Top eigenvalue of a random matrix: Large deviations

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First Appearence of Random Matrices

Biometrika, 20, 32-52 (1928)

THE GENERALISED PRODUCT MOMENT DISTRIBUTION IN SAMPLES FROM A NORMAL MULTIVARIATE POPULATION.

By JOHN WISHART, M.A., B.Sc. Statistical Department, Rothamsted Experimental Station.

1. *Introduction.*

For some years prior to 1915, various writers struggled with the problems that arise when samples are taken from uni-variate and bi-variate populations, assumed in most cases for simplicity to be normal. Thus "Student," in 1908*, by considering the first four moments, was led by K. Pearson's methods to infer the distribution of standard deviations, in samples from a normal population. His results, for comparison with others to be deduced later, will be stated in the form

$$dp = \frac{1}{\Gamma\left(\frac{N-1}{2}\right)} A^{\frac{N-1}{2}} . e^{-Aa} . a^{\frac{N-3}{2}} da \quad \text{.....................(1)},$$

where N is the size of the sample, and

$$A = \frac{N}{2\sigma^2}, \quad a = s^2,$$

σ being the standard deviation of the sampled population, and s that estimated from the sample. Thus, if $x_1, x_2, \ldots x_N$ are the sample values,

$$N\bar{x} = \sum_1^N (x),$$

and

$$Ns^2 = \sum_1^N (x - \bar{x})^2.$$

When bi-variate populations were considered, other problems arose, such as the distribution of the correlation coefficient and of the regression coefficient in samples. These problems, taken by themselves, were found to be difficult, and only approximative results had been reached, when, in 1915, R. A. Fisher† gave a formula for the simultaneous distribution of the three quadratic statistical derivatives, namely the two variances (squared standard deviations) and the product moment coefficient. Thus, let $x_1, x_2, \ldots x_N$ represent the sample values of the

Covariance Matrix

phys. math

$$X = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \begin{vmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \\ X_{31} & X_{32} \end{vmatrix}$$

in general (MxN)

$$X^t = \begin{vmatrix} X_{11} & X_{21} & X_{31} \\ X_{12} & X_{22} & X_{32} \end{vmatrix}$$

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$$W = X^t X = \begin{vmatrix} X_{11}^2 + X_{21}^2 + X_{31}^2 & X_{11}X_{12} + X_{21}X_{22} + X_{31}X_{32} \\ X_{12}X_{11} + X_{22}X_{21} + X_{32}X_{31} & X_{12}^2 + X_{22}^2 + X_{32}^2 \end{vmatrix}$$

→ (NxN) COVARIANCE MATRIX (unnormalized)

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Null model → random data: X → random $(M \times N)$ matrix

→ $W = X^t X$ → random $N \times N$ matrix (Wishart, 1928)

RMT in Nuclear Physics: Eugene Wigner

SESSION IIB

INTERPRETATION OF LOW ENERGY NEUTRON SPECTROSCOPY

CHAIRMAN—W. W. Havens, Jr.

IIB1. DISTRIBUTION OF NEUTRON RESONANCE LEVEL SPACING.

E. P. WIGNER, *Princeton University*
Presented by E. P. Wigner

The problem of the spacing of levels is neither a terribly important one nor have I solved it. That is really the point which I want to make very definitely. As we go up in the energy scale it is evident that the detailed analyses which we have seen for low energy levels is not possible, and we can only make

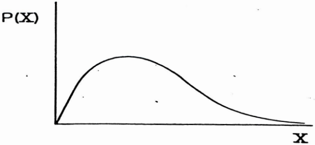

Fig. IIB1-1. Probability of a level spacing X.

ɔur, that is a much more serious deviation and much ɜss probable statistically.

Let me say only one more word. It is very likely ıat the curve in Figure I is a universal function. ı other words, it doesn't depend on the details of ıe model with which you are working. There is one articular model in which the probability of the ıergy levels can be written down exactly. I men-oned this distribution already in Gatlinburg. It is alled the Wishart distribution. Consider a set of ymmetric matrixes in such a way that the diagonal .ement m_{11} has a distribution $\exp(-m_{11}^2/4)$. In other ords, the probability that this diagonal element ıall assume the value m_{11} is proportional to ɛp$(-m_{11}^2/4)$. Then as I mentioned, and this was ıown a long time ago by Wishart, the probability for ıe characteristic roots to be $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$, if ıis is an n dimensional matrix, is given by the ɛpression:

$$^{-1/2}(\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2)\,[(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\ldots\ldots(\lambda_{n-1} - \lambda_n)].$$

bility that two successive roots have a distance X, then you have to integrate over all of them except two. This is very easy to do for the first integration, possible to do for the second integration, but when you get to the third, fourth and fifth, etc., integrations you have the same problem as in statistical mechanics, and presumably the solution of the problem will be accomplished by one of the methods of statistical mechanics. Let me only mention that I did integrate over all of them except one, and the result is $\frac{1}{2\pi}\sqrt{4n - \lambda^2}$. This is the probability that the root shall be λ. All I have to do is to integrate over one less variable than I have integrated over, but this I have not been able to do so far.

DISCUSSION

W. HAVENS: Where does one find out about a Wishart distribution?

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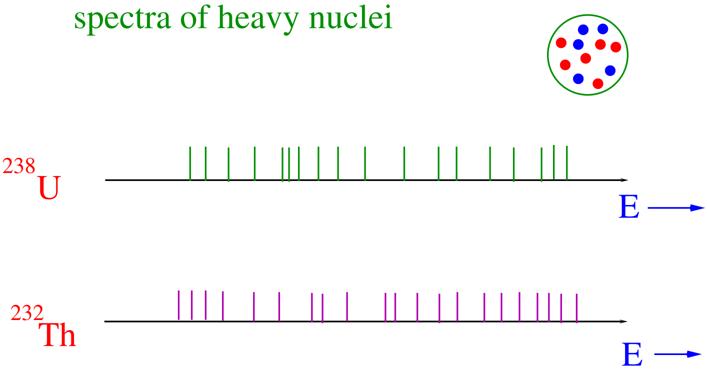
W. HAVENS: Where does one find out about a Wishart distribution?

E. WIGNER: A Wishart distribution is given in S. S. Wilks book about statistics and I found it just by accident.

Random Matrices in Nuclear Physics



WIGNER ('50) : replace complex H by random matrix

DYSON, GAUDIN, MEHTA, ...

Applications of Random Matrices

Physics: nuclear physics, quantum chaos, disorder and localization, mesoscopic transport, optics/lasers, quantum entanglement, neural networks, gauge theory, QCD, matrix models, cosmology, string theory, statistical physics (growth models, interface, directed polymers...), ...

Mathematics: Riemann zeta function (number theory), free probability theory, combinatorics and knot theory, determinantal points processes, integrable systems, ...

Statistics: multivariate statistics, principal component analysis (PCA), image processing, data compression, Bayesian model selection, ...

Information Theory: signal processing, wireless communications, ..

Biology: sequence matching, RNA folding, gene expression network ...

Economics and Finance: time series analysis,....

Recent Ref: The Oxford Handbook of Random Matrix Theory
ed. by G. Akemann, J. Baik and P. Di Francesco (2011)

Spectral Statistics in Random Matrix Theory (RMT)

Working model: real, symmetric $N \times N$ Gaussian random matrix

$$J = \begin{pmatrix} J_{11} & J_{12} & \dots & J_{1N} \\ J_{12} & J_{22} & \dots & J_{2N} \\ \dots & \dots & \dots & \dots \\ J_{1N} & J_{2N} & \dots & J_{NN} \end{pmatrix}$$

$$\mathrm{Prob.}[J] \propto \exp\left[-\frac{1}{2}\sum_{i,j} J_{ij}^2\right]$$

$$= \exp\left[-\frac{1}{2}\,\mathrm{Tr}\left(J^2\right)\right]$$

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N real eigenvalues: $\lambda_1, \lambda_2, \dots, \lambda_N \longrightarrow$ strongly correlated

Spectral statistics in RMT $\Rightarrow$ statistics of $\{\lambda_1, \lambda_2, \dots, \lambda_N\}$

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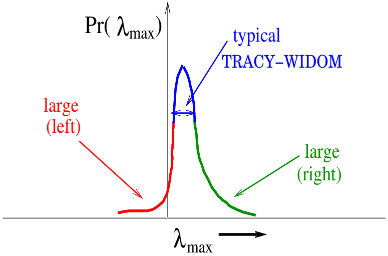

Typical fluctuations (small)
$\Rightarrow$ Tracy-Widom distribution
$\rightarrow$ ubiquitous

[directed polymer, random permutation, growth models, KPZ equation, sequence alignment, ...]

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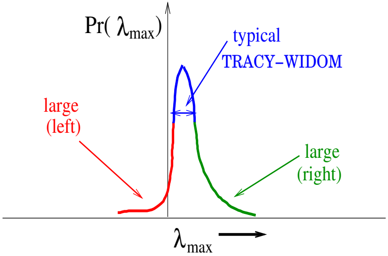

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This talk $\Rightarrow$ Atypical rare fluctuations $\Rightarrow$ large deviation functions

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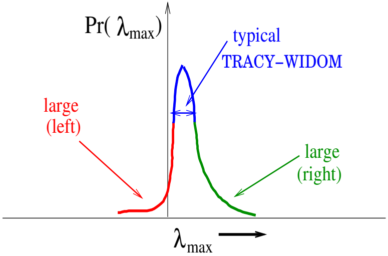

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$\Rightarrow$ 3-rd order phase transition

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- Summary and Generalizations

I. Why $\lambda_{\max}$?

R.M. May, Nature, 238, 413 (1972)

GENERAL

Will a Large Complex System be Stable?

Gardner and Ashby[1] have suggested that large complex systems which are assembled (connected) at random may be expected to be stable up to a certain critical level of connectance, and then, as this increases, to suddenly become unstable. Their conclusions were based on the trend of computer studies of systems with 4, 7 and 10 variables.

Here I complement Gardner and Ashby's work with an analytical investigation of such systems in the limit when the number of variables is large. The sharp transition from stability to instability which was the essential feature of their paper is confirmed, and I go further to see how this critical transition point scales with the number of variables n in the system, and with the average connectance C and interaction magnitude α between the various variables. The object is to clarify the relation between stability and complexity in ecological systems with many interacting species, and some conclusions bearing on this question are drawn from the model. But, just as in Gardner and Ashby's work, the formal development of the problem is a general one, and thus applies to the wide range of contexts spelled out by these authors.

Specifically, consider a system with n variables (in an ecological application these are the populations of the n interacting species) which in general may obey some quite nonlinear set of first-order differential equations. The stability of the possible equilibrium or time-independent configurations of such a system may be studied by Taylor-expanding in the neighbourhood of the equilibrium point, so that the stability of the possible equilibrium is characterized by the equation

$$\mathbf{dx}/\mathrm{d}t = \mathbf{Ax} \quad (1)$$

Here in an ecological context $\mathbf{x}$ is the $n\times 1$ column vector of the disturbed populations x_j, and the $n\times n$ interaction matrix $\mathbf{A}$ has elements a_{jk} which characterize the effect of species k on species j near equilibrium[2,3]. A diagram of the trophic web immediately determines which a_{jk} are zero (no web link), and the type of interaction determines the sign and magnitude of a_{jk}.

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$J_{ij} \to (N \times N)$ random interaction matrix
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- Question: What is the probabality that the system remains stable once the interaction is switched on?

(R.M. May, Nature, 238, 413, 1972)

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- Prob.(the system is stable)=Prob.$[\lambda_{\max} < w] = P(w, N)$

Cumulative distribution of the top eigenvalue

Stable-Unstable Phase Transition as $N \to \infty$

- Assuming that the interaction matrix $J_{ij} \to$ Real Symmetric Gaussian

$$\mathrm{Prob.}[J_{ij}] \propto \exp\left[-\frac{N}{2}\sum_{i,j} J_{ij}^2\right] \propto \exp\left[-\frac{N}{2}\mathrm{Tr}(J^2)\right]$$

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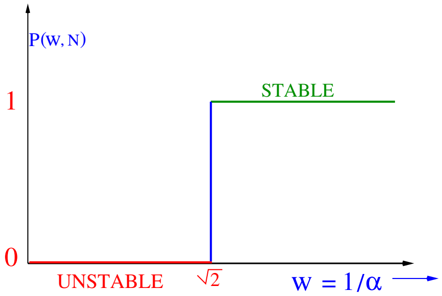
- May observed a sharp phase transition as $N \to \infty$:

$$w = \frac{1}{\alpha} > \sqrt{2} \Rightarrow \text{Stable (weakly interacting)}$$
$$w = \frac{1}{\alpha} < \sqrt{2} \Rightarrow \text{Unstable (strongly interacting)}$$

Prob.(the system is stable)=Prob.$[\lambda_{\max} < w]$ $= P(w, N)$



Finite but Large N:

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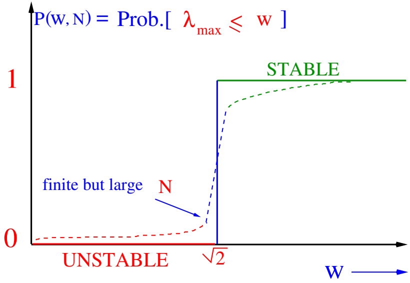
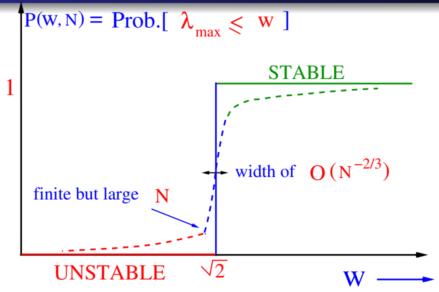


- Is there any thermodynamic sense to this phase transition?
- What is the analogue of free energy?
- What is the order of this phase transition?

II. Summary of Results

For Large but Finite N: Summary of Results



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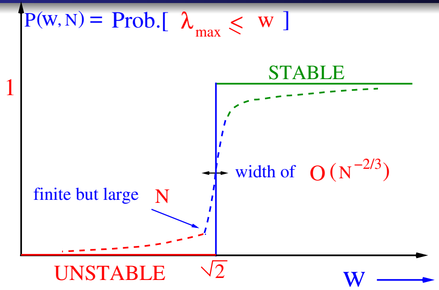

$$P(w,N) \sim \exp\left[-N^2\Phi_-(w)+\ldots\right] \quad \text{for} \quad \sqrt{2}-w\sim O(1)$$

$$\sim F_1\left[\sqrt{2}\,N^{2/3}\left(w-\sqrt{2}\right)\right] \quad \text{for} \quad |w-\sqrt{2}|\sim O(N^{-2/3})$$

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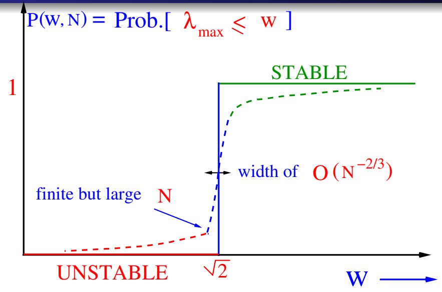

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\end{aligned}
$$

Crossover function: $F_1(z) \to$ Tracy-Widom (1994)

Exact tail functions: $\Phi_{\mp}(w)$ (Dean & S.M., 2006, S.M. & Vergassola, 2009)

Higher order corrections: (Borot, Eynard, S.M., & Nadal 2011, Nadal & S.M., 2011)

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- Left large deviation function:

$$\Phi_-(w) = \frac{1}{108}\left[36w^2 - w^4 - (15w + w^3)\sqrt{w^2+6} + 27\left(\ln(18) - 2\ln(w + \sqrt{6+w^2})\right)\right] \quad \text{where} \quad w < \sqrt{2}$$

[D. S. Dean & S.M., PRL, 97, 160201 (2006)]

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[S.M. & M. Vergassola, PRL, 102, 060601 (2009)]

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[S.M. & M. Vergassola, PRL, 102, 060601 (2009)]

As $w \to \sqrt{2}$ (from right), $\Phi_+(w) \to \frac{2^{7/4}}{3}(w-\sqrt{2})^{3/2}$

Large Deviation Functions

These large deviation functions $\Phi_{\pm}(w)$ have been found useful in a large variety of problems:

[Fyodorov 2004, Fyodorov & Williams 2007, Bray & Dean 2007, Auffinger, Ben Arous & Cerny 2010, Fydorov & Nadal 2012.... —— stationary points on random Gaussian surfaces and spin glass landscapes]

[Cavagna, Garrahan, Giardina 2000,... —— Glassy systems]

[Susskind 2003, Douglas et. al. 2004, Aazami & Easther 2006, Marsh et. al. 2011, ...—— String theory & Cosmology]

[Beltrani 2007, Dedieu & Malajovich, 2007, Houdre 2011...——Random Polynomials, Random Words (Young diagrams)

3-rd Order Phase Transition

$$P(w,N) \approx \begin{cases} \exp\left\{-N^2\Phi_-(w)+\ldots\right\} & \text{for } w<\sqrt{2} \qquad \text{(unstable)} \\ 1-\exp\left\{-N\Phi_+(w)+\ldots\right\} & \text{for } w>\sqrt{2} \qquad \text{(stable)} \end{cases}$$

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$$\lim_{N\to\infty} -\frac{1}{N^2}\ln\left[P(w,N)\right] = \begin{cases} \Phi_-(w) \sim (\sqrt{2}-w)^3 & \text{as } w\to\sqrt{2}^- \\ 0 & \text{as } w\to\sqrt{2}^+ \end{cases}$$

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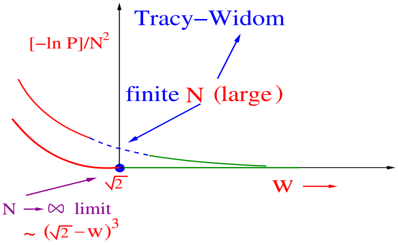

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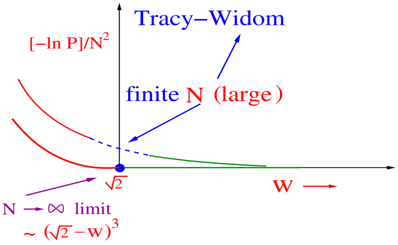


3-rd derivative $\to$ discontinuous

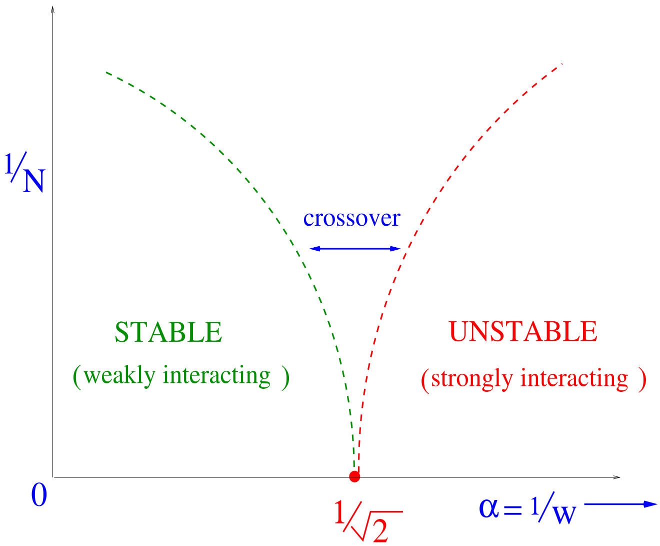
Crossover: $N\to\infty$, $w\to\sqrt{2}$ keeping

$(w-\sqrt{2})\,N^{2/3}$ fixed

$P(w,N) \to F_1\left[\sqrt{2}\,N^{2/3}\,\left(w-\sqrt{2}\right)\right]$

$\to$ Tracy-Widom

Large N Phase Transition: Phase Diagram



PHYSICAL REVIEW D VOLUME 21, NUMBER 2 15 JANUARY 1980

Possible third-order phase transition in the large-N lattice gauge theory

David J. Gross
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540

Edward Witten
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 10 July 1979)

The large-N limit of the two-dimensional U(N) (Wilson) lattice gauge theory is explicitly evaluated for all fixed $\lambda = g^2 N$ by steepest-descent methods. The λ dependence is discussed and a third-order phase transition, at $\lambda = 2$, is discovered. The possible existence of such a weak- to strong-coupling third-order phase transition in the large-N four-dimensional lattice gauge theory is suggested, and its meaning and implications are discussed.

Volume 93B, number 4 PHYSICS LETTERS 30 June 1980

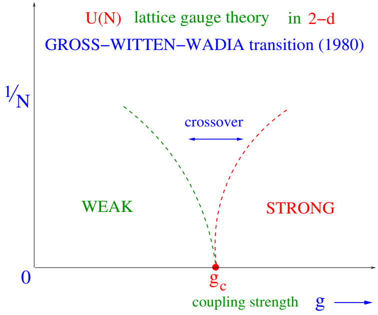
$N = \infty$ PHASE TRANSITION IN A CLASS OF EXACTLY SOLUBLE MODEL LATTICE GAUGE THEORIES ★

Spenta R. WADIA
The Enrico Fermi Institute, University of Chicago, Chicago, IL 60637, USA

Received 27 March 1980

A nice review of large-N gauge theory: M. Marino, arXiv:1206.6272

Large N Phase Transition: Phase Diagram



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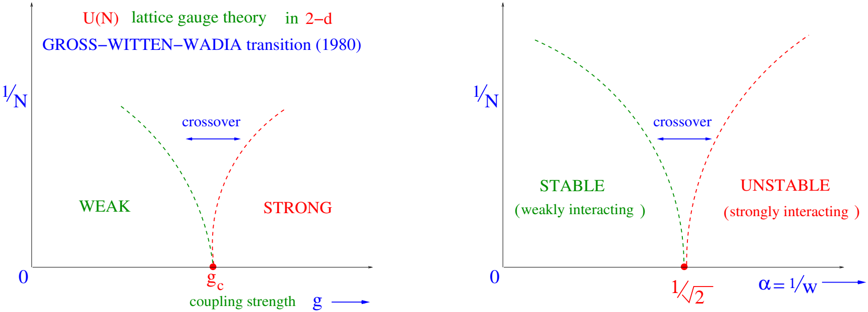

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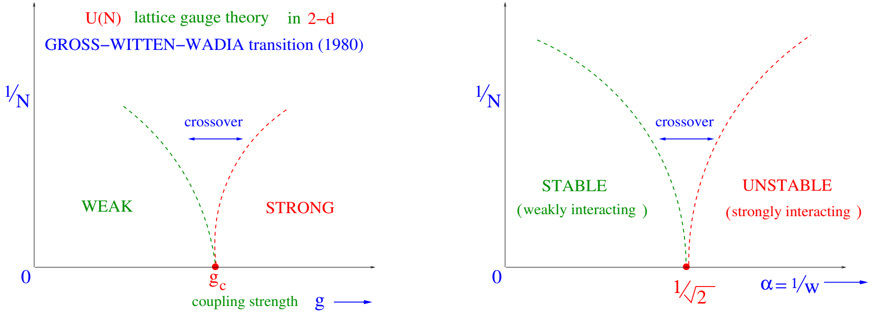

Similar 3-rd order phase transition in $U(N)$ lattice-gauge theory in 2-d

Unstable phase $\equiv$ Strong coupling phase of Yang-Mills gauge theory
Stable phase $\equiv$ Weak coupling phase of Yang-Mills gauge theory

Tracy-Widom $\Rightarrow$ crossover function in the double scaling regime (for finite but large N)

III. Coulomb Gas

Gaussian Random Matrices

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- Ensembles: Orthogonal (GOE), Unitary (GUE) or Symplectic (GSE)

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- Joint distribution of eigenvalues (Wigner, 1951)

$$P(\lambda_1, \lambda_2, \ldots, \lambda_N) = \frac{1}{Z_N} \exp\left[-\frac{\beta}{2} N \sum_{i=1}^{N} \lambda_i^2\right] \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

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- $Z_N =$ Partition Function

$$= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \{\prod_i d\lambda_i\} \exp\left[-\frac{\beta}{2}\, N \sum_{i=1}^{N} \lambda_i^2\right] \prod_{j<k} |\lambda_j - \lambda_k|^{\beta}$$

Coulomb Gas Interpretation

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$$\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \{\prod_i d\lambda_i\} \exp\left[-\frac{\beta}{2}\left\{\sum_{i=1}^{N} N\,\lambda_i^2 - \sum_{j\neq k} \log|\lambda_j - \lambda_k|\right\}\right]$$

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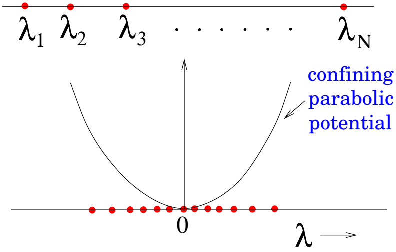

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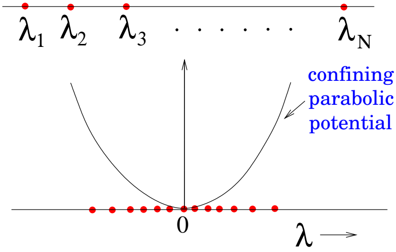

- Balance of energy $\Rightarrow N^2\,\lambda^2 \sim N^2$
- Typical eigenvalue: $\lambda_{\rm typ} \sim O(1)$ for large N

Spectral Density: Wigner's Semicircle Law

- Av. density of states: $\rho(\lambda, N) = \langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \rangle$

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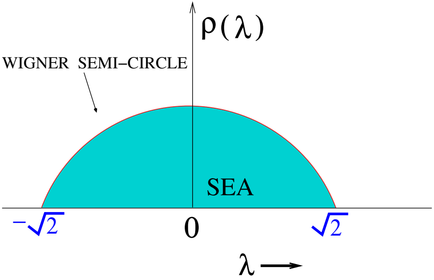

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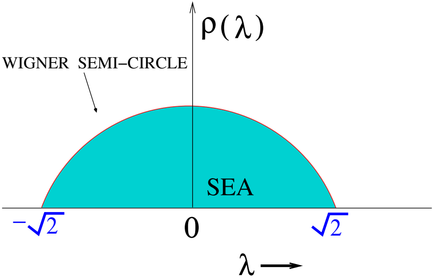
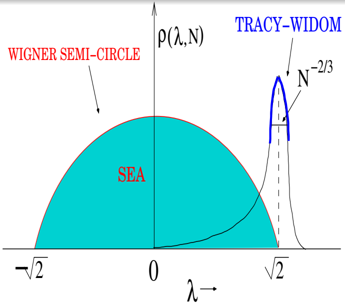
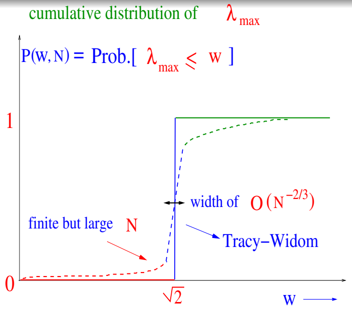


- $\langle \lambda_{\max} \rangle = \sqrt{2}$ for large N.
- $\lambda_{\max}$ fluctuates from one sample to another. $\text{Prob}[\lambda_{\max}, N] = ?$

Tracy-Widom distribution for $\lambda_{\max}$





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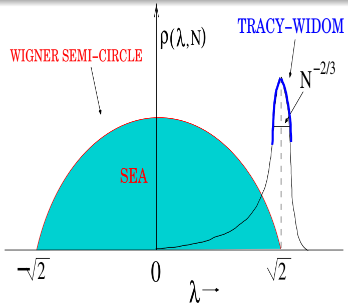

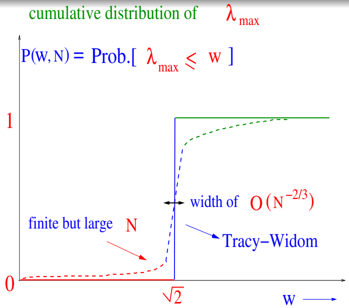

- $\langle \lambda_{\max} \rangle = \sqrt{2}$; typical fluctuation: $\int_{\lambda_{\max}}^{\infty} \rho(\lambda)\, d\lambda \sim 1/N$

Using $\rho(\lambda) \sim (\sqrt{2} - \lambda)^{1/2} \Rightarrow |\lambda_{\max} - \sqrt{2}| \sim N^{-2/3} \rightarrow$ small

[Bowick & Brezin '91, Forrester '93]

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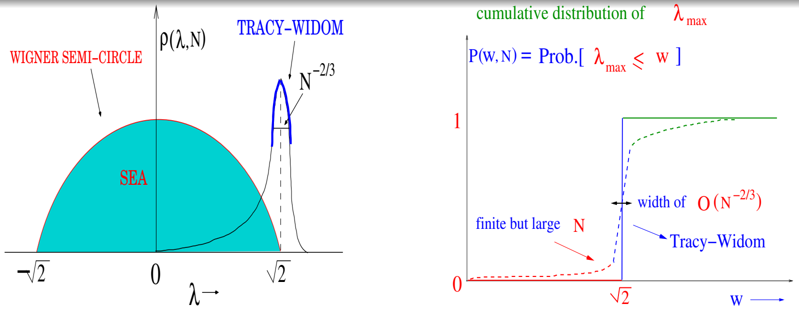

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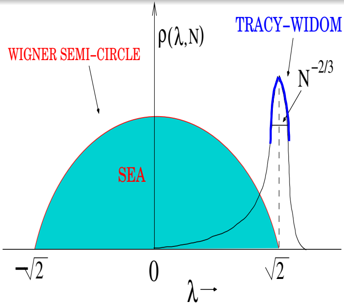

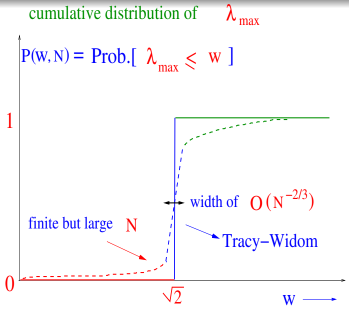

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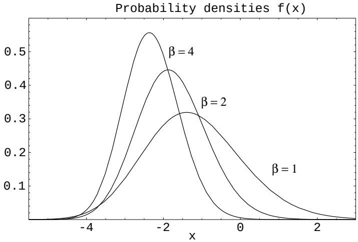
Tracy-Widom Distribution for $\lambda_{\max}$



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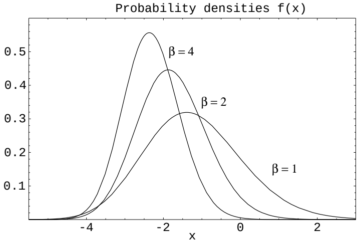

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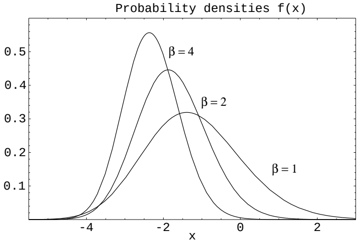

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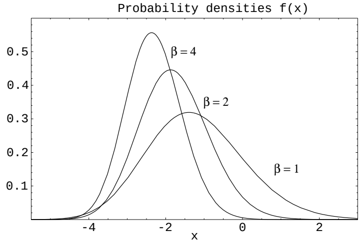

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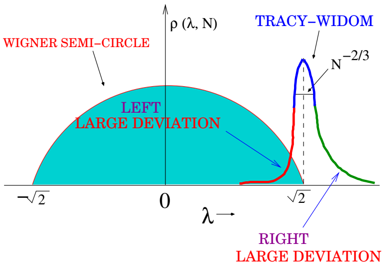
Applications: Growth models, Directed polymer, Sequence Matching .....

(Baik, Borodin, Calabrese, Comtet, Corwin, Deift, Dotsenko, Dumitriu, Edelman, Ferrari, Forrester, Johansson, Johnstone, Le doussal, Nadal, Nechaev, O'Connell, Péché, Prähofer, Quastel, Rains, Rambeau, Rosso, Sano, Sasamoto, Schehr, Spohn, Takeuchi, Virag, ...)

Probability of Large Deviations of $\lambda_{\max}$:



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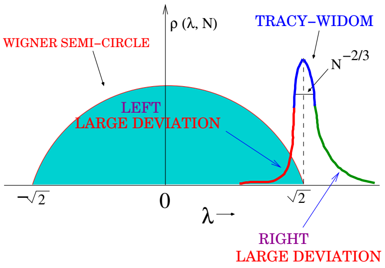

- Tracy-Widom law $\mathrm{Prob}[\lambda_{\max} \le w, N] \to F_\beta\left(\sqrt{2}\, N^{2/3}\,(w - \sqrt{2})\right)$ describes the prob. of typical (small) fluctuations of $\sim O(N^{-2/3})$ around the mean $\sqrt{2}$, i.e., when $|\lambda_{\max} - \sqrt{2}| \sim N^{-2/3}$

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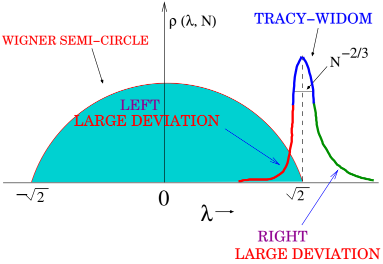

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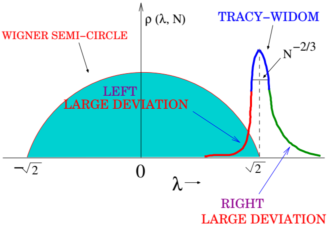
- Q: How to describe the prob. of large (atypical) fluctuations when

$$|\lambda_{\max} - \sqrt{2}| \sim O(1) \to \text{Large deviations from mean}$$

Large Deviation Tails of $\lambda_{\max}$



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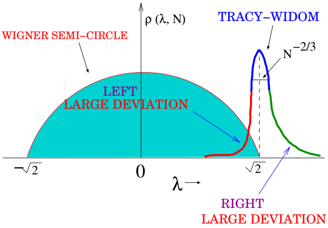

Prob. density of the top eigenvalue: $\mathrm{Prob.}\left[\lambda_{\max} = w, N\right]$ behaves as:

$$\sim \quad \exp\left[-\beta N^2 \Phi_-(w)\right] \qquad \text{for} \quad \sqrt{2} - w \sim O(1)$$

$$\sim \quad N^{2/3} f_\beta\left[\sqrt{2}\, N^{2/3}\left(w - \sqrt{2}\right)\right] \qquad \text{for} \quad |w - \sqrt{2}| \sim O(N^{-2/3})$$

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IV. Saddle Point Method

Distribution of $\lambda_{\max}$: Saddle Point Method

$$\mathrm{Prob}[\lambda_{\max} \le w,\, N] = \mathrm{Prob}[\lambda_1 \le w, \lambda_2 \le w, \ldots, \lambda_N \le w] = \frac{Z_N(w)}{Z_N(\infty)}$$

$$Z_N(w) = \int_{-\infty}^{w} \ldots \int_{-\infty}^{w} \{\prod_i d\lambda_i\} \exp\left[-\frac{\beta}{2}\left\{N\sum_{i=1}^{N}\lambda_i^2 - \sum_{j\neq k}\log|\lambda_j - \lambda_k|\right\}\right]$$

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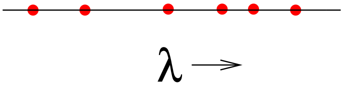
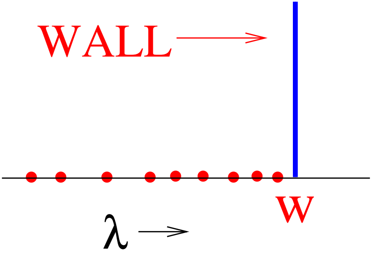
denominator



numerator



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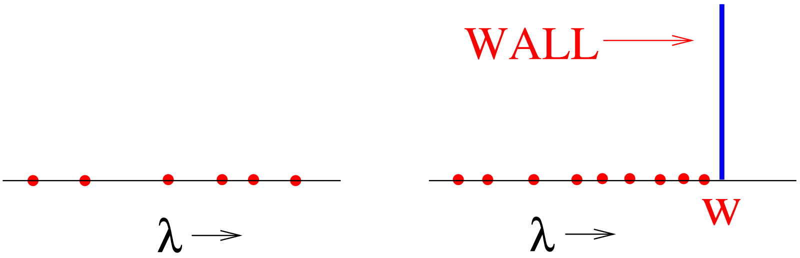
denominator numerator



Setting up the Saddle Point Method

- $$\boxed{Z_N(w) \propto \int_{-\infty}^{w} \prod_i d\lambda_i \exp\left[-\beta N^2 E\left(\{\lambda_i\}\right)\right]}$$

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$$\boxed{\lambda = \mathcal{P}\int_{-\infty}^{w} \frac{\rho_w(\lambda')\, d\lambda'}{\lambda - \lambda'}}$$ for $\lambda \in [-\infty, w] \rightarrow$ Semi-Hilbert transform

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Exact solution for all w :

[D. S. Dean & S.M., PRL, 97, 160201 (2006); PRE, 77, 041108 (2008)]

Exact Saddle Point Solution

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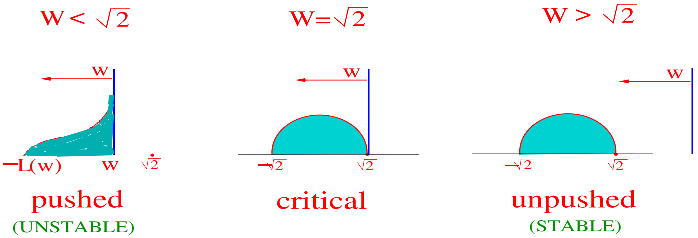

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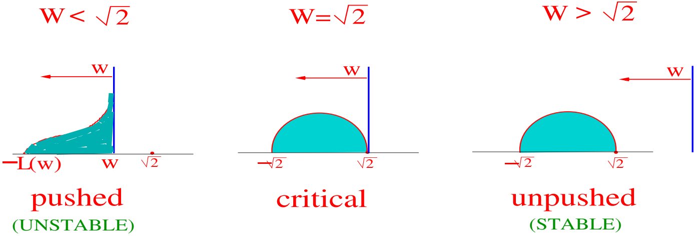

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Left Large Deviation Function

$$\mathrm{Prob}[\lambda_{\max} \leq w, N] = \frac{Z_N(w)}{Z_N(\infty)} \sim \exp\left[-\beta N^2 \left\{S[\rho_w(\lambda)] - S[\rho_\infty(\lambda)]\right\}\right]$$

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$$\Phi_-(w) = \frac{1}{108}\Big[36w^2 - w^4 - (15w + w^3)\sqrt{w^2+6}$$

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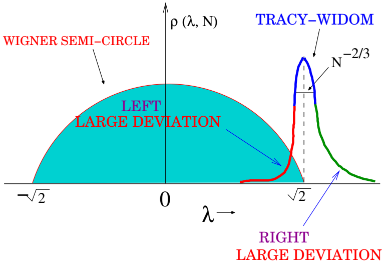
Note also that $\Phi_-(w) \approx \frac{1}{6\sqrt{2}}(\sqrt{2}-w)^3$ as $w \to \sqrt{2}$ from below

Matching with the left tail of Tracy-Widom:



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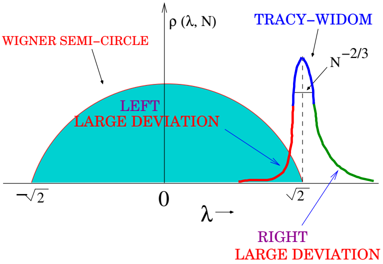

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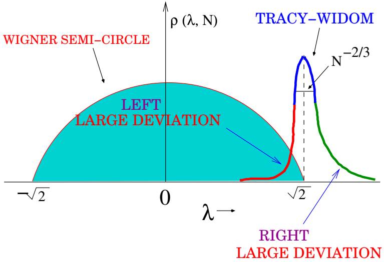

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recovers the left tail of TW: $f_\beta(x) \sim \exp[-\frac{\beta}{24}\,|x|^3]$ as $x \to -\infty$

Right Large Deviation Function: $w > \sqrt{2}$

• For $w \geq \sqrt{2}$, saddle point solution of the charge density $\rho_w(\lambda)$ sticks to the semi-circle form: $\rho_{\rm sc}(\lambda) = \frac{1}{\pi}\sqrt{2-\lambda^2}$ for all $w \geq \sqrt{2}$

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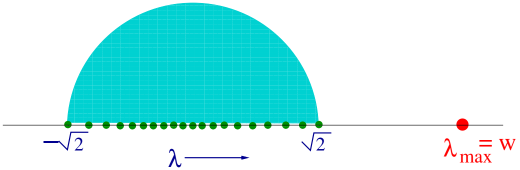
$\longrightarrow$ $(N-1)$-fold integral

WIGNER SEMI–CIRCLE



Pulled Coulomb gas

$$p(w,N) \propto e^{-\beta N w^2/2} \left\langle e^{\beta \sum_j \ln|w-\lambda_j|} \right\rangle$$

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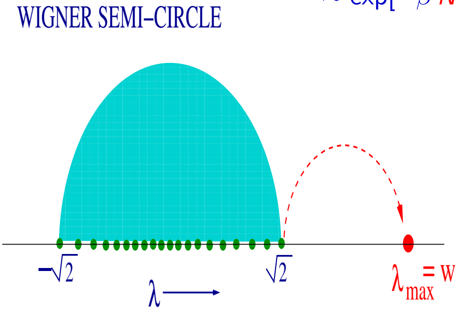

$N\,\Phi_+(w) = \Delta E(w)$

$= \frac{w^2}{2} - \int_{-\sqrt{2}}^{\sqrt{2}} \ln(w-\lambda)\, \rho_{\rm sc}(\lambda)\, d\lambda$

$\Rightarrow$ energy cost in pulling a charge out of the Wigner sea

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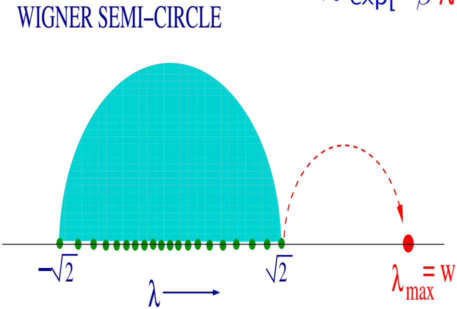

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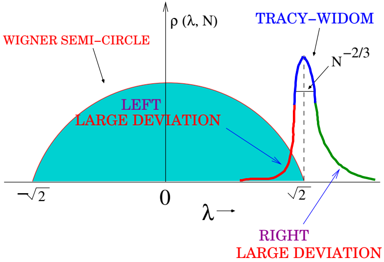
$$\Rightarrow \boxed{\Phi_+(w) = \frac{1}{2} w\sqrt{w^2-2} + \ln\left[\frac{w-\sqrt{w^2-2}}{\sqrt{2}}\right]} \quad (w > \sqrt{2})$$

[S.M. & Vergassola, PRL, 102, 160201 (2009)]

Matching with the right tail of Tracy-Widom



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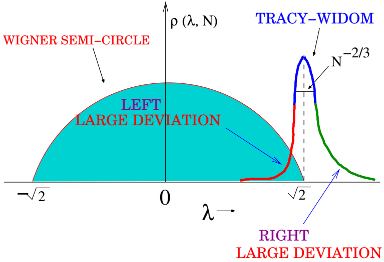

As $w \to \sqrt{2}$ from above, $\Phi_+(w) \to \frac{2^{7/4}}{3}(w-\sqrt{2})^{3/2}$

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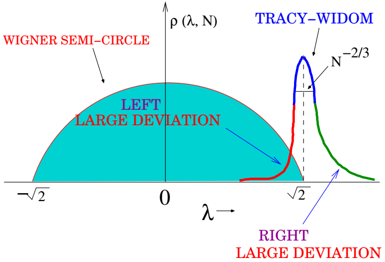

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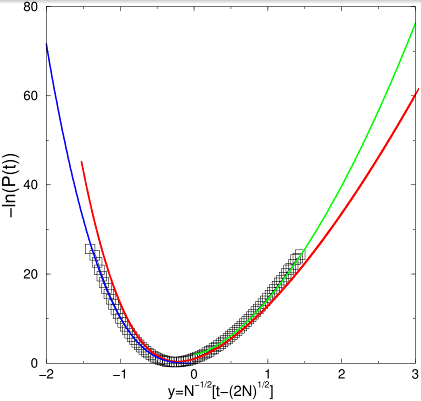
Comparison with Simulations:



$N \times N$ real Gaussian matrix ($\beta = 1$): $N = 10$

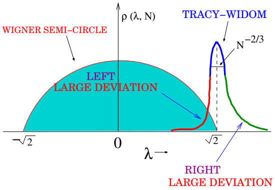
squares $\rightarrow$ simulation points

red line $\rightarrow$ Tracy-Widom

blue line $\rightarrow$ left large deviation function ($\times N^2$)

green line $\rightarrow$ right large deviation function ($\times N$).

Summary and Generalizations



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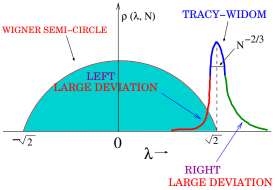

Prob. density of the top eigenvalue: $\mathrm{Prob.}\,[\lambda_{\max} = w, N]$ behaves as:

$$\sim \quad \exp\left[-\beta N^2 \Phi_-(w)\right] \qquad \text{for} \quad \sqrt{2} - w \sim O(1)$$

$$\sim \quad N^{2/3} f_\beta \left[\sqrt{2}\, N^{2/3} \left(w - \sqrt{2}\right)\right] \qquad \text{for} \quad |w - \sqrt{2}| \sim O(N^{-2/3})$$

$$\sim \quad \exp\left[-\beta N \Phi_+(w)\right] \qquad \text{for} \quad w - \sqrt{2} \sim O(1)$$

3-rd Order Phase Transition

Cumulative prob. of $\lambda_{\max}$:

$$P\left(\lambda_{\max} \le w, N\right) \approx \begin{cases} \exp\left\{-\beta N^2 \Phi_-(w) + \ldots\right\} & \text{for} \quad w < \sqrt{2} \\ 1 - A \exp\left\{-\beta N \Phi_+(w) + \ldots\right\} & \text{for} \quad w > \sqrt{2} \end{cases}$$

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- Left $\longrightarrow$ strong-coupling phase $\rightarrow$ perturbative
  higher order corrections ($1/N$ expansion) to free energy

  [Borot, Eynard, S.M., & Nadal 2011]

- Right $\longrightarrow$ weak-coupling phase $\rightarrow$ non-perturbative
  higher order corrrections [Nadal & S.M. 2011, Borot & Nadal, 2012]

3-rd order transition $\rightarrow$ ubiquitous

- $\lambda_{\rm max}$ for other matrix ensembles: Wishart: $W = X^{\dagger} X \rightarrow (N \times N)$
  $\rightarrow$ covariance matrix

Typical: Tracy-Widom [Johansson 2000, Johnstone 2001]

Large deviations: Exact rate functions

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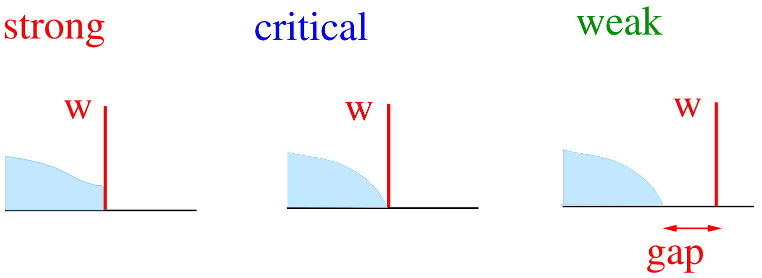
- Conductance and Shot Noise in Mesoscopic Cavities
- Entanglement entropy of a random pure state in a bipartite system
- Maximum displacement in Vicious walker problem
- Distribution of Wigner time-delay . . .

[Bohigas, Comtet, Forrester, Nadal, Schehr, Texier, Vergassola, Vivo,..+S.M. (2008-2013)]

Basic mechanism for 3-rd order transition



Gap between the soft edge (square-root signularity) of the Coulomb droplet and the hard wall vanishes as a control parameter g goes through a critical value g_c:

$$\text{gap} \longrightarrow 0 \text{ as } g \to g_c$$

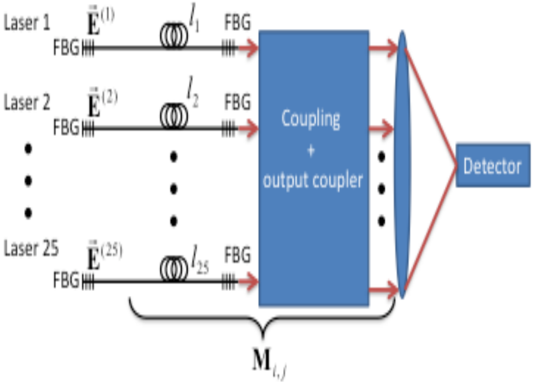
Experimental Verification with Coupled Lasers

Measuring maximal eigenvalue distribution of Wishart random matrices with coupled lasers

Moti Fridman, Rami Pugatch, Micha Nixon, Asher A. Friesem, and Nir Davidson*
Weizmann Institute of Science, Dept. of Physics of Complex Systems, Rehovot 76100, Israel
(Dated: May 30, 2011)

We determined the probability distribution of the combined output power from twenty five coupled fiber lasers and show that it agrees well with the Tracy-Widom, Majumdar-Vergassola and Vivo-Majumdar-Bohigas distributions of the largest eigenvalue of Wishart random matrices with no fitting parameters. This was achieved with 500,000 measurements of the combined output power from the fiber lasers, that continuously changes with variations of the fiber lasers lengths. We show experimentally that for small deviations of the combined output power over its mean value the Tracy-Widom distribution is correct, while for large deviations the Majumdar-Vergassola and Vivo-Majumdar-Bohigas distributions are correct.

Experimental Verification with Coupled Lasers

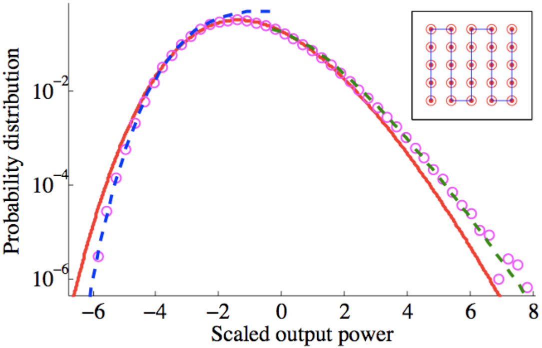


combined output power from fiber lasers $\propto \lambda_{\max}$

$\lambda_{\max} \to$ top eigenvalue of the Wishart matrix $W = X^t X$

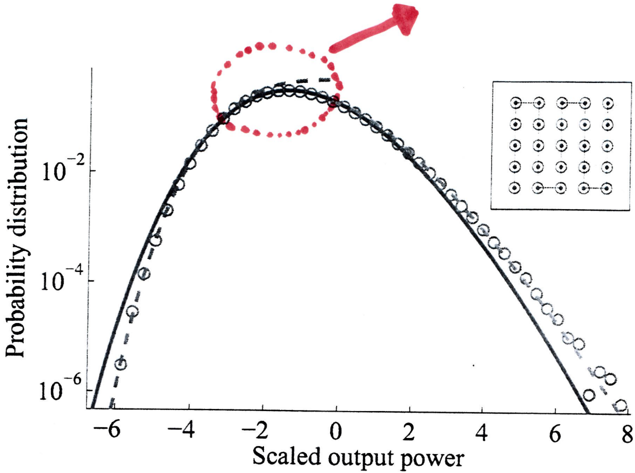
where $X \to$ real symmetric Gaussian matrix $(\beta = 1)$

Experimental Verification with Coupled Lasers



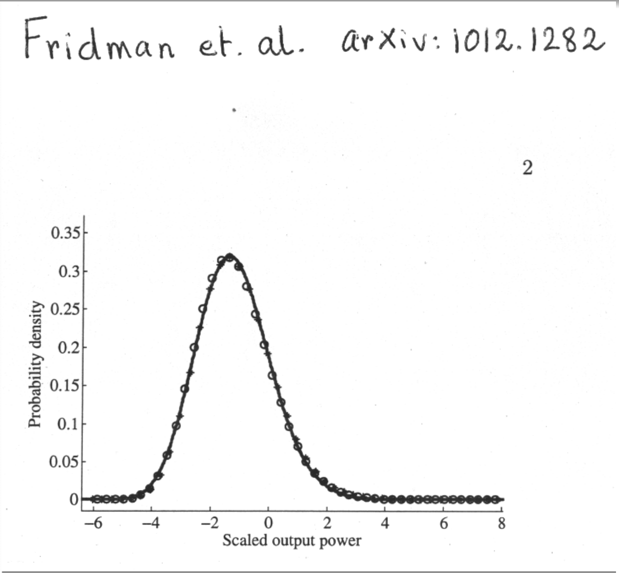
(a)

Experimental Verification with coupled lasers



(a)

Tracy-Widom density with $\beta = 1$



Collaborators

Students: C. Nadal (Oxford Univ., UK)
J. Randon-Furling (Univ. Paris-1, France)
R. Marino (LPTMS, Orsay)

Postdoc: D. Villamaina (ENS, Paris, France)

Collaborators:

- O. Bohigas, A. Comtet, G. Schehr, C. Texier, P. Vivo (LPTMS, Orsay, France)
- R. Allez (Paris-Dauphin, France)
- G. Borot (Geneva, Switzerland)
- J.-P. Bouchaud (CFM, Ecole Polytechnique, Paris, France)
- B. Eynard (Saclay, France)
- K. Damle, V. Tripathi (Tata Institute, Bombay, India)
- D.S. Dean (Bordeaux, France)
- P. J. Forrester (Melbourne, Australia)
- A. Lakshminarayan (IIT Madras, India)
- A. Scardichio (ICTP, Trieste, Italy)
- S. Tomsovic (Washington State Univ., USA)
- M. Vergassola (Institut Pasteur, Paris, France)

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• G. Borot, B. Eynard, S.N. Majumdar and C. Nadal, "Large Deviations of the Maximal Eigenvalue of Random Matrices", J. Stat. Mech., P11024 (2011)

• C. Texier and S. N. Majumdar, " Wigner time-delay distribution in chaotic cavities and freezing transition", Phys. Rev. Lett. 110, 250602 (2013).

Recent review

Recent review: S.M. & G. Schehr, arXiv: 1311.0580

J. Stat. Mech. P01012 (2014)

Tracy-Widom distributions (1994)

The scaling function $F_\beta(x)$ has the expression:

- $\beta = 1$: $F_1(x) = \exp\left[-\frac{1}{2}\int_x^\infty \left[(y-x)q^2(y) + q(y)\right]\, dy\right]$

- $\beta = 2$: $F_2(x) = \exp\left[-\int_x^\infty (y-x)q^2(y)\, dy\right]$

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$\frac{d^2 q}{dy^2} = 2\, q^3(y) + y\, q(y)$ with $q(y) \to \mathrm{Ai}(y)$ as $y \to \infty \to$ Painlevé-II

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$$

where explicit expressions for $\Phi_1(w)$, ϕ_β and $\Phi_2(\beta, w)$ were obtained recently (Borot, Eynard, S.M., & Nadal, JSTAT, P11024 (2011))

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The constant τ_β

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$$+ \int_0^\infty dx \left[\frac{6\,x\,\coth(x/2) - 12 - x^2}{12x^2(e^{\beta\,x/2}-1)}\right]$$

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For $\beta = 1$, 2 and 4

$\rightarrow$ agrees with Baik, Buckingham and DiFranco (2008)

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Close to $w \to \sqrt{2}^+$, this gives

$\mathrm{Prob.}\left[\lambda_{\max} < \sqrt{2} + 2^{-1/2} N^{-2/3} x\right] \to -\frac{1}{16\pi x^{3/2}} e^{-(4/3)x^{3/2}}$

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- As a bonus, our method also provides a 'simpler' derivation of TW distribution for $\beta = 2$ (Nadal and S.M., 2011)

A simple example of large deviation tails

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$$P(M = Nx, N) \sim \exp\left[-N\Phi(x)\right] \qquad \text{where}$$

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- Clearly $P(M, N) = \binom{N}{M} 2^{-N}$ $(M = 0, 1, \ldots, N) \to$ binomial distribution

  with mean$= \langle M \rangle = \frac{N}{2}$ and variance$=\sigma^2 = \langle \left(M - \frac{N}{2}\right)^2 \rangle = \frac{N}{4}$
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- Setting $M/N = x$ and using Stirling's formula $N! \sim N^{N+1/2} e^{-N}$ gives

$$P(M = Nx, N) \sim \exp\left[-N\Phi(x)\right] \qquad \text{where}$$

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- $\Phi(x) \to$ symmetric with a minimum at $x = 1/2$ and for small arguments $|x - 1/2| << 1$, $\Phi(x) \approx 2(x - 1/2)^2$

  $\to$ recovers the Gaussian form near the peak

Covariance Matrix

$$X = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{cc} \text{phys.} & \text{math} \\ \left| \begin{array}{cc} X_{11} & X_{12} \\ X_{21} & X_{22} \\ X_{31} & X_{32} \end{array} \right| \end{array}$$

in general (MxN)

$$X^t = \left| \begin{array}{ccc} X_{11} & X_{21} & X_{31} \\ X_{12} & X_{22} & X_{32} \end{array} \right|$$

in general (NxM)

$$W = X^t X = \left| \begin{array}{cc} X_{11}^2 + X_{21}^2 + X_{31}^2 & X_{11}X_{12} + X_{21}X_{22} + X_{31}X_{32} \\ X_{12}X_{11} + X_{22}X_{21} + X_{32}X_{31} & X_{12}^2 + X_{22}^2 + X_{32}^2 \end{array} \right|$$

→ (NxN) COVARIANCE MATRIX (unnormalized)

Principal Component Analysis

Consider N students and $M = 2$ subjects (phys. and math.)

$X \rightarrow (N \times 2)$ matrix and $W = X^t X \rightarrow 2 \times 2$ matrix

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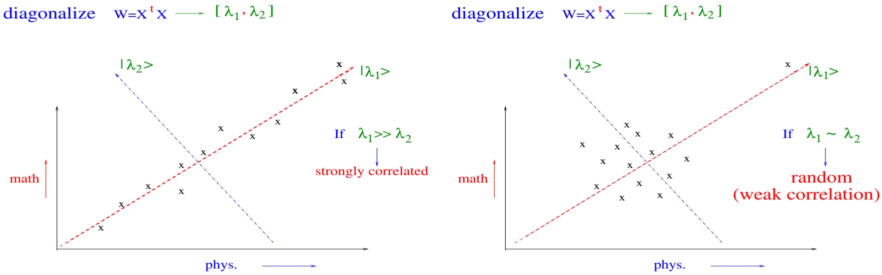

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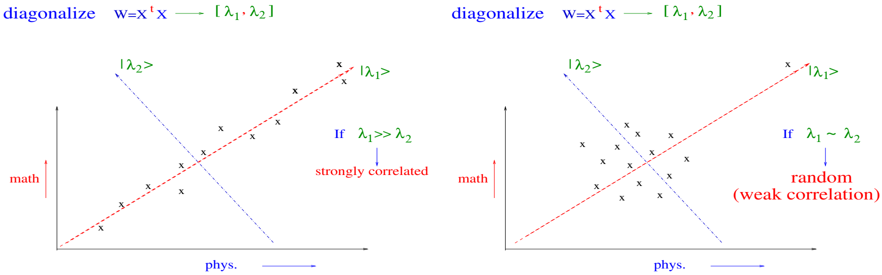

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$\Rightarrow$ practical method for image compression in computer vision

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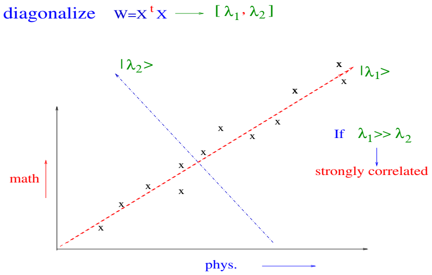

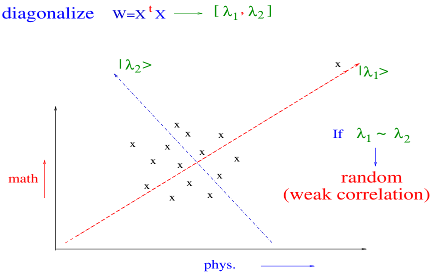

data compression via 'Principal Component Analysis' (PCA)

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Null model $\to$ random data: $X \to$ random $(M \times N)$ matrix

$\to W = X^t X \to$ random $N \times N$ matrix (Wishart, 1928)

Generalization to Wishart Matrices

- $W = X^{\dagger}X \to (N \times N)$ square covariance matrix (Wishart, 1928)
- Entries of X Gaussian: $\mathrm{Pr}[X] \propto \exp\left[-\frac{\beta}{2} N\, \mathrm{Tr}(X^{\dagger}X)\right]$

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- Average density of states for large N: Marcenko-Pastur (1967)

$$\rho(\lambda, N) = \langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \rangle \xrightarrow[N \to \infty]{} \rho(\lambda) = \frac{1}{2\pi} \sqrt{\frac{4-\lambda}{\lambda}}$$

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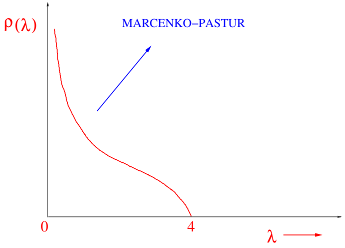
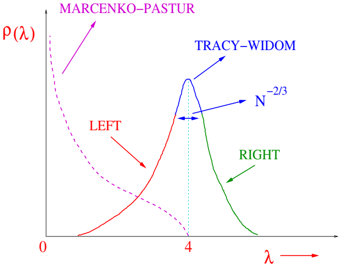


Distribution of $\lambda_{\max}$



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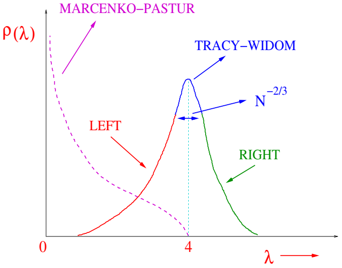

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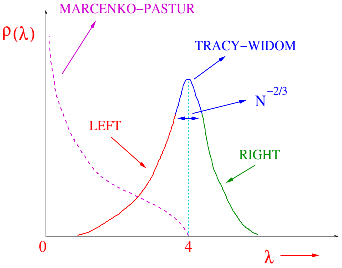

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$$P\left(\lambda_{\max} = w, N\right) \approx \begin{cases} \exp\left\{-\beta N^2 \Psi_-(w)\right\} & \text{for} \quad w < 4 \\ \exp\left\{-\beta N \Psi_+(w)\right\} & \text{for} \quad w > 4 \end{cases}$$

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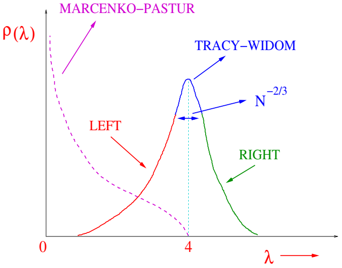

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- $\Psi_-(w)$ and $\Psi_+(w)$ $\to$ computed exactly

(Vivo, S.M. & Bohigas 2007, S.M. & Vergassola 2009)

Exact Left and Right Large Deviation Functions

Using Coulomb gas + Saddle point method for large N:

- Left large deviation function:

$$\Psi_-(w) = \ln\left[\frac{2}{w}\right] - \frac{w-4}{8} - \frac{(w-4)^2}{64} \quad w \leq 4$$

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- Right large deviation function:

$$\Psi_+(w) = \sqrt{\frac{w(w-4)}{4}} + \ln\left[\frac{w-2-\sqrt{w(w-4)}}{2}\right] \quad w \geq 4$$

(S.M. and Vergassola, 2009)

Other Problems with 3-rd Order Phase Transitions

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- Bipartite Entanglement of a Random Pure State

Probability distribution of entanglement entropy

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- Non-Intersecting Brownian Motions and Random Matrices

  $\rightarrow$ relation to 2-d Yang-Mills gauge theory

  Schehr, S.M., Comtet, Randon-Furling, PRL, 101, 150601 (2008)

  Forrester, S.M. & Schehr, Nucl. Phys. B 844, 500 (2011), J. Stat. Phys. (2013)