A brief introduction to Montgomery Conjecture (Pair correlation of zeros of ζ)

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ζ

$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ is defined for $s = \sigma + i\gamma$ for $\sigma > 1$.

ζ

To extend ζ meromorphically to $\mathbb{C}$ we use the formula:

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)}\{\frac{1}{s(s-1)} + \int_1^\infty (x^{1/2s-1} + x^{-1/2s-1}) \sum_{n=1}^\infty e^{-n^2\pi x} dx\}$$

and observe the right–hand side integral represents an entire function of s.

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To prove the previous formula we use

$$\Gamma(\frac{s}{2}) = \int_0^{\infty} e^{-t} t^{s/2-1} dt = n^s \pi^{s/2} \int_0^{\infty} e^{-n^2 \pi x} x^{1/2s-1} dx.$$

Some properties of ζ

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$\zeta(s) \neq 0$ for $\sigma > 1$. This follows from the convergence of the Euler product formula:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \Pi_{p \text{ prime}} \left(1 - \frac{1}{p^s} \right).$$

Some properties of ζ

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$\zeta(s)$ has simple zeros at $0, -2, -4, \cdots$. Because $\Gamma(s/2)$ has simple poles at $0, -2, -4, \cdots$ and

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Some properties of ζ

$\zeta(s) \neq 0$ for $\sigma = 1$ (result of Hadamard and De la Vallée Poussin).

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Zeros of ζ are symmetric respect to $\sigma = 1/2$ for $0 \leq \sigma \leq 1$.Because $\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-\frac{(1-s)}{2}}\Gamma(\frac{(1-s)}{2})\zeta(1-s)$
Hence $\zeta(s) \neq 0$ for $\sigma = 0$.

Some properties of ζ

From
$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)}\{\frac{1}{s(s-1)} + \int_1^{\infty}(x^{1/2s-1} + x^{-1/2s-1})\sum_{n=1}^{\infty} e^{-n^2\pi x} dx\}$
we can see zeros of ζ are symmetric respect the real axis because conjugates of zeros are also zeros.

Some properties of ζ

The zeros of ζ are symmetric respect to $s = 1/2$.

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The zeros of ζ are symmetric respect to $s = 1/2$. Because $\xi(s) = \frac{1}{2}s(s-1)\pi^{-1/2s}\Gamma(s/2)\zeta(s)$ satisfies $\xi(s) = \xi(1-s)$ and the function $\frac{1}{2}s\Gamma(s/2)$ has no zeros.

Some properties of ζ

Riemann conjecture: All the non–trivial zeros of ζ are contained in the line $\sigma = 1/2$.

Some properties of ζ

$N(T)$ the number of zeros in the critical line such that, $0 \leq \gamma < T$ then

$$N(T) = \frac{T}{2\pi} log(\frac{T}{2\pi}) - \frac{T}{2\pi} + O(log(T)).$$

Montgomery conjecture

Montgomery conjecture

(1973) Montgomery Pair Correlation Conjecture: Assume the Riemann hypothesis. For fixed $0 < a < b < \infty$ as $T \to \infty$,

$$\sum_{(\gamma,\gamma')\in[0,T]^2 : a \leq (\gamma-\gamma')\frac{log(T)}{2\pi} \leq b} 1 \sim \frac{T}{2\pi} log(T) \int_a^b \left(1 - (\frac{sin(\pi u)}{\pi u})^2\right) du.$$

Gaussian Unitary Ensemble

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Definition

A **Gaussian Unitary Ensemble** *is a set of* $N \times N$ *Hermitian matrices* $H := (a_{ij})$ *such that:*

- *The real and imaginary parts of the entries* a_{ij} *of* H *are independent random variables.*
- $P(H)dH = P(H')dH'$ *where* $H' = U^{-1}HU$ *where* U *is unitary.*

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H is GUE then a_{ij} have Gaussian distributions

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Let $R(x_1, x_2)dx_1dx_2$ denotes the pair correlation of eigenvalues. Intuitively the probability that there are pairs of eigenvalues in $[x_1, x_1 + dx_1] \times [x_2, x_2 + dx_2]$

Asimptotics of GUE pair correlation distribution of eigenvalues

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Theorem

Let $R(x_1, x_2)$ denotes the pair correlation of eigenvalues. Then,

$$\frac{1}{\alpha_1\alpha_2}R(x_1, x_2) \sim 1 - (\frac{\sin(\pi u)}{\pi u})^2$$

as $N \to \infty$ where $u = |x_1/\alpha_1 - x_2/\alpha_2|$ and $\alpha_j = \frac{\pi}{\sqrt{2N - x_j^2}}$ is the mean local spacing of eigenvalues at x_j, $j = 1, 2$.

Convolution formula

Convolution formula

Theorem

We have

$$\sum_{(\gamma,\gamma')\in[0,T]^2} r\left((\gamma-\gamma')\frac{\log(T)}{2\pi}\right)\omega(\gamma-\gamma') \sim \frac{T}{2\pi}\log(T)\int_{\infty}^{\infty} F(u)\hat{r}(u)du$$

Convolution formula

Observe that if $r(u) = \chi_{[a,b]}(u)$ and $\omega(\gamma - \gamma') \to 1$ when $(\gamma - \gamma') \to 0$ we will have a tool for motivating Montgomery conjecture!

Motivation of M. Conjecture

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Suppose we have already F and have proved the previous convolution formula...

Motivation of M. Conjecture

Montgomery conjectured furthermore:

$$F(\alpha) = \begin{cases} 1 + o(1) & \text{for } |\alpha| \geq 1 \\ (1 + o(1))T^{-2|\alpha|}\log(T) + |\alpha| + o(1) & \text{for } |\alpha| < 1. \end{cases}$$

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$$= \frac{T}{2\pi}\log(T)\int_{\infty}^{\infty} \hat{F}(u)r(u)du$$

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$= \frac{T}{2\pi}\log(T)\int_{\infty}^{\infty} \hat{F}(u)r(u)du$
If we had a function F such that

$$\hat{F}(u) = 1 - (\frac{sin(\pi u)}{\pi u})^2 + \delta_0$$

Motivation of M. Conjecture

$$\sum_{(\gamma,\gamma')\in[0,T]^2} r((\gamma-\gamma')\frac{\log(T)}{2\pi})\omega(\gamma-\gamma')$$

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$$= \frac{T}{2\pi}\log(T)\int_{\infty}^{\infty} (1-(\frac{sin(\pi u)}{\pi u})^2+\delta_0)r(u)du.$$

Motivation of M. Conjecture

Finally

$$\frac{T}{2\pi}\log(T)\int_{\infty}^{\infty}(1-(\frac{sin(\pi u)}{\pi u})^2+\delta_0)r(u)du =$$
$$\frac{T}{2\pi}\log(T)\int_{a}^{b}(1-(\frac{sin(\pi u)}{\pi u})^2 du.$$

Remark

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Using the previous approach for $r(u) := r_1(u) := \frac{\sin(2\pi a u)}{\pi a u}$, it is possible to prove that $2/3$ of the zeros of the critical line are simple.

What is missing

What is missing

- F satisfies convolution formula and $\hat{F}(u) = 1 - (\frac{sin(\pi u)}{\pi u})^2 + \delta_0$.
- ω.
- Motivates

$$F(\alpha) = \begin{cases} 1 + o(1) & \text{for } |\alpha| \geq 1 \\ (1 + o(1))T^{-2|\alpha|}\log(T) + |\alpha| + o(1) & \text{for } |\alpha| < 1. \end{cases}.$$

What is missing

It is enough to consider $\omega(u) = \frac{4}{4+u^2}$. Roughly because, for $T >> 0$, $a \leq (\gamma - \gamma')\frac{\log(T)}{2\pi} \leq b$ only if $\gamma - \gamma'$ is small, hence $\omega(\gamma - \gamma') \sim 1$.

What is missing

$$F(u) := F(u,T) := \left(\frac{T}{2\pi}\log(T)\right)^{-1} \sum_{(\gamma,\gamma')\in[0,T]^2} T^{iu(\gamma-\gamma')}\omega(\gamma-\gamma').$$

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$$=\sum_{(\gamma,\gamma')\in[0,T]^2}\omega(\gamma-\gamma')r(\alpha(\gamma-\gamma')\frac{\log(T)}{2\pi}).\square$$

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Proposition

We have:

$$\hat{F}(u) = 1 - \left(\frac{\sin(\pi u)}{\pi u}\right)^2 + \delta_0,$$

for $u < 1$.

What is missing

Assuming Riemann hypothesis Montgomery proves:

$$F(u) = (1 + o(1))T^{-2u}\log(T) + u + o(1),$$

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Since $T^{-2u}\log(T)$ behaves like δ_0 when $T \to \infty$, we can deduce that in the limit $F(u) = |u| + \delta_0$.
We know that if $f(u) := \left(\frac{\sin(\pi u)}{\pi u}\right)^2$ then $\hat{f}(u) = (1 - |u|)\chi_1(u)$.
The proposition follows from $F(u) = (1 - \hat{f}(u)) + \delta_0(u)$ because $\hat{\delta}_0 = 1$.

Numerical motivation

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Odlyzko in 1987 obtained many zeros in the critical line with very height heights to empirically test the Montgomery conjecture.

Thank you!