WEAK COUPLINGS IN THE ONE-COMPONENT PLASMA IN TWO DIMENSIONS FOR A DISK WITH A CENTERED IMPENETRABLE NEUTRALIZING CHARGE

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Problem statement

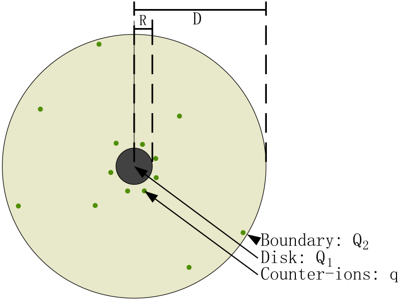

Figure : The 2D cell model. The disk with charge Q_1 and radius R is surrounded by counter-ions of charge $-q$ enclosed by an exterior boundary at D with charge Q_2.

ion–disk: Manning parameter

$$\xi = \frac{1}{2}\beta Q_1 q \propto \frac{N}{T}. \quad (1)$$

ion–ion: coupling parameter Γ

$$\Xi = \frac{1}{2}\beta q^2 \propto \frac{1}{T}. \quad (2)$$

Often recalled Γ in the literature

ion–boundary:

$$\xi_B = \frac{1}{2}\beta Q_2 q \propto \frac{N}{T}. \quad (3)$$

Same meaning as the Manning parameter

lateral extension parameter

$$\Delta = \log \frac{D}{R} \quad (4)$$

Problem statement

$$\beta H = 2\xi \sum_{j=1}^{N} \log \left| \frac{\mathbf{r}_j}{\tilde{R}} \right| - 2\Xi \sum_{1 \leqslant j < k \leqslant N} \log \left| \frac{\mathbf{r}_j - \mathbf{r}_k}{R} \right| + \underbrace{\frac{[N\Xi - \xi]^2}{\Xi} \Delta + N\Xi \log \frac{R}{L}}_{\tilde{E}_B} \qquad (1)$$

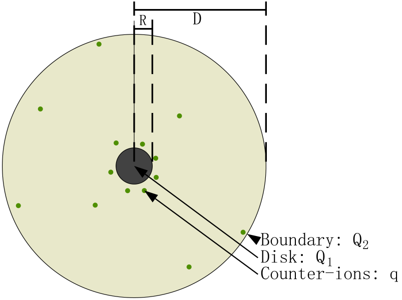

Figure : The 2D cell model. The disk with charge Q_1 and radius R is surrounded by counter-ions of charge $-q$ enclosed by an exterior boundary at D with charge Q_2.

Neutrality and the thermodynamic limit

Neutrality $N\Xi = \xi + \xi_B$.
The thermodynamic limit, or $N \to \infty$, is equivalent to $\Xi \to 0$ at constant ξ and ξ_B

Different cases

- $\Xi \to 0$
- $\Xi \to \infty$
- Ξ is a whole number

Characteristics

Mean field theory, to this problem often attributed to Fuoss et al. (1951), Katchalsky et al. (1953). To our problem it is equivalent to $\Xi \to 0$
Method: Poisson-Boltzmann equation

Different cases

- $\Xi \to 0$
- $\Xi \to \infty$
- Ξ is a whole number

Characteristics

The strong coupling regime (Šamaj and Trizac 2011a;b)
Method: Wigner Strong Coupling approach

Different cases

- $\Xi \to 0$
- $\Xi \to \infty$
- Ξ is a whole number

Characteristics

Method: Analytic

- $\Xi = 1$: Free fermion (Deutsch and Lavaud 1974, Deutsch et al. 1979, Jancovici 1981)
- $\Xi = 2, 3, \ldots$ method proposed by Šamaj et al. (1994)

Different cases

- □ $\Xi \to 0$
- □ $\Xi \to \infty$
- ‡ Ξ is a whole number
- ⋆ Ξ is small

Objectives

The weak coupling regime (Burak and Orland 2006)

- Determine $\mathcal{Z}$
- Derive the profile
- Recover mean field results
- Determine condensation (MF $\to$ $f_M = 1 - 1/\xi$)

The partition function $\mathcal{Z}$

$$\mathcal{Z}_N = \int \mathsf{d}^{2N} r \, e^{-\beta H(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)}. \tag{2}$$

Using a complex variable notation the Hamiltonian reads,

$$\beta H = 2\xi \sum_{j=1}^{N} \log \|z_j\| - \Xi \sum_{1 \leqslant j < k \leqslant N} \log\left[(z_j - z_k)(\bar{z}_j - \bar{z}_k)\right] + \widetilde{E}_B, \tag{3}$$

and the partition function is rewritten as,

$$\mathcal{Z}_N \propto \int \mathsf{D}^N z \left[\prod_{1 \leqslant j < k \leqslant N} \|z_j - z_k\|^2 \right]^{\Xi} \prod_{j=1}^{N} \|z_j\|^{-2\xi}. \tag{4}$$

The partition function $\mathcal{Z}$

$$\mathcal{Z}_N \propto \int \mathsf{D}^N z \left[\prod_{1 \leqslant j < k \leqslant N} \|z_j - z_k\|^2 \right]^{\Xi} \prod_{j=1}^{N} \|z_j\|^{-2\xi} \,. \tag{2}$$

Which brings us back to $\Xi \in \{1, 2, \dots\}$ since,

$$\left[\prod_{1 \leqslant j < k \leqslant N} (z_j - z_k) \right] = \mathsf{Det}\left[\mathsf{V}_{N \times N}\right], \tag{3}$$

with

$$\mathsf{V}_{N \times N} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ z_1 & z_2 & z_3 & \dots & z_N \\ z_1^2 & z_2^2 & z_3^2 & \dots & z_N^2 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ z_1^{N-1} & z_2^{N-1} & z_3^{N-1} & \dots & z_N^{N-1} \end{bmatrix} \tag{4}$$

Weak coupling approximation

Proposed by Burak and Orland (2006),

$$2\log|\mathbf{r}_1-\mathbf{r}_2| = \log\|\mathbf{r}_1\| + \log\|\mathbf{r}_2\| + \log\left[2\cosh\left(\log\|\mathbf{r}_1\| - \log\|\mathbf{r}_2\|\right) - 2\cos\theta_{12}\right] \approx 2\log|\mathbf{r}_>|\,, \quad (5)$$

transforming the Hamiltonian to,

$$\beta H \approx \frac{2\xi}{\Xi}\sum_{j=1}^{N} y_j - 2\sum_{1\leqslant j<k\leqslant N} y_>^{(j,k)} + \tilde{E}_B\,, \quad (6)$$

with $y = \Xi\log(r/R)$ (a.k.a. *centrifugal variables*).

- Distance between particles is large
- Dropped angular correlations
- Ξ needs to be small
- 2D $\rightarrow$ 1D.
- $y_>$ suggests arrangement

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- ⋆ Distance between particles is large
- ⋆ Dropped angular correlations
- ⋆ Ξ needs to be small
- ⋆ 2D $\rightarrow$ 1D.
- ⋆ $y_>$ suggests arrangement

Departing from the base order, denoted by **[BO]**, $y_1 < y_2 < \cdots < y_N$, the N-dimensional phase space is mapped entirely from permutations of such arrangement.

Weak coupling $\mathcal{Z}$

Written in terms of the $\{y_k\}$ variables,

$$\mathcal{Z}(\Xi, \xi, N, \Delta) = \left(\frac{2\pi R^2}{\Xi}\right)^N \int \mathrm{d}^N y \, e^{-\beta H + \frac{2}{\Xi} \sum_{j=1}^N y_j} , \tag{7}$$

transforms to,

$$\mathcal{Z}(\Xi, \xi, N, \Delta) = e^{-\widetilde{E}_B} \left(\frac{2\pi R^2}{\Xi}\right)^N N! \int_0^{\Xi\Delta} \mathrm{d}y_N \int_0^{y_N} \mathrm{d}y_{N-1} \ldots \int_0^{y_2} \mathrm{d}y_1 \, e^{-\mathcal{H}}, \tag{8}$$

with

$$\mathcal{H} = \frac{2(\xi - 1)}{\Xi} \sum_{j=1}^N y_j - 2 \sum_{1 \leqslant j < k \leqslant N} y_k = \sum_{j=1}^N \left[\frac{2(\xi - 1)}{\Xi} - 2(j-1)\right] y_j \tag{9}$$

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with

$$\mathcal{H} = \sum_{j=0}^{N} a_j (y_{j+1} - y_j) - a_N \Xi\Delta, \tag{9}$$

choosing $y_0 = 0$ and $y_{N+1} = \Xi\Delta$, and,

$$a_j = \left[j - \left(\frac{\xi - 1}{\Xi} + \frac{1}{2}\right)\right]^2 = \left[j - \left(\frac{f_M}{1 + \frac{\xi_B}{\xi}} N + \frac{1}{2}\right)\right]^2 = [j - \sqrt{a_0}]^2 \tag{10}$$

Rationale: Use the Laplace transformation to find the partition function Burak and Orland (2006).

Weak coupling $\mathcal{Z}$

Using the notation $f_j(x) = e^{-a_j x}$,

$$\mathcal{Z}(\Xi, \xi, N, \Delta) = \left(\frac{2\pi R^2}{\Xi}\right)^N N!\, e^{a_N \Xi\Delta - \widetilde{E}_B} \times \underbrace{\int_0^{\Xi\Delta} \mathrm{d}y_N \int_0^{y_N} \mathrm{d}y_{N-1} \dots \int_0^{y_2} \mathrm{d}y_1 \prod_{j=0}^{N} f_j\left(y_{j+1} - y_j\right)}_{\left[f_N \otimes f_{N-1} \otimes \cdots \otimes f_1 \otimes f_0\right](\Xi\Delta)} . \tag{11}$$

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$$\mathcal{T}^{[\Xi\Delta]}_{\{f_N \otimes \cdots \otimes f_0\}}(s) = \prod_{j=0}^{N} \frac{1}{s + a_j} \quad (12)$$

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$$\mathcal{Z}(\Xi, \xi, N, \Delta) = \left(\frac{2\pi R^2}{\Xi}\right)^N N!\, e^{a_N \Xi\Delta - \widetilde{E}_B} \times \underbrace{\int_0^{\Xi\Delta} \mathrm{d}y_N \int_0^{y_N} \mathrm{d}y_{N-1} \ldots \int_0^{y_2} \mathrm{d}y_1 \prod_{j=0}^{N} f_j\left(y_{j+1} - y_j\right)}_{[f_N \otimes f_{N-1} \otimes \cdots \otimes f_1 \otimes f_0](\Xi\Delta)}. \quad (11)$$

$$\mathcal{T}^{[\Xi\Delta]}_{\{f_N \otimes \cdots \otimes f_0\}}(s) = \prod_{j=0}^{N} \frac{1}{s + a_j} \quad (12)$$

Leading to the inverse

- Anticipating, $\mathcal{Z} \propto \sum_j e^{-a_j \Xi\Delta}$
- Δ needs to be large
- $\mathcal{Z} \sim e^{-a_{j^\star} \Xi\Delta}$
- $\{a_j\}$'s may be degenerate

Who is $j^\star$

$j^\star$ is the integer closes to $\sqrt{a_0}$.

$$\sqrt{a_0} = \frac{\xi - 1}{\Xi} + \frac{1}{2} = \frac{f_M}{1 + \frac{\xi_B}{\xi}} N + \frac{1}{2}$$

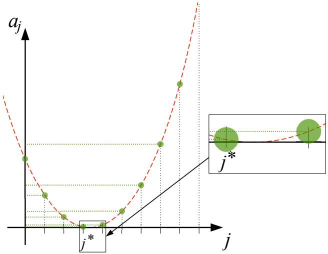

Figure : Artistic representation of a_j as a function of j with $j^\star$ the location of the minimum

Who is $j^\star$

$j^\star$ is the integer closes to $\sqrt{a_0}$.

$$j^\star = \left\lceil \frac{\xi - 1}{\Xi} \right\rceil = \left\lceil \frac{f_M N}{1 + \frac{\xi_B}{\xi}} \right\rceil . \qquad (13)$$

Condensation

Notice how

$$\frac{j^\star}{N} \to \frac{f_M}{1 + \frac{\xi_B}{\xi}}$$

in the thermodynamic limit

Transitions

Since $\mathcal{Z} \sim e^{-a_{j^\star}\Xi\Delta}$ then at a change in $j^\star$ the behavior of the system will change!

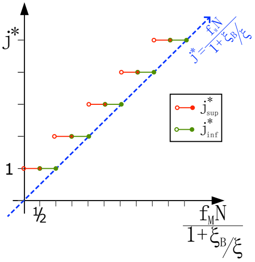

Figure : The value for $j^\star$ as a function of $f_M N/(1 + \xi_B/\xi)$.

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Degeneracy

Depends if

$$\frac{2}{\Xi}\left(1 + \xi_B\right) \in \mathbb{N},$$

which will be even or odd

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$$\mathcal{T}^{[\Xi\Delta]}_{\{f_N \otimes \cdots \otimes f_0\}}(s) = \prod_{j=0}^{N} \frac{1}{s + a_j}$$

Non-degenerate case

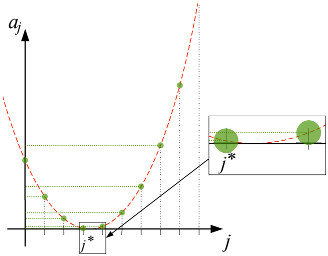

Figure : Artistic representation of a_j as a function of j with $j^\star$ the location of the minimum

$$\mathcal{Z}(\Xi, \xi, N, \Delta) = \left(\frac{2\pi R^2}{\Xi}\right)^N N!\, e^{a_N \Xi\Delta - \widetilde{E}_B} \sum_{j=0}^{N} \underbrace{\left[\prod_{k=0, k\neq j}^{N} \frac{1}{a_k - a_j}\right]}_{\text{Defined as } C_{0,N;j}} e^{-a_j \Xi\Delta} . \qquad (14)$$

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$j^\star$ is the integer closes to $\sqrt{a_0}$.

$$j^\star = \left\lceil \frac{\xi - 1}{\Xi} \right\rceil = \left\lceil \frac{f_M N}{1 + \frac{\xi_B}{\xi}} \right\rceil . \qquad (13)$$

Degeneracy

Depends if

$$\frac{2}{\Xi}\left(1 + \xi_B\right) \in \mathbb{N},$$

which will be even or odd

Even degenerate case

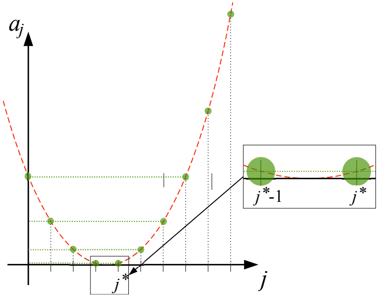

Figure : Artistic representation of a_j as a function of j with $j^\star$ the location of the minimum with $2(1+\xi_B)/\Xi$ an even number.

Who is $j^\star$

$j^\star$ is the integer closes to $\sqrt{a_0}$.

$$j^\star = \left\lceil \frac{\xi - 1}{\Xi} \right\rceil = \left\lceil \frac{f_M N}{1 + \frac{\xi_B}{\xi}} \right\rceil . \qquad (13)$$

Degeneracy

Depends if

$$\frac{2}{\Xi}\left(1 + \xi_B\right) \in \mathbb{N},$$

which will be even or odd

Odd degenerate cases

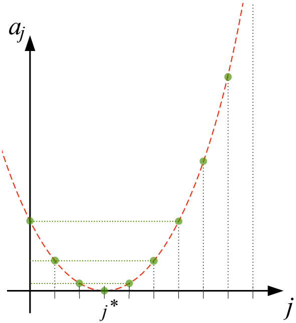

Figure : Artistic representation of a_j as a function of j with $j^\star$ the location of the minimum with $2(1 + \xi_B)/\Xi$ an odd number

Who is $j^\star$

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$$j^\star = \left\lceil \frac{\xi - 1}{\Xi} \right\rceil = \left\lceil \frac{f_M N}{1 + \frac{\xi_B}{\xi}} \right\rceil . \qquad (13)$$

Degeneracy

Depends if

$$\frac{2}{\Xi}(1 + \xi_B) \in \mathbb{N},$$

which will be even or odd

Degenerate cases Due to the continuity of the free energy → the partition function, they are a limiting behavior of the non-degenerate scenario.

The free energy

The excess free energy is,

$$\beta F_{exc}(N,\Delta,\xi,\Xi) \ = \ -\log\left[\frac{1}{V}\mathcal{Z}(\Xi,\xi,N,\Delta)\right], \tag{14}$$

keeping in mind,

$$\mathcal{Z}(\Xi,\xi,N,\Delta) = \underbrace{\left(\frac{2\pi\, R^2}{\Xi}\right)^N N!\, e^{a\,N\,\Xi\Delta - \widetilde{E}_B}}_{1} \underbrace{\sum_{j=0}^{N} C_{0,N;j}\, e^{-a_j\,\Xi\Delta}}_{2}. \tag{15}$$

Dominant behavior

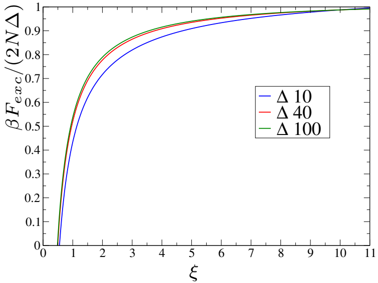

Figure : The excess free energy dominant term coming from 1 for $N = 10$ and $\xi_B = 0$

Subdominant behavior

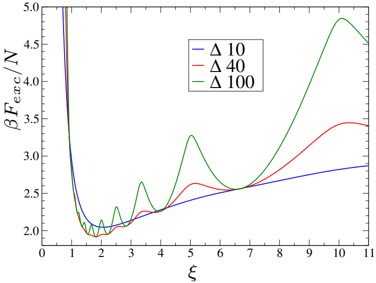

Figure : The excess free energy sub-dominant term coming from 2 for $N = 10$ and $\xi_B = 0$

Density profile

As for the profile,

$$\rho = \frac{N}{\mathcal{Z}(\Xi, \xi, N, \Delta)} \int \mathsf{d}^N \mathbf{r}\, \delta\left(\mathbf{r} - \mathbf{r}_1\right)\, e^{-\beta H} = \frac{\Xi}{2\pi R^2 e^{2y/\Xi}} \rho_y \,, \tag{16}$$

with

$$\rho_y := N \left\langle \delta\left(y - y_1\right) \right\rangle_{\{y_j\}} \,. \tag{17}$$

Since the average must be consistent with the **[BO]**. Hence,

$$\rho_y = \left\langle \delta\left(y - y_1\right) \right\rangle^T_{\{y_j\}} + \left\langle \delta\left(y - y_2\right) \right\rangle^T_{\{y_j\}} + \cdots + \left\langle \delta\left(y - y_N\right) \right\rangle^T_{\{y_j\}} \,, \tag{18}$$

and,

$$\left\langle \delta\left(y - y_k\right) \right\rangle^T_{\{y_j\}} = \left(\frac{2\pi\, R^2}{\Xi} \right)^N \frac{e^{-\widetilde{E}_B}\, N!}{\mathcal{Z}(\Xi, \xi, N, \Delta)} \int_{\textbf{[BO]}} \mathsf{d}^N y\, \delta\left(y - y_k\right) e^{-\mathcal{H}'} \tag{19}$$

Density profile

As for the profile,

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with

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and,

$$\langle \delta\left(y - y_k\right) \rangle^T_{\{y_j\}} = \frac{\left\{\sum_{j=k}^{N} C_{k,N;j}\, e^{-a_j(\Xi\Delta - y)}\right\} \left\{\sum_{j=0}^{k-1} C_{0,k-1;j}\, e^{-a_j y}\right\}}{\sum_{j=0}^{N} C_j\, e^{-a_j \Xi\Delta}} \tag{18}$$

Density profile

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Characteristics

- ⋆ Upfront decay of $1/r^2$ (MF)
- ⋆ Behavior near R
- ⋆ Behavior near D
- ▸ Condensation $\Rightarrow j^\star$

Density profile

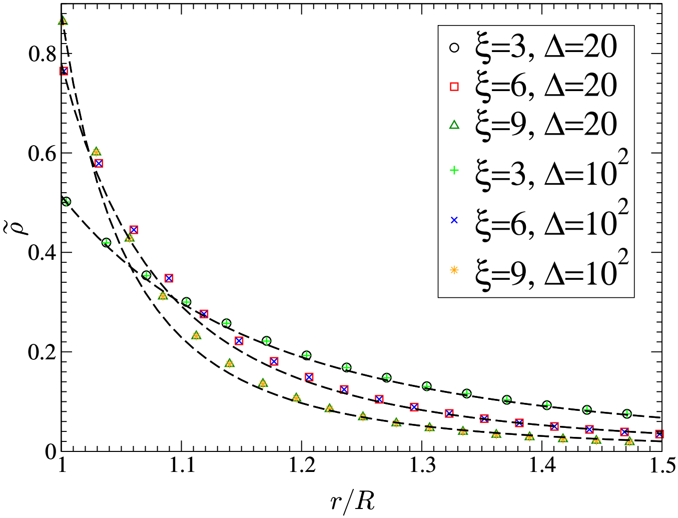

Figure : The density profile $\tilde{\rho} = 2\pi R^2\, \rho/(N\xi)$ near the charged disk for different values of the Manning parameter for $N = 10$ and $\xi_B = 0$.

Density profile at infinite dilution

Near R, ($k \leqslant j^\star$)

$$\langle \delta\left(y - y_k\right)\rangle^T_{\{y_j\}} \simeq \sum_{j=0}^{k-1} \frac{C_{0,k-1;j}}{C_{0,k-1;j^\star}}\, e^{-(a_j - a_{j^\star})y} + \mathcal{O}\left(e^{-\Xi\Delta}\right), \tag{19}$$

and for the exterior shell ($k > j^\star$),

$$\langle \delta\left(y - y_k\right)\rangle^T_{\{y_j\}} \simeq \sum_{j=k}^{N} \frac{C_{k,N;j}}{C_{k,N;j^\star}}\, e^{-(a_j - a_{j^\star})(\Xi\Delta - y)} + \mathcal{O}\left(e^{-\Xi\Delta}\right). \tag{20}$$

Density profiles near mean field

At infinite dilution, using $\tilde{\rho} = 2\pi R^2 \rho/(N\xi)$,

$$\tilde{\rho} = \frac{\Xi\, e^{-2y/\Xi}}{N\xi} \sum_{k=1}^{j^\star} \left\{ \sum_{j=0}^{k-1} \frac{C_{0,k-1;j}}{C_{0,k-1;j^\star}} \right\} e^{-(a_j - a_{j^\star})y} \to \frac{f_M^2}{1+\frac{\xi_B}{\xi}} \left(\frac{\widetilde{R}}{r}\right)^2 \frac{1}{\left(1+(\xi-1)\log\frac{\tilde{r}}{\widetilde{R}}\right)} \tag{21}$$

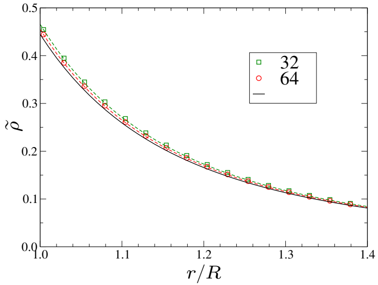

Figure : Density profile $\tilde{\rho} = 2\pi R^2\, \rho/(N\xi)$ for $\xi = 3$ and $\Delta = 100$.

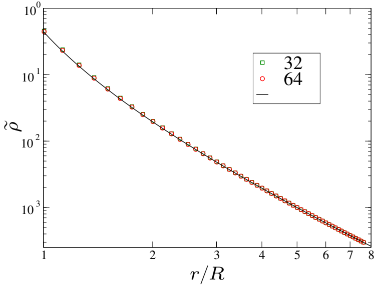

Figure : Same as the figure to the left with logarithmic scales on both axis.

Contact densities

As for the value of the densities at contact, at R it gives

$$\tilde{\rho}|_{y=0} = \left(f_M - \left[f_M - \frac{j^\star}{N} \right] \right) \left(f_M + \left[f_M - \frac{j^\star}{N} \right] + \frac{1}{N} \right), \tag{22}$$

and for D,

$$e^{2\Delta} \tilde{\rho}|_{y=\Delta} = \frac{1}{N\,\xi} (a_N - a_{j^\star}) = \left(\frac{1}{\xi} - \left[f_M - \frac{j^\star}{N} \right] - \frac{1}{N} \right) \left(\frac{1}{\xi} + \left[f_M - \frac{j^\star}{N} \right] \right), \tag{23}$$

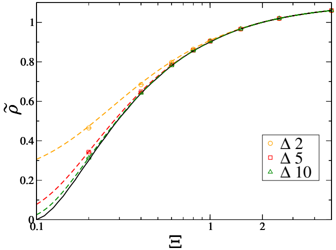

Figure : The density $\tilde{\rho}$ at contact in $r = R$ for $N = 10$ and $\xi_B = 0$.

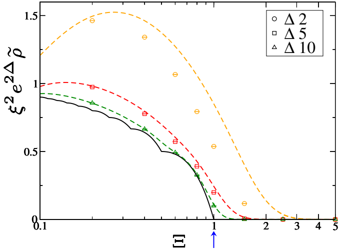

Figure : The density $\tilde{\rho}$ at contact in $r = D$ for $N = 10$ and $\xi_B = 0$.

Integrated charge

- Mean field behavior at large distances?
- Condensation
- Linear form at the transitions
- Full condensation for $\Xi > 1 \Rightarrow$ Strong couplings

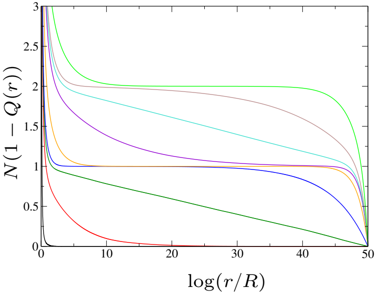

Figure : The integrated charge $N(1 - Q(r))$ as a function of the logarithmic distance for $\Delta = 10^2$, $\xi_B = 0$ and $N = 10$ for various Ξ. The plots read for the coupling parameter from top to bottom $\Xi = \frac{2}{5}, \frac{10}{21}, \frac{1}{2}, \frac{10}{19}, \frac{2}{3}, \frac{10}{11}, 1, \frac{10}{9}$ and 2.

Integrated charge

- Mean field behavior at large distances?
- Condensation
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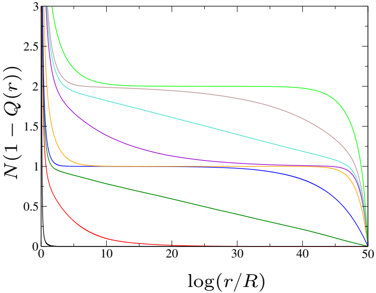
$$f_{2D} = \frac{j^\star}{N} \tag{24}$$



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Integrated charge

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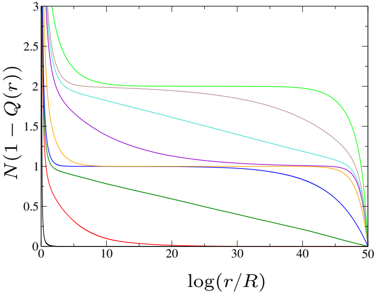
At the transition the $j^{\star\text{th}}$ particle is **not** condensed



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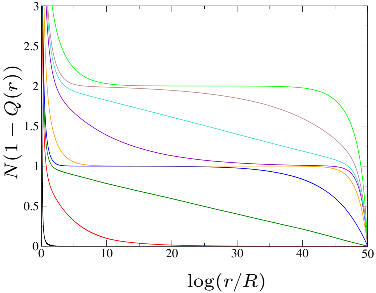
Integrated charge

- Mean field behavior at large distances?
- Condensation
- Linear form at the transitions
- Full condensation for $\Xi > 1 \Rightarrow$ Strong couplings
- Mean field result is recovered in the thermodynamic limit



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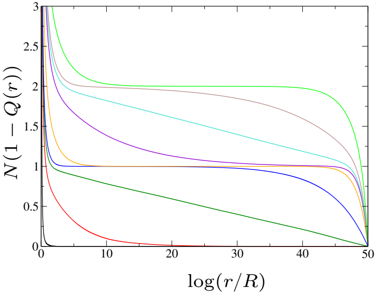
Integrated charge

- Mean field behavior at large distances?
- Condensation
- Linear form at the transitions
- Full condensation for $\Xi > 1 \Rightarrow$ Strong couplings

$$f_{2D} \to f_M \tag{24}$$



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Conclusions

- Showed an equivalent 1D problem of the 2D system
- With some effort we computed $\mathcal{Z}$ and ρ
- Recovered mean field results from our assumptions
- Determined condensation

References I

Yoram Burak and Henri Orland. Manning condensation in two dimensions. *Phys. Rev. E*, 73:010501, Jan 2006. doi: 10.1103/PhysRevE.73.010501. URL http://link.aps.org/doi/10.1103/PhysRevE.73.010501.

C. Deutsch and M. Lavaud. Equilibrium properties of a two-dimensional coulomb gas. *Phys. Rev. A*, 9: 2598–2616, Jun 1974. doi: 10.1103/PhysRevA.9.2598. URL http://link.aps.org/doi/10.1103/PhysRevA.9.2598.

C. Deutsch, H. E. Dewitt, and Y. Furutani. Debye thermodynamics for the two-dimensional one-component plasma. *Phys. Rev. A*, 20:2631–2633, Dec 1979. doi: 10.1103/PhysRevA.20.2631. URL http://link.aps.org/doi/10.1103/PhysRevA.20.2631.

Raymond M. Fuoss, Aharon Katchalsky, and Schneor Lifson. The Potential of an Infinite Rod-Like Molecule and the Distribution of the Counter Ions. *Proceedings of the National Academy of Sciences*, 37(9):579–589, 1951. URL http://www.pnas.org/content/37/9/579.short.

L. Šamaj and E. Trizac. Counter-ions at charged walls: Two-dimensional systems. *The European Physical Journal E: Soft Matter and Biological Physics*, 34(2):1–14, 2011a.

Ladislav Šamaj and Emmanuel Trizac. Wigner-crystal formulation of strong-coupling theory for counterions near planar charged interfaces. *Phys. Rev. E*, 84:041401, Oct 2011b. doi: 10.1103/PhysRevE.84.041401. URL http://link.aps.org/doi/10.1103/PhysRevE.84.041401.

B. Jancovici. Exact Results for the Two-Dimensional One-Component Plasma. *Phys. Rev. Lett.*, 46:386–388, Feb 1981. doi: 10.1103/PhysRevLett.46.386. URL http://link.aps.org/doi/10.1103/PhysRevLett.46.386.

A. Katchalsky, S. Lifson, and J. Mazur. The electrostatic free energy of polyelectrolyte solutions. I. Randomly kinked macromolecules. *Journal of Polymer Science*, 11(5):409–423, 1953. ISSN 1542-6238. doi: 10.1002/pol.1953.120110503. URL http://dx.doi.org/10.1002/pol.1953.120110503.

L. Šamaj, J. K. Percus, and M. Kolesík. Two-dimensional one-component plasma at coupling Γ=4: Numerical study of pair correlations. *Phys. Rev. E*, 49:5623–5627, Jun 1994. doi: 10.1103/PhysRevE.49.5623. URL http://link.aps.org/doi/10.1103/PhysRevE.49.5623.

Thank you for your attention!