On the partial relation between random matrices and some non-equilibrium systems

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Outline

Introduction

Main Idea

Consider a large matrix whose elements are random variables with given probability laws. There are several ensembles of random matrices, the most popular are the Gaussian and the circular ensembles (unitary, orthogonal and symplectic)

Main Objective

The main objective is to calculate the statistical behavior of the eigenvalues and eigenvectors.

Gaussian Orthogonal Ensemble (GOE)

The Gaussian orthogonal ensemble E_{1G} is defined in the space of real symmetric matrices of order $N \times N$ by two requirements.

1. The ensemble is invariant under every automorphism:

$$P(S)dS = P(S')dS' \quad S \longrightarrow W^T S W,$$

of E_{1c} into itself where W is any real $N \times N$ orthogonal matrix.

2. The probability that a matrix from the ensemble E_{1G} lies between S and $S + dS$ is

$$P_N(S)dS = \prod_{i \leq j} \underbrace{f_{ij}(S_{ij})}_{\text{Gaussian distribution}} dS_{ij}.$$

The elements of S are statistically independent and satisfy $\langle S_{i,j} \rangle = 0$ and $\left\langle |S_{i,j}|^2 \right\rangle = N$.

Gaussian Unitary Ensemble (GUE)

The Gaussian unitary ensemble E_{2G} is defined in the space of Hermitian matrices of order $N \times N$ by two requirements.

1. The ensemble is invariant under every automorphism:

$$P(S)dS = P(S')dS' \quad S \longrightarrow W^{-1}\,S\,W,$$

of E_{2c} into itself where W is any unitary $N \times N$ matrix.

2. The probability that a matrix from the ensemble E_{2G} lies between S and $S + dS$ is

$$P_N(S)dS = \prod_{i \leq j} \underbrace{f_{ij}^{(0)}\left(S_{ij}^{(0)}\right)}_{\text{Gaussian distribution}} dS_{ij}^{(0)} \prod_{i<j} \underbrace{f_{ij}^{(1)}\left(S_{ij}^{(1)}\right)}_{\text{Gaussian distribution}} dS_{ij}^{(1)}.$$

The elements of S are statistically independent.

Joint Probability density

It is possible to show that the join probability density to have eigenvalues around $x_1, x_2, \cdots, x_N$ is given by

$$P_{N\beta G}(x_1, \cdots, x_N) = \frac{1}{Z_N(\beta)} e^{-\sum_{i=1}^{N} x_i^2} \prod_{i<j} |x_i - x_j|^{\beta},$$

with $\beta = 1$ for the orthogonal ensemble and $\beta = 2$ for the unitary ensemble.

Physical Interpretation

If we compare $P_{N\beta G}(x_1, \cdots, x_N)$ with the Boltzmann factor, it is possible to see that Gaussian ensembles are statistically equivalent to an equilibrium many particle system where the particles interact according to the potential

$$V = \sum_{i=1}^{N} \frac{x_i^2}{\beta} - \sum_{i<j} \ln |x_i - x_j|$$

with

$$\beta = \frac{1}{K_B T}$$

There are several interactions potentials which leads to the same $P^{(0)}(s)$ and $g(r)$. Then, in general it is not possible to calculate unambiguously the interaction potential from these distribution.

It is possible to find several different systems which share the same spacing distribution functions $P^{(0)}(s)$.

Basic Statistical Properties

The scaled density of eigenvalues $\rho(r)$ is given by $\rho(r) = \frac{1}{2\pi}(4-r^2)^{\frac{1}{2}}$ with $r = x/\sqrt{N/2}$

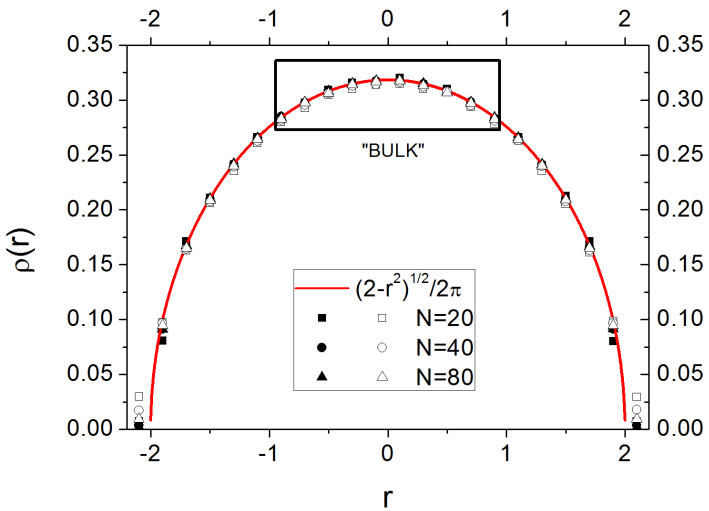
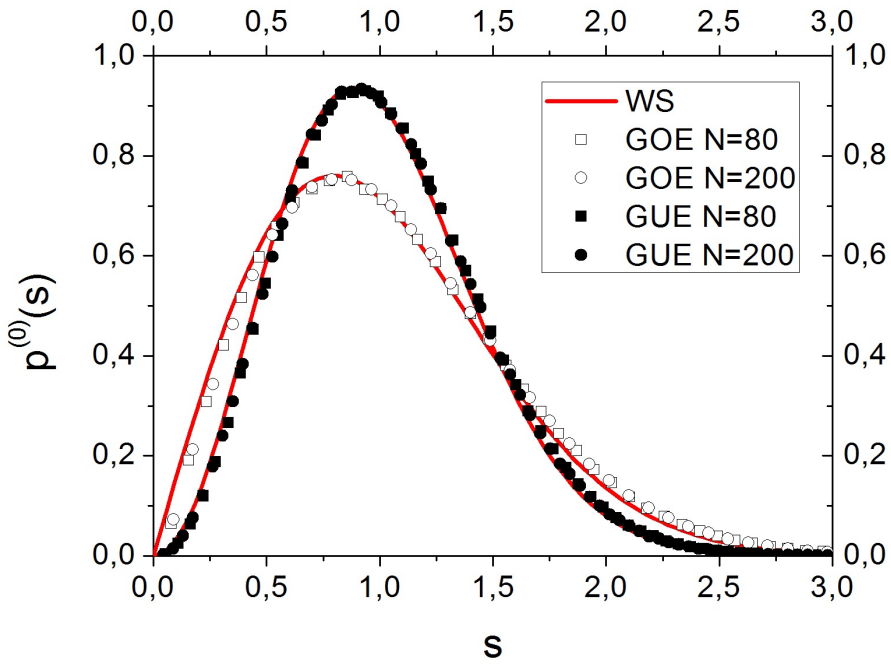


Figure: Semi-circle law, open and filled symbols correspond to GOE and GUE respectively.

$$r(x_1) \propto \int dx_2 \cdots \int dx_N P_{N\beta G}(x_1, \cdots, x_N)$$

Nearest-Neighbor distribution

In the "bulk", the scaled nearest-neighbor distribution, $p^{(0)}(s)$, is given approximately by the Wigner surmise



$$P_\beta(s) = a_\beta s^\beta e^{-b_\beta s^2}$$

$$b_\beta = \left[\frac{\Gamma\left(\frac{\beta+2}{2}\right)}{\Gamma\left(\frac{\beta+1}{2}\right)}\right]^2$$

$$a_\beta = \frac{2b_\beta^{(\beta+1)/2}}{\Gamma\left(\frac{\beta+1}{2}\right)}$$

Figure: Nearest-neighbor distribution and WS. As usual $S_i = x_{i+1} - x_i$ and $s = S/\langle S \rangle$.

Nearest-Neighbor distribution

The distribution of the largest eigenvalue $Q(x_{max})$ is given by

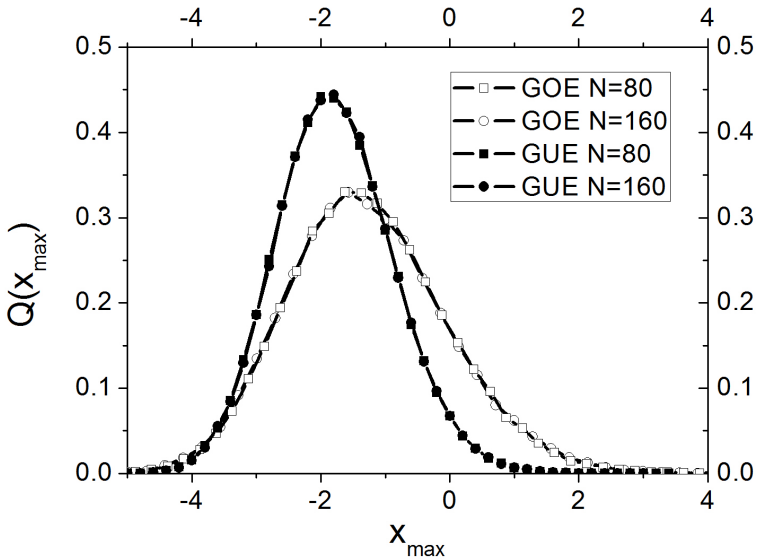


Figure: Distribution of the largest eigenvalue.

For the GUE, $Q(x_{max}) = \frac{dF_2(x)}{dx}$ with $F_2(x) = e^{-\int_s^\infty (x-s)u(x)^2 dx}$ and $\frac{d^2u}{dx^2} = 2u^3 + x\,u$. It is also possible find similar expressions for the GOE (1994 Tracy and Widom).

Terrace-Step-Kink (TSK) Model

1. The molecular-beam epitaxy (MBE) is commonly used in the construction of nano and micro-electronic devices.
2. Steps are used to have some control of the morphology of the surface during growth.
3. Steps act as a sink (most favorable place for attachment), then, nucleation on terraces may be avoided providing some control on the growth (temperature not too low and interstep distance not too large).

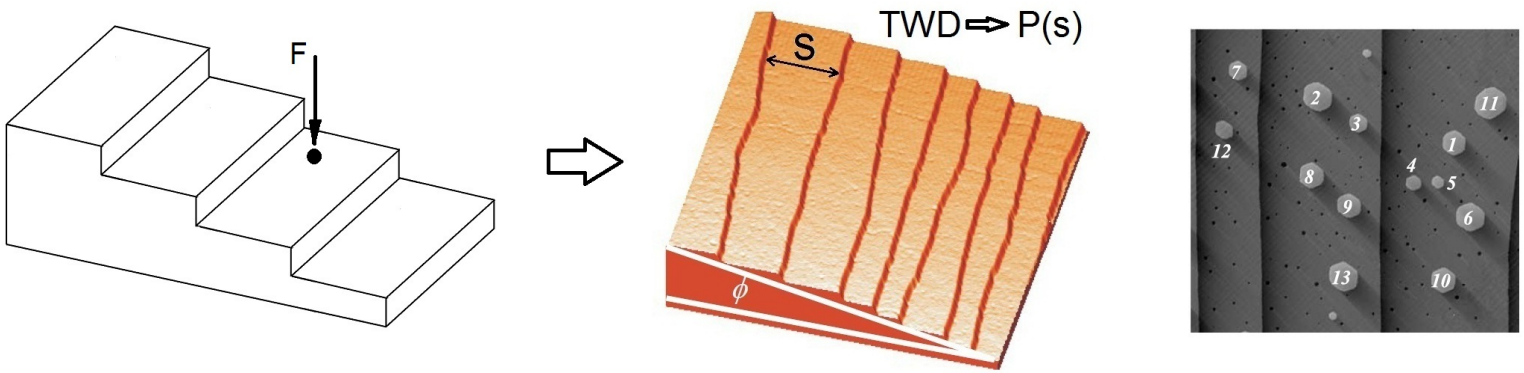
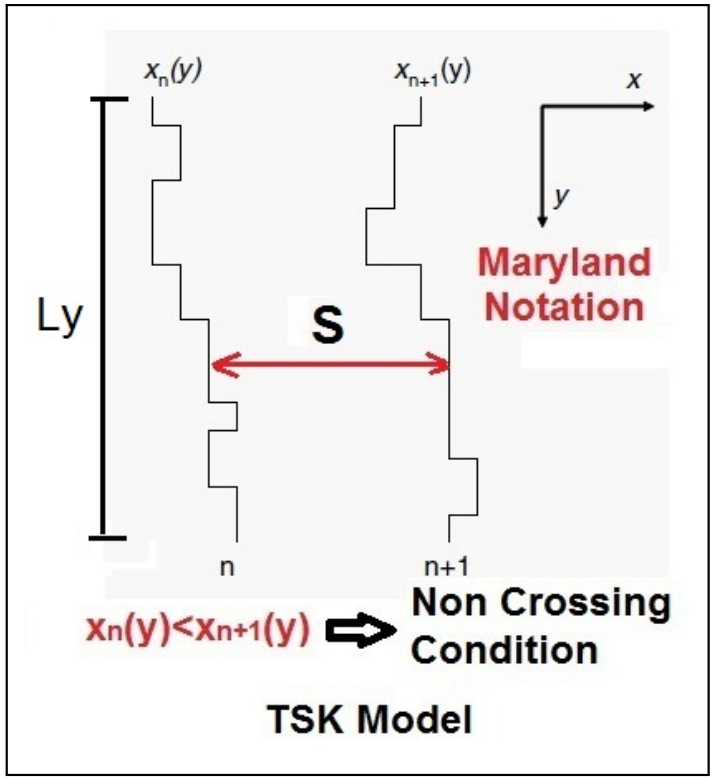


Figure: STM images of thermally equilibrated steps on Al/Si(111) surface. E. D. Williams (left) and M. Giensen (rigth). Note that the steps cannot cross each other.

Terrace-Step-Kink (TSK) Model

The TSK is one of the simplest models to describe fluctuations on steps.

1. At low T, kinks are the dominant excitation. In the TSK, the only excitations taken into account are the kinks.
2. In particular vacancies and adatoms on terraces are neglected.
3. These simplifications restrict the applicability of the TSK to low temperatures where the predominant thermal excitations are the kinks.



Terrace-Step-Kink (TSK) Model

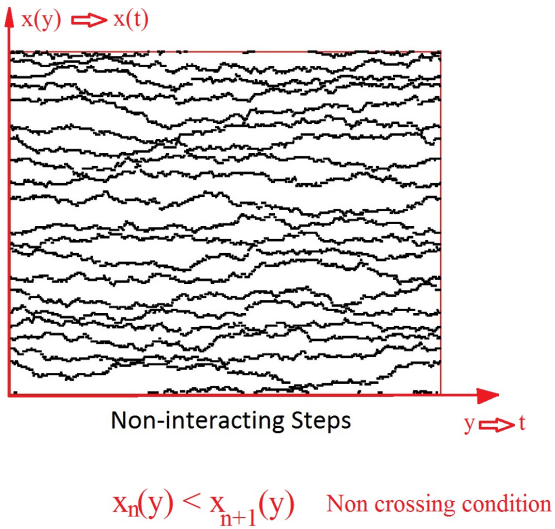
The Hamiltonian of the system can be written as

$$H = \sum_{y_n=1}^{Ly} \left(\sum_{i=1}^{N} \overbrace{\epsilon_k \ |x_i(y_{n+1}) - x_i(y_n)|}^{kink \quad energy} + \sum_{i=1}^{N} \sum_{j=1}^{q} \underbrace{v(|x_i(y_n) - x_j(y_n)|)}_{\substack{\text{“instantaneous”} \\ interaction \ among \ steps}} \right)$$

where ϵ_k is the energy required to form a simple kink, N is the number of steps, q is the number of interacting neighbors, L_y is the width of the terrace and $v(\zeta)$ is the interaction potential between steps.

Interacting Steps with NN Interactions

The TSK model is suitable to MC simulations[1].



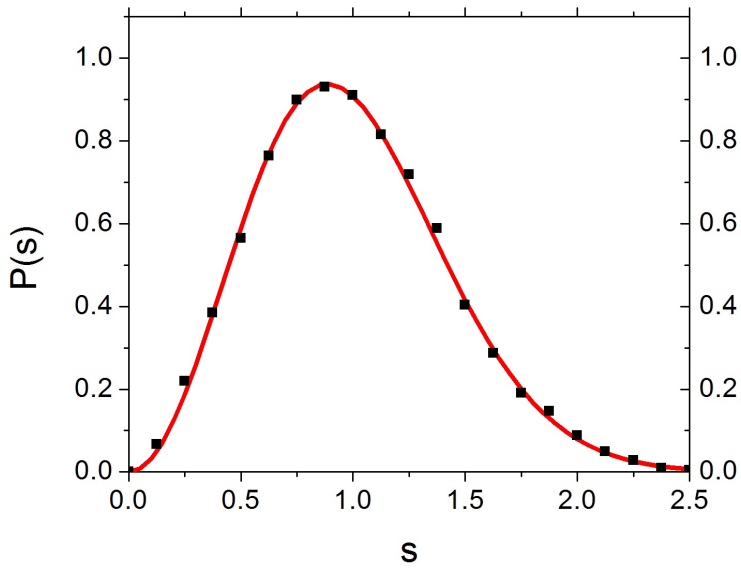


Figure: Typical configurations obtained through Monte Carlo simulations.

[1] Is is possible to show that the TSK model is equivalent to the vicious random walk problem and to the 1D fermions at zero temperature.

Coalescing Random Walk (CRW)

The one-dimensional single species coalescing random walk process has the following properties:

1. Is a collection of particles executing independent random walks on an infinite one-dimensional lattice.
2. When two particles meet they suffer a fusion interaction

$$A + A \rightarrow A$$

3. The fusion is irreversible and instantaneous.
4. Each lattice site could be empty or occupied by only one particle at time t.

Coalescing Random Walk (CRW)

Let be $P(n,t)$ the probability to have to nearest neighbors at a distance n at time t. The master equation for $P(n,t)$ is given by:

$$\frac{dP(n,t)}{dt} = \frac{1}{2}P(n+1,t) + \frac{1}{2}P(n+1,t) - 2\,P(n,t)$$

with boundary conditions $P(0,t) = 0$ and $P(\infty,t) = 0$. In the continuum limit, the solution for the initial condition $P(S,0) = \delta(S - S_0)$, is given by

$$P(S,t) = \frac{1}{\pi\, t}\,\sinh\frac{S\,S_0}{2\,t}\,e^{-\frac{S^2+S_0^2}{4\,t}}$$

In the limit $t \to \infty$ we have

$$P(S,t) = \frac{1}{\pi\, t}\,\frac{S\,S_0}{t}\,e^{-\frac{S^2}{4\,t}}$$

Coalescing Random Walk (CRW)

Some results are

1. The system is homogeneous, i.e., the density is given by $\rho(x,t) \equiv \rho(t)$.
2. There is dynamical scaling because it is possible to define a scaled inter-particle distance $s = S(t)/\langle S(t)\rangle$ with $\langle S(t)\rangle = \sqrt{2\pi t}$.
3. The distribution of s is given by

$$P(s) = \frac{\pi}{2} s\, e^{-\frac{\pi}{4}s^2} \quad \text{WS for GOE}$$

Coalescing Random Walk (CRW)

The scaling property allow one to compare the scaled distribution $P(s)$ of the CRW with the one of any equilibrium system

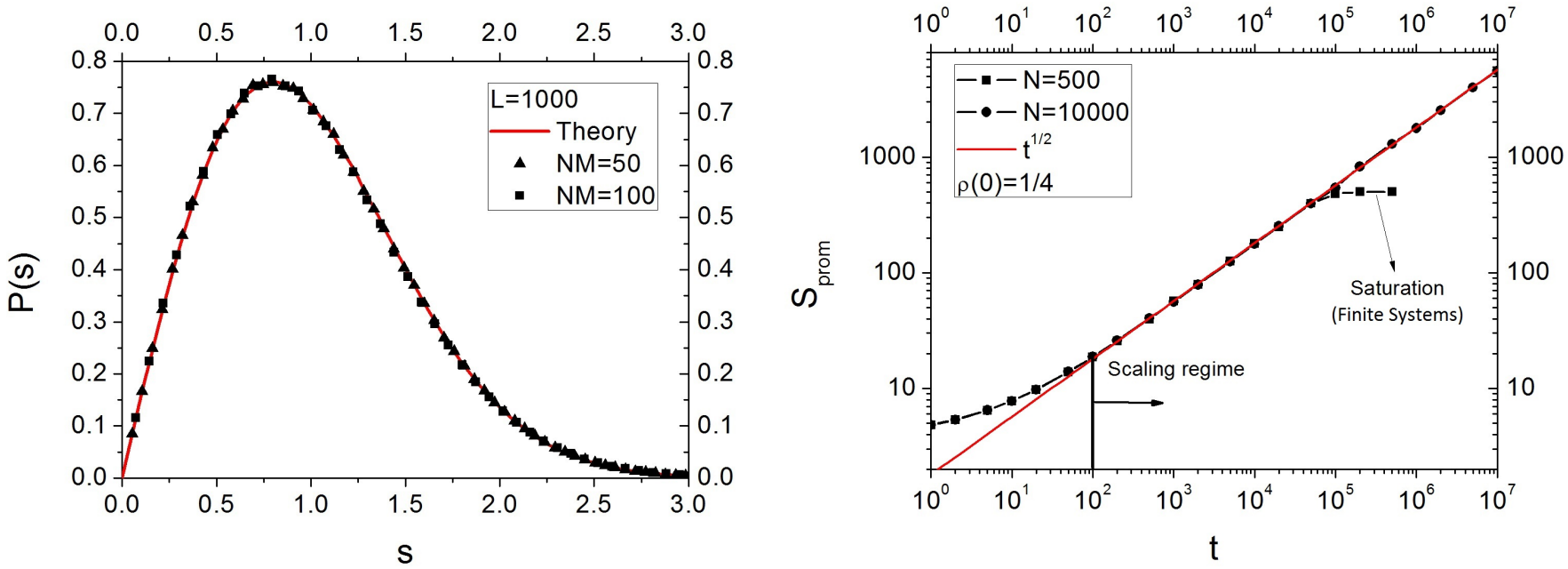
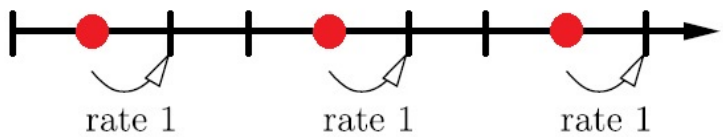


Figure: Numerical results for $P(s)$ at two different times. The analytical solution is also included.

Totally asymmetric simple exclusion process (TASEP)

Lets consider a 1D system with the following conditions:

1. The particles jump on their right neighbor site with rate one if the site is empty.



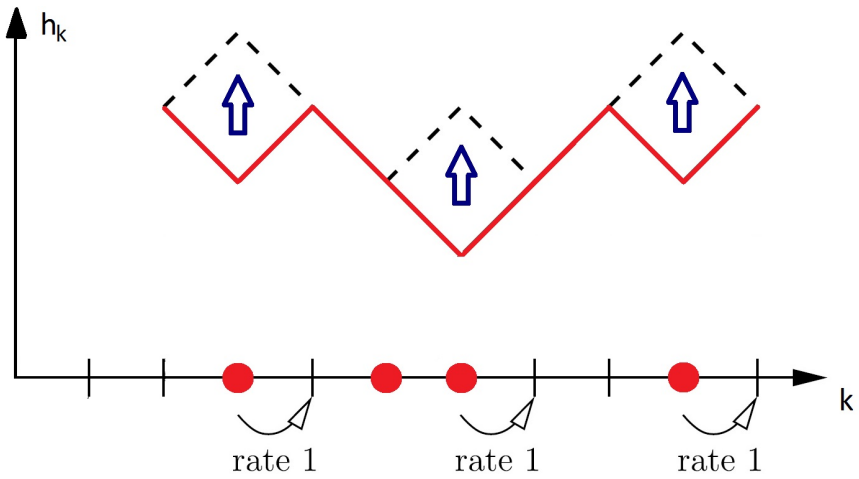
2. Periodic boundary conditions are imposed on a lattice with N sites.

Totally asymmetric simple exclusion process (TASEP)

It is possible to map this system onto a growth model by assigning a vertical position for each lattice site. For example

$$h_k = \frac{N}{2} + \sum_{j=-N/2}^{k} m_j$$

where $m_l = -1$ if there is a particle in the l-th site and $m_l = 1$ otherwise.



Totally asymmetric simple exclusion process (TASEP)

Lets consider a particular initial condition. The step initial condition, is defined by $m_j = -1$ for $-\frac{N}{2} \leq j < 0$ and $m_j = 1$ for $0 \leq j \leq \frac{N}{2}$. Then, the initial configuration is given by

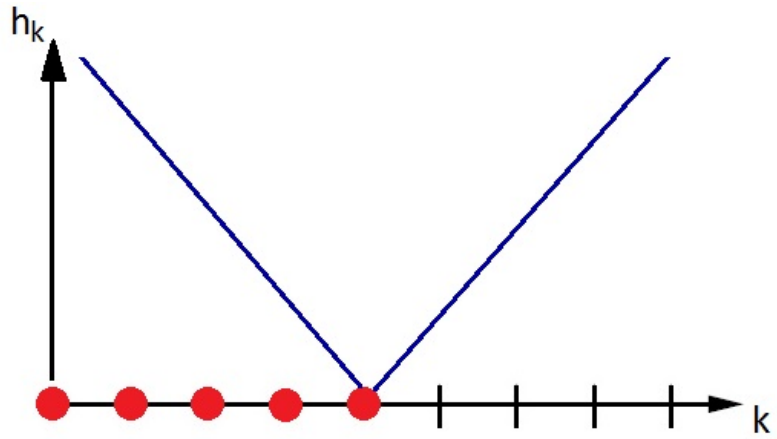


Figure: Initial configuration of the particle positions.

Totally asymmetric simple exclusion process (TASEP)

The scaled height at the origin, $h_0^{sc} = \frac{h_0 - t/2}{-t^{1/3}}$, follows the same distribution of the largest eigenvalue of the GUE as shown below

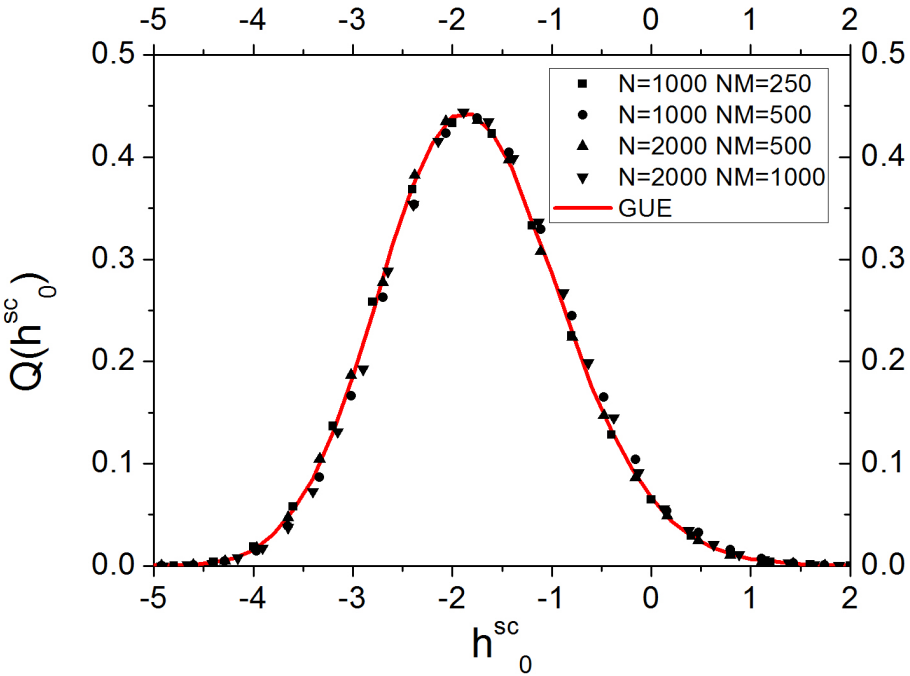


Figure: Connection between the TASEP and the GUE.

Systems with domain formation

In domains systems the analog of the nearest-neighbor distribution is the domain size distribution.

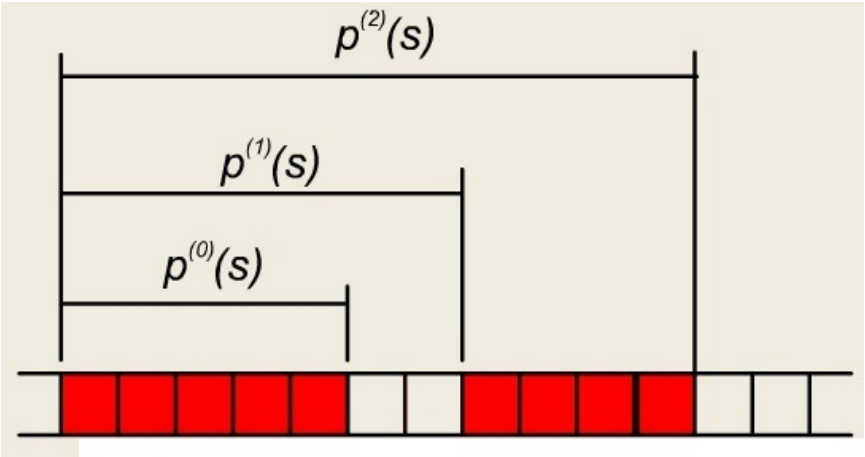


Figure: Definition of $p^{(0)}(s)$ for domain systems.

Quasi one-dimensional gas

1. Fully periodic $2 \times L$ rectangular lattice half filled with two equal number of two types of particles (labeled by their charge $+$ or $-$).
2. An infinite external field, $\vec{E}$, drives the two species in opposite directions along the x axis (long axis).
3. each lattice site can be occupied at most by only one particle.

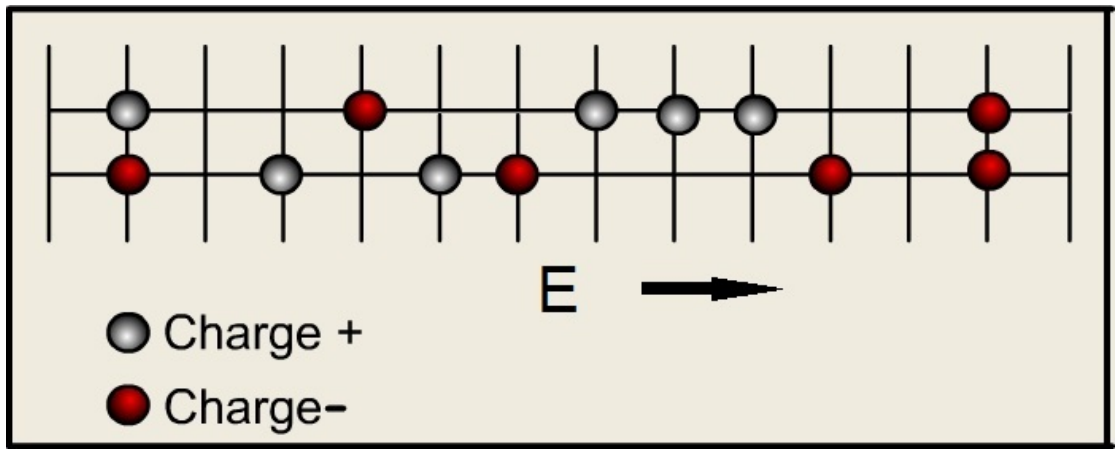


Figure: Quasi one-dimensional gas proposed by J. T. Mettetal, B. Schmittmann and R. K. P. Zia Europhys. Lett. **58**, 653 (2002).

Quasi one-dimensional gas

The system evolves in time according to the following dynamical rules:

1. L particles are randomly inserted in a $2 \times L$ rectangular lattice, $\frac{L}{2}$ particles $(+)$ and $\frac{L}{2}$ particles $(-)$, the remaining sites are empty. Periodic boundary conditions are imposed in both directions of the lattice.
2. Two neighbor sites are chosen at random.
3. The contents of the sites are exchanged with probability 1 if the neighbors sites are particle-hole
4. If they are particle-particle the content are exchanged with probability γ.
5. The exchanges which result in $+/-$ particles moving in the positive/negative x direction are forbidden due to the action of $\vec{E}$.
6. A time unit correspond to $2L$ attempts of exchange.

Quasi one-dimensional gas

Some results are

- For low γ values the system remains homogeneous, i.e., the system evolves without domain formation.
- For high γ values the system evolves as follows.
  - For early times small domains form everywhere due to mutual obstruction of the opposite species.
  - In the low density region the traveler particles leak out of the domain at one end and later rejoin it at the other end.
  - The charge distribution of this macroscopic domain is not trivial.
  - For intermediate times there is dynamical scaling.
  - After a long time, there is a non-equilibrium steady state (NESS) in which only one macroscopic domain survives

Quasi one-dimensional gas

Distribution of the domain sizes

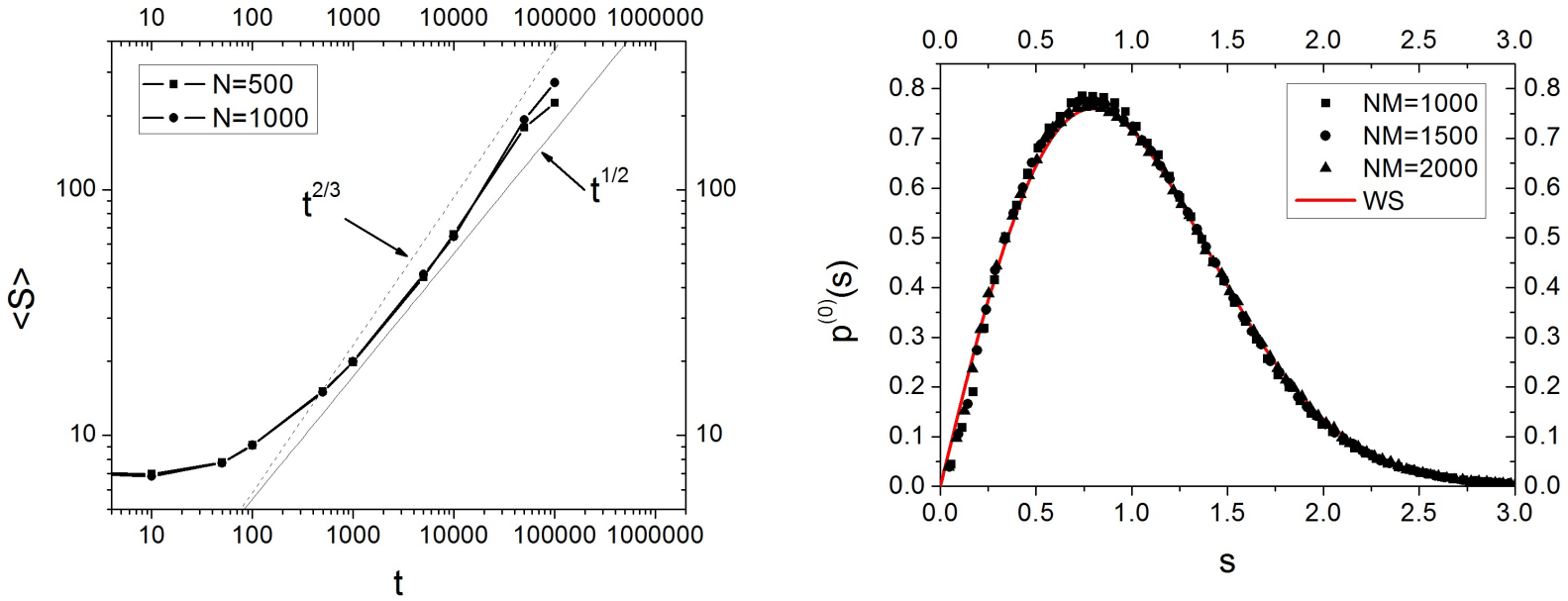


Figure: Domain size distribution.

One-dimensional spin system

1. The lattice has a length N with $N\mu$ spins up ("+") and $N(1-\mu)$ spins down ("−") with $0 < \mu < 1$.
2. Periodic boundary conditions are imposed.
3. An external field which drives the up ("+") spins to the right and the down ("−") spins to the left

The microscopic dynamics of the lattice of spins may be mapped onto one for an array domain dynamics, which provides a good approximation in the regime $T \ll E \ll J$.

One-dimensional spin system

The algorithm used for the numerical simulation is the following:

1. Set up a random array of alternating down and up spins, with μN spins up and $(1-\mu)\,N$ spins down.
2. Choose a domain at random.
3. If the domain is down, move it to the left (i.e., reduce the size of its left neighbor by one and increase its right neighbor by one), otherwise move it to the right.
4. If one of the neighbor domains is zero size, then the domain is removed.
5. Update the clock by 1/number of domains.
6. Repeat steps 2–6.

One-dimensional spin system

From the simulation, the following qualitative results are observed:

- For early times, little domains form everywhere.
- At intermediate times, some domains disappear while other domains grow. In this time regime, the system shows dynamical scaling.
- For later times, only two macroscopic domains remain, which move in opposite directions. The system falls in a non-equilibrium steady state.

One-dimensional spin system

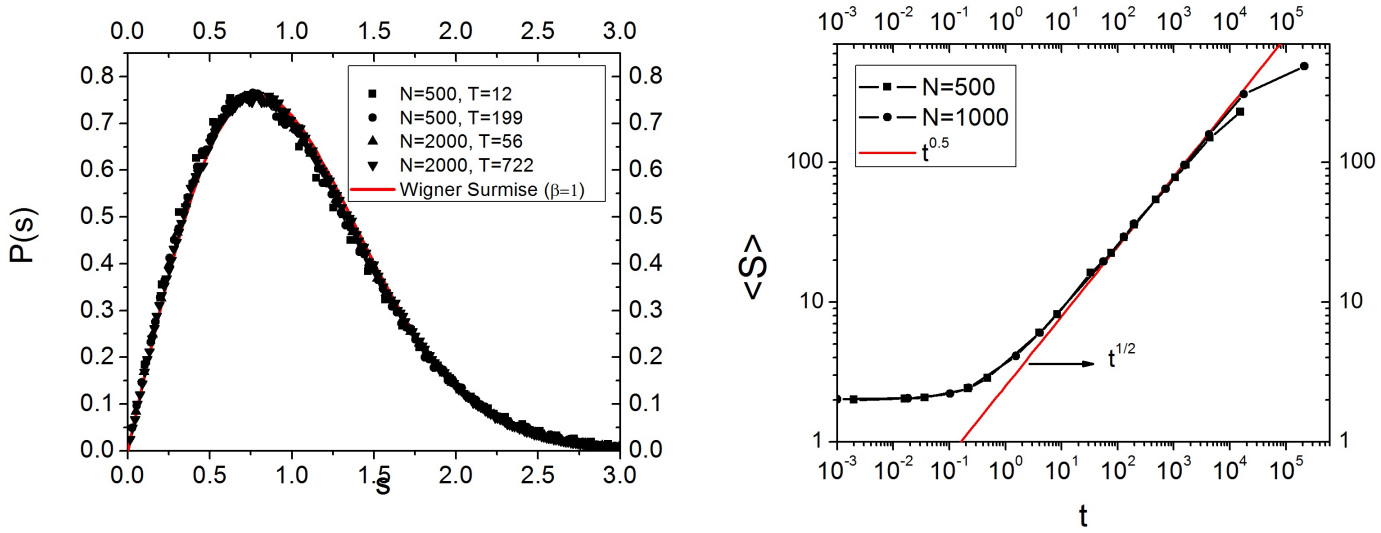


Figure: One-dimensional spin chain proposed by Stephen J. Cornell and Alan J. Bray PRE **54**, 1153 (1996).

Conclusions and final remarks

- Out of equilibrium system exhibits many interesting and novel features as dynamical scaling, NESS, order disorder transitions, etc.
- Even exactly solvable cases such as the TASEP shows interesting features (The height fluctuation at one point strongly depends on the initial and boundary conditions).
- Systems with different microscopical features can share similar fluctuations. In this sense those fluctuations are universal.
- In many cases, changes on the details on the model do not change features of the system. However, often the solution method changes drastically.

Conclusions and final remarks

- The connection is just partial. For example, many systems can share the same gap size distribution with different statistical behaviors

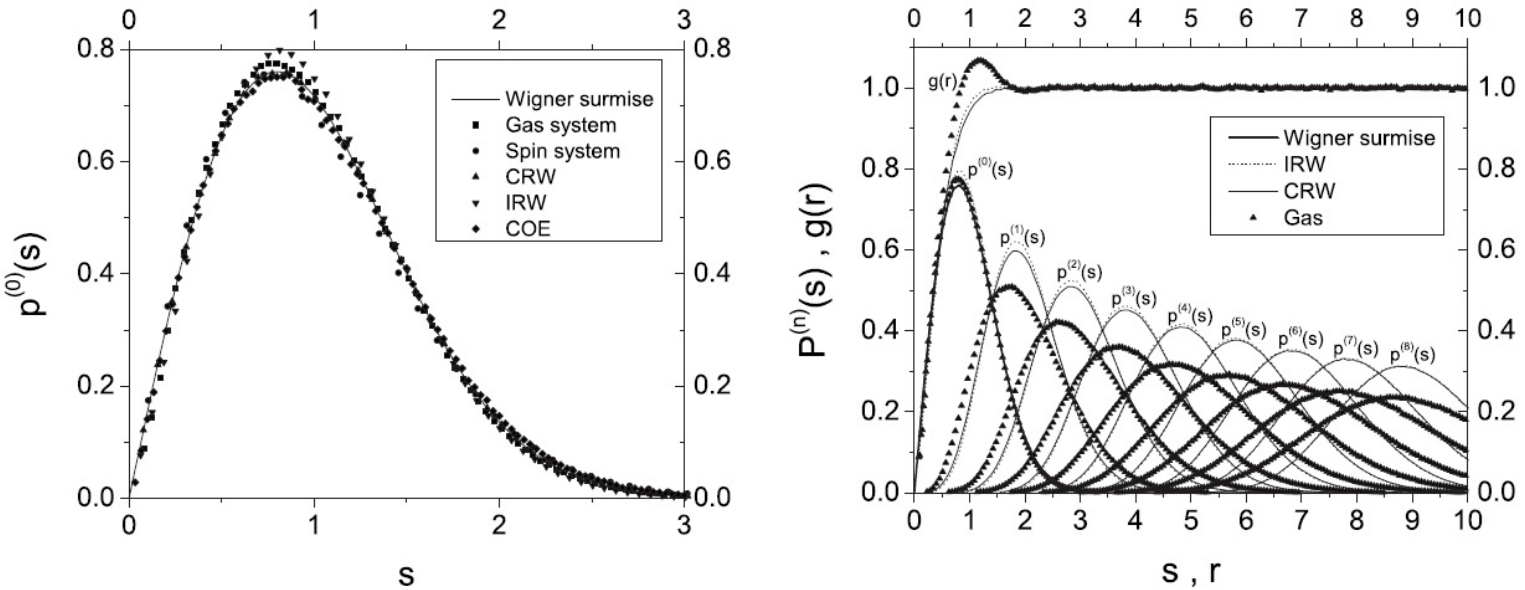


Figure: González and Téllez PRE **76**, 011126 (2007).

Acknowledgments

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