle C_t \rangle_{\mathcal{G}} = \frac{V \cdot d \cdot (d-1)^{t-1} \cdot \frac{d-1}{dV}}{2t} = \frac{(d-1)^t}{2t}$$

Hence, short t - perioic cycles with $t < \log_{d-1} V$ are **rare** .

The R-neighbourhood of every vertex for $R \leq log_{d-1}\frac{V}{2}$ is almost surely a d-regular tree.

The diameter of a $G(V,d)$ graph, i.e. the maximal distance between vertices in G, is given by

$$\mathrm{diam}(G) = \log_{d-1}(V \log_{d-1} V) + O(1)$$

Therefore, the typical distance between vertices along the boundary of the 'local tree' is of the same magnitude as the distance between two arbitrary vertices in G.

Denote by C_t the number of t- cycles (prinitive, non backscatter, non self intersecting t-periodic orbits).

For $t < log_{d-1}\frac{V}{2}$, the C_t distribute as independent Poisson variables with a mean: $\langle C_t \rangle_{\mathcal{G}} = \frac{(d-1)^t}{2t}$

Reminder : For d-regular graphs the spectra of A and L are the same but for a shift by d and sign change.

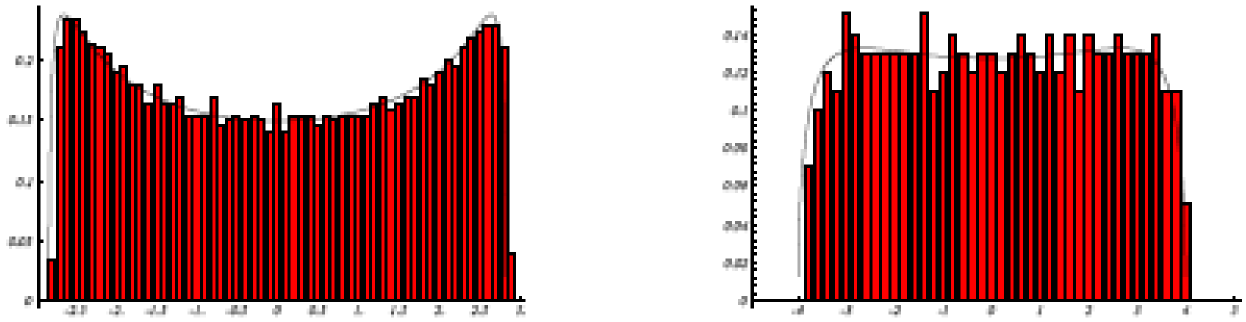
Spectral properties of d-regular graphs

The spectral density $\rho(\lambda) = \frac{1}{V-1} \sum_{k=1}^{V-1} \delta(\lambda - \lambda_k)$ (The trivial eigenvalue is excluded)

The mean spectral density for d regular graphs, $(V \to \infty\ ,\ d = const)$

Kesten MacKay limit distribution: (Supported in $|\lambda| < 2\sqrt{d-1}$):

$$\rho_{KM}(\lambda) = \lim_{V\to\infty} \frac{1}{V-1} \left\langle \sum_{k=1}^{V-1} \delta(\lambda - \lambda_k) \right\rangle_{\mathcal{G}} = \frac{d}{2\pi} \frac{\sqrt{4(d-1) - \lambda^2}}{d^2 - \lambda^2}$$



(a) Cubic graph on 2000 vertices. (b) 5-valent graph on 500 vertices.

Figure 1. Eigenvalue distributions of random graphs *vs* McKay's law

EIGENVALUE SPACINGS FOR REGULAR GRAPHS

DMITRY JAKOBSON, STEPHEN D. MILLER,
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Unfolding the spectrum with the Kesten-McKay density using $\mathcal{N}_{KM}$: The mean spectral counting function.

$$s_j = \mathcal{N}_{KM}(\lambda_j) \quad ; \quad \frac{\mathrm{d}\mathcal{N}_{KM}}{\mathrm{d}\lambda} = \rho_{KM}(\lambda) \;\; ; \; \mathrm{d}s = \rho_{KM}(\lambda)\mathrm{d}\lambda = \frac{\mathrm{d}\lambda}{\langle \mathrm{d}\lambda \rangle}$$

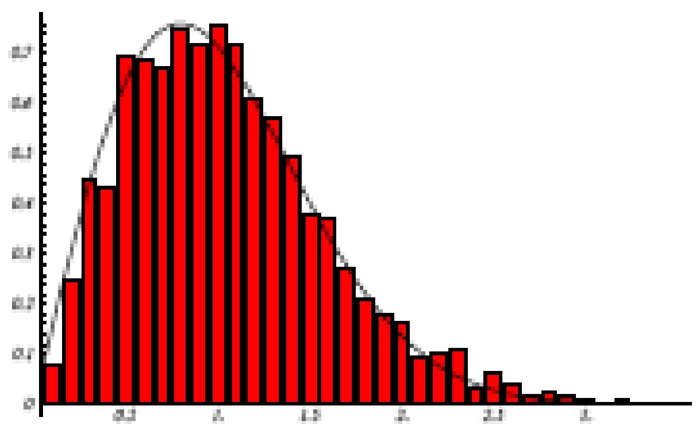


Figure 2. Level spacing distribution of a cubic graph on 2000 vertices *vs* GOE

2. The trace formula

Purpose: express the spectral density in terms of "periodic orbits"

Connectivity of directed edges

$$B_{e,e'} = \delta_{o(e),\tau(e')}$$

$$J_{e,e'} = \delta_{o(e),\tau(e')}\delta_{o(e'),\tau(e)}$$

$$Y = B - J$$

Y excludes back tracking connections!

Note: The number of non-backtracking periodic walks $= \mathrm{tr}Y^t$

However: The number of all periodic walks $= \mathrm{tr}A^t = \mathrm{tr}B^t$

Bartholdi's Identity for d-regular graphs

Define the two $2E \times V$ rectangular matrices

$$B^{(+)}_{e,i} := \begin{cases} 1 & \text{if } t(e) = i \\ \\ 0 & \text{otherwise} . \end{cases} \quad ; \quad B^{(-)}_{e,i} := \begin{cases} 1 & \text{if } o(e) = i \\ \\ 0 & \text{otherwise} . \end{cases}$$

Denoting by $\widetilde{X}$ the transpose of X, one can easily prove that

$$\begin{aligned} B^{(+)}\widetilde{B^{(-)}} = B \quad &; \quad \widetilde{B^{(-)}}B^{(+)} = A \\ \widetilde{B^{(+)}}B^{(+)} = dI^{(V)} \; &; \quad B^{(+)}\widetilde{B^{(+)}} = YJ + I^{(2E)} . \end{aligned} \tag{1}$$

For arbitrary complex s, w construct the two $(2E+V)\times(2E+V)$ square matrices

$$L = \begin{bmatrix} (1-w^2s^2)I^{(V)} & -\widetilde{B^{(-)}} + ws\widetilde{B^{(+)}} \\ \\ 0 & I^{(2E)} \end{bmatrix} \; ; \; M = \begin{bmatrix} I^{(V)} & \widetilde{B^{(-)}} - ws\widetilde{B^{(+)}} \\ \\ sB^{(+)} & (1-w^2s^2)I^{(2E)} \end{bmatrix}$$

Using the identities (1) one can compute the matrices LM and ML, and since their determinants are equal, one finally gets Bartholdi's identity.

$$\det(I^{(2E)} - s(B - wJ)) = (1 - w^2s^2)^{E-V} \det(I^{(V)}(1 + w(d-w)s^2) - sA) .$$

2.a The "canonical " trace formula $(w = 1)$

Bass' (Bartholdi's) identity :

$$\det(\eta I^{(2E)} - Y) = (\eta^2 - 1)^{E-V} \det(I^{(V)}(\eta^2 + (d-1)) - \eta A) \ .$$

η arbitrary complex number.

$I^{(2E)}$ and $I^{(V)}$ unit matrix in $2E$ and V dimensions.

$A_{i,j}$: Adjacency Matrix, $\dim A = V$; $B_{e',e} = \delta_{o(e'),\tau(e)}$, $\dim B = 2E$.

$J_{e',e} = \delta_{e',\hat{e}}$; $Y = B - J$: The Hashimoto connectivity matrix.

$$\sharp\{t\text{-periodic walks }\} = \ \mathrm{tr}B^t \ = \ \mathrm{tr}A^t.$$

$$\sharp\{t\text{-periodic non-backscattering walks }\} = \ \mathrm{tr}Y^t$$

The spectrum of A can be used to count n.b. periodic walks

and *vice versa*

The counts of n.b. periodic walks can be used to compute the spectrum of A

Bartholdi again : $\det(\eta I^{(2E)} - Y) = (\eta^2-1)^{E-V} \det(I^{(V)}(\eta^2+(d-1)) - \eta A)$.

The spectrum of Y: $\sigma(Y)=$

$$\left\{1\times(E-V), (-1)\times(E-V), \{\eta_l^{\pm} \ : \ \eta_l^{\pm\,2} + (d-1) - \lambda_l \eta_l^{\pm} = 0 \ , \ l = 1,\cdots,V\}\right\}$$

$$\eta_l^{\pm} = \frac{1}{2}\left(\lambda_l \pm i\sqrt{4(d-1)-\lambda_l^2}\right) \ .$$

$$\lambda_0 = d \quad \rightarrow \quad \eta_0^{\pm} = (d-1), 1$$

Assume Ramanujan: $\eta_l^{\pm} = \sqrt{d-1}\, \mathrm{e}^{\pm i\phi_l} \ ; \ \phi_l = \mathrm{arcos}\frac{\lambda_l}{2\sqrt{d-1}}$

$l = 1,\cdots,V-1$

The number of t-periodic non backscattering walks :

$$\mathrm{tr}\ Y^t = (d-1)^t + (E-V)(1+(-1)^t) + 1 + 2(d-1)^{\frac{t}{2}} \sum_{l=1}^{V-1} \cos t\phi_l \ .$$

Remember $E = \frac{Vd}{2}$ and use Chebyshev Polynomials : $T_n(x) = \cos\left(n \ \mathrm{arcos}(x)\right)$

$$\mathrm{tr}\ Y^t = (d-1)^t + V(\frac{d}{2}-1)(1+(-1)^t) + 1 + 2(d-1)^{\frac{t}{2}} \sum_{l=1}^{V-1} T_t\left(\frac{\lambda_l}{2\sqrt{d-1}}\right) \ .$$

$$y_t = \frac{1}{V}\frac{\mathrm{tr}\ Y^t - (d-1)^t - 1}{(d-1)^{\frac{t}{2}}} = \frac{(\frac{d}{2}-1)}{(d-1)^{\frac{t}{2}}}(1+(-1)^t) + \frac{2}{V}\sum_{l=1}^{V-1} T_t\left(\frac{\lambda_l}{2\sqrt{d-1}}\right).$$

Multiply by $\frac{1}{\pi(1+\delta_{0,n})}\ T_t\left(\frac{\lambda}{2\sqrt{d-1}}\right)$, and sum over t:

$$\frac{1}{\pi}\sum_{t=0}^{\infty}\frac{1}{1+\delta_{t,0}}\ T_t\left(\frac{\lambda}{2\sqrt{d-1}}\right)\ y_t =$$

$$= \frac{d-2}{2\pi}\sum_{t=0}^{\infty}\frac{1}{1+\delta_{t,0}}\frac{(1+(-1)^t)}{(d-1)^{\frac{t}{2}}}T_t\left(\frac{\lambda}{2\sqrt{d-1}}\right) + \frac{1}{V}\sum_{l=1}^{V-1}\delta_T\left(\frac{\lambda-\lambda_l}{2\sqrt{d-1}}\right).$$

$$\text{Where}:\ \delta_T(x-y) = \frac{2}{\pi}\sum_{t=0}^{\infty}\frac{1}{1+\delta_{t,0}}T_t(x)T_t(y)$$

$$\int_{-1}^{1}\frac{\mathrm{d}x}{\sqrt{1-x^2}}\delta_T(x-y)f(x) = f(y)$$

Note:

$\mathrm{tr}\ Y^0 = 2E \quad \rightarrow \quad y_0 = \frac{2E-1}{V} = d - \frac{1}{V}$

$\mathrm{tr}\ Y^1 = \mathrm{tr}\ Y^2 = 0 \quad \rightarrow \quad y_t = -\frac{1}{V}(d-1)^{\frac{t}{2}} \quad ; \quad t = 1, 2$

The trace formula:

$$\rho(\lambda) = \frac{d}{2\pi}\frac{\sqrt{4(d-1)-\lambda^2}}{d^2-\lambda^2} + \frac{1}{\pi}\mathcal{R}e\left[\sum_{t=3}^{\infty}\frac{y_t}{\sqrt{4(d-1)-\lambda^2}}\ \mathrm{e}^{it\arccos\frac{\lambda}{2\sqrt{d-1}}}\right]$$

Kesten McKay *Sum over t-periodic non-backscattering walks.*

$$y_t \;=\; \frac{1}{V}\frac{[\text{Number of t periodic n.b. walks}]-[(d-1)^t-1]}{(d-1)^{\frac{t}{2}}}.$$

y_t : The deviation of the number of t-periodic non-backscattering walks from its mean. (properly normalized)

Note: $\langle y_t\rangle_{\mathcal{G}} = 0$ for $t \le \log_{d-1} V$ (Bollobas, McKay, Wormald)

"Action" per single step on the orbit: $\arccos\frac{\lambda}{2\sqrt{d-1}}$

Alternatively:

$$\rho(\lambda) = \frac{d}{2\pi}\frac{\sqrt{4(d-1)-\lambda^2}}{d^2-\lambda^2} + \frac{1}{\pi}\sum_{t=3}^{\infty}\frac{y_t}{\sqrt{4(d-1)-\lambda^2}}\ T_t\left(\frac{\lambda}{2\sqrt{d-1}}\right)$$

$T_l(x)$: Chebyshev Polynomials of the first kind.

$$\mathcal{R} = \{\lambda : |\lambda| < 2(d-1)^{1/2}\ \}$$

Non Ramanujan graphs : $\mathcal{R}^c$ not empty.

$$\mathrm{tr}\, Y^t = (d-1)^t + (E-V)(1+(-1)^t) + 1 + 2(d-1)^{\frac{t}{2}} \left[\sum_{\lambda_l \in \mathcal{R}} \cos t\phi_l + \sum_{\lambda_l \in \mathcal{R}^c} \cosh t\psi_m \right] .$$

$$\psi_m = \mathrm{arcosh} \frac{|\lambda_m|}{2\sqrt{d-1}}$$

$$\mathrm{tr}\, Y^t = (d-1)^t + 2(d-1)^{\frac{t}{2}} \sum_{\lambda_l \in \mathcal{R}^c} \cosh t\psi_m + V(\frac{d}{2}-1)(1+(-1)^t) + 1 + 2(d-1)^{\frac{t}{2}} \sum_{\lambda_l \in \mathcal{R}} T_t \left(\frac{\lambda_l}{2\sqrt{d-1}} \right) .$$

$$y_t = \frac{1}{V} \frac{[\text{Number of t periodic n.b. walks}] - \left[(d-1)^t \left(1 + \frac{2}{(d-1)^{t/2}} \sum_{\lambda_l \in \mathcal{R}^c} \cosh t\psi_m \right) - 1 \right]}{(d-1)^{\frac{t}{2}}} .$$

y_t : The deviation of the number of t-periodic non-backscattering walks from its exponentially growing part (properly normalized).

$$[\cdots] \approx (d-1)^t \left(1 + \exp\left[-t \left(\frac{1}{2} \ln(d-1) - \left(\frac{V^{-0.6}}{\sqrt{d-1}} \right)^{\frac{1}{2}} \right) \right] \right) - 1$$

2.b A continuous family of trace formulas

Back to Bartholdi's identity :

$$\det(\eta I^{(2E)} - (B - wJ)) = (\eta^2 - w^2)^{E-V} \det(I^{(V)}(\eta^2 + w(d-w)) - \eta A) \, .$$

The spectrum of $Y(w) = B - wJ$ reads:

$$(d-w), \;\; w, \;\; (w, -w) \times (E - V), \;\; \left(\sqrt{w(d-w)} \; \mathrm{e}^{\pm i\phi_k} \right)_{k=1,V-1}$$

For $1 < w < d-1, \;\; 2\sqrt{d-1} \; < 2\sqrt{w(d-w)} \; .$

Choose w such that $\lambda_1 < 2\sqrt{w(d-w)} \;\; \rightarrow \;\; \phi_k = \arccos \frac{\lambda_k}{2\sqrt{w(d-w)}}$ are real.

Y(*w*) gives a weight (1-*w*) to back scattering!

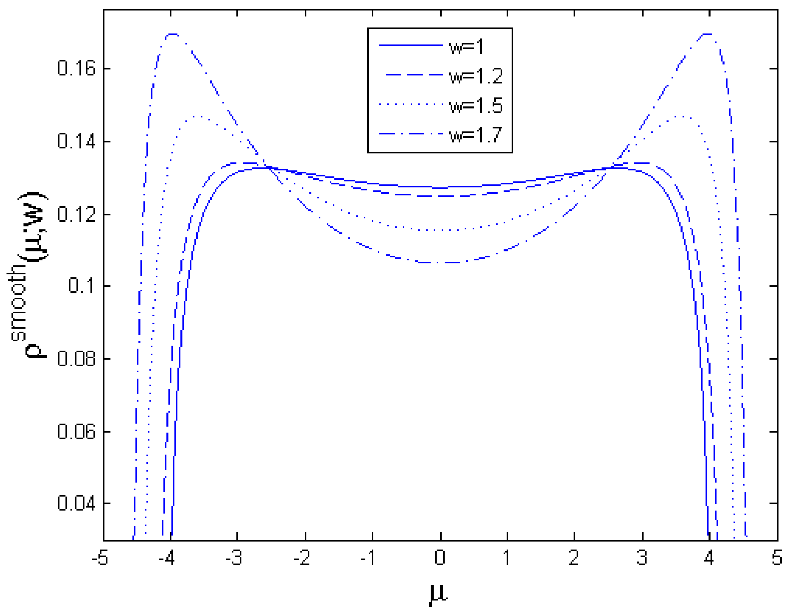
$$\text{Define :} \qquad y_t(w) \;=\; \frac{1}{V}\frac{\mathrm{tr}\; Y(w)^t - ((d-w)^t + w^t)}{(w(d-w))^{\frac{t}{2}}}$$

The w - trace formula:

$$\rho(\lambda) = \rho^{smooth}(\lambda; w) + \frac{1}{\pi}\mathcal{R}e\left[\sum_{t=3}^{\infty} \frac{y_t(w)}{\sqrt{4w(d-w) - \lambda^2}}\; e^{it\frac{\lambda}{2\sqrt{w(d-w)}}}\right]$$

$$\rho^{smooth}(\lambda; w) = \frac{d}{2\pi\sqrt{4w(d-w) - \lambda^2}} \cdot \left(1 - \frac{(d-2w)(d-2)}{d^2 - \lambda^2} + 2\frac{(w-1)^2(2\lambda^2 - 1)}{w(d-w)}\right)$$

The *w* - trace formula includes counting of backscattering walks weighted by (1-*w*) per backscatter!



2.c The spectrum of equilateral quantum graphs from Bartholdi

For an equilateral d-regular quantum graph with $L = \pi$

$U = e^{ik\pi}\left(\frac{2}{d}B - J\right)$.

Taking $\eta = \frac{d}{2}e^{-ik\pi}$, $w = \frac{d}{2}$, Bartholdi gives:

$\det(I^{(2E)} - U(k)) = C(1 - e^{2ik\pi})^{E-V}\prod_{l=1}^{V}(\cos k\pi - \frac{\lambda_l}{d})$

$$\frac{1}{V}\rho_Q(k) = \frac{1}{2}\frac{\sqrt{4(d-1) - d^2\cos^2(k\pi)}}{|\sin(k\pi)|}\cdot \mathbb{I}_R + (\frac{d-2}{2})\delta(k-1)|$$

$$\mathbb{I}_R = \begin{cases} 1 & \text{if } k \in [\frac{1}{\pi}\arccos(\frac{2\sqrt{d-1}}{d}), 1 - \frac{1}{\pi}\arccos(\frac{2\sqrt{d-1}}{d})] \\ 0 & \text{otherwise} \end{cases} .$$

The spectrum for the equi lateral graph is periodic consisting of two components.

When the lengths are randomized $l_{ij} = \pi(1 + \epsilon r_{ij})$ with $|r_{ij}| \leq 1$ random uniformly distributed, the numerical distribution is: