Lower Bounds for singular solutions to the Navier-Stokes equations in $\dot{H}^s(\mathbb{T}^3)$

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Abstract

It is strongly believed that solutions to *Navier-Stokes equations* are unique. Uniqueness of solutions to these equations is closely related to their regularity. One way to prove regularity is give a characterization of singular solutions and show that one of these conditions fails to be true. A characterization of these singular solutions can be formulated in the following statement:

Let u be a weak solution to the *Navier-Stokes equations* with maximal time of regularity $T < \infty$ there exist some constants c_s, $s = 3/2, 5/2$ and some $\rho > 0$ such that

$$\sup_{t\in[T-\rho,T)} \|u(\cdot,t)\|_{\dot{H}^s(\mathbb{T}^3)} \geq \frac{c_s}{|\log(T-t)|^{\frac{2s-1}{4}}(T-t)^{\frac{2s-1}{4}}}.$$

These are rates at which Sobolev spaces norms must blow-up.

Introduction

The Navier-Stokes equations model the evolution of the velocity field of an incompressible fluid. The take the form

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u = -\nabla p + \nu\triangle u + f, \quad \nabla\cdot u = 0, \tag{NS}$$

where u is a velocity field in $\mathbb{R}^n$, p is the fluid's pressure and ν the kinematic viscosity.

These equations were written in the 19th Century, however many natural questions about their solutions remain open. It is not known whether any physically reasonable initial state always leads, under this system, to a physically reasonable solution (state). The two dimensional case $\mathbb{R}^2$ was solved by O. Ladyzhenskaya in [5]. The case of $\mathbb{R}^3$ remains open, it is one of the ***Millenium prize problems***. This problem can also be formulated for periodic functions in which the domain of the velocity field u can be taken as $\mathbb{T}^3 = [0,1]^3$ (which will be our case).

Partial Results(Towards regularity)

- In the seminal paper *'Sur le mouvement d'un liquide visqueux emplissant l'espace'*[6] Leray proved the existence of an initial interval of regularity: Given some initial data $\phi \in (C^\infty(\mathbb{R}^3)\cap L^2(\mathbb{R}^3))^3$, there exists $\eta > 0$, and a weak solution $u(x,t) \in (C^\infty(\mathbb{R}^3)\cap L^2(\mathbb{R}^3))^3$, for all $t \in [0,\eta)$.
- In the same paper Leray states without a proof the follwing blow-up rates, if T the maximal time of regularity for a weak solution u, there exist some universal constants, $c_p > 0$ for $p > 3$ such that
$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^3)} \geq \frac{c_p}{(T-t)^{\frac{p-3}{2p}}}, \tag{1}$$
- In 1984 Kato [4] extends these results to $p = 3$: there exists ϵ_3 such that if
$$\|u(\cdot,t)\|_{L^3(\mathbb{R}^3)} \leq \epsilon_3 \quad \forall t > 0, \tag{2}$$
then u remains regular at all times.
- In 2003 Escauriaza, Seregin and Šverák [3] greatly improve Kato's result to any constant ϵ_3.
- The continuous embeddings in Sobolev spaces $\dot{H}^s \hookrightarrow L^{\frac{3}{6-2s}}$ imply by (1) the blow-up rates
$$\|u(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} \geq \frac{c_s}{(T-t)^{\frac{2s-1}{4}}}, \tag{3}$$
for $1/2 < s < 3/2$ and Kato's result implies $s = 1/2$.
- Robinson, Sadowski and Silva extend (3) in [7] for the range $3/2 < s < 5/2$.
- It is proven in [2] the extrapolation for the cases $s = 3/2, 5/2$, where a logarithmic correction had to be added,
$$\|u(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} \geq \frac{c_s}{|\log(T-t)|^{\frac{2s-1}{4}}(T-t)^{\frac{2s-1}{4}}}, \tag{4}$$
- The weakest formulation of (4): Let T be the maximal time of existence of a weak solution u then there exist a sequence of times $t_j \to T$ and some bounds c_s, such that
$$\|u(\cdot,t_j)\|_{\dot{H}^s(\mathbb{T}^3)} \geq \frac{c_s}{|\log(T-t_j)|^{\frac{2s-1}{4}}(T-t_j)^{\frac{2s-1}{4}}},$$
was the first result, which gave the right exponent to (4).

Framework

The problem fits nicely in Sobolev spaces. Namely in $\dot{H}^s$, they are Hilbert spaces, here one can take limits. And the Fourier series are used, they translate the problem into frequency modes. Here one has an integral formulation which is the starting point for the proof of the blow-up rates.

Sobolev spaces

We define weak derivatives

Definition 1. *g is the weak derivative of f if it satisfies*

$$\int_\Omega f\phi dx = -\int_\Omega g\phi' dx, \quad \forall\phi \in C_0^\infty(\Omega).$$

For $\alpha \in \mathbb{N}^n$ a multinidex we call $D^\alpha f = \frac{\partial^{|\alpha|}f}{\partial^{\alpha_1}x_1\partial^{\alpha_2}x_2\ldots\partial^{\alpha_n}x_n}$, $\quad |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$.

Definition 2. *The Sobolev $H^s(\mathbb{T}^3)$ is the vector space*

$$H^s(\mathbb{T}^n) := \{f \in L^2(\mathbb{T}^n) | D^\alpha f \in L^2(\mathbb{T}^n), |\alpha| \leq s\}.$$

$H^s(\mathbb{T}^n)$ is a Hilbert space.

Fourier Series

Definition 3. *The Fourier transform $\mathcal{F}$, is the linear map*

$$\mathcal{F}: L^2(\mathbb{T}^n) \to L^2(\mathbb{Z}^n),$$
$$f(x) \longmapsto \hat{f}_\xi := \tfrac{1}{(2\pi)^{n/2}}\textstyle\int_{\mathbb{T}^n} f(x)e^{-i\langle x,\xi\rangle}dx, \quad \xi \in \mathbb{Z}^n.$$

$\mathcal{F}$ is an isometry, with inverse $\mathcal{F}^{-1}$:

$$\mathcal{F}^{-1}(\hat{f}_\xi)(x) := \frac{1}{(2\pi)^{n/2}}\sum_{\xi\in\mathbb{Z}^n}\hat{f}_\xi e^{i\langle\xi,x\rangle} \quad x \in \mathbb{T}^n.$$

Derivation translates into multiplication, in Fourier space

$$\frac{\partial f}{\partial x_j} = \sum_{\xi\in\mathbb{Z}^n}\frac{\partial}{\partial x_j}\hat{f}_\xi e^{i\langle\xi,x\rangle} = i\sum_{\xi\in\mathbb{Z}^n}\xi_j\hat{f}_\xi e^{i\langle\xi,x\rangle}.$$

And so we can define $H^s(\mathbb{T}^n) = \{f \in L^2(\mathbb{T}^n) |\ |\xi|^s(\mathcal{F}f)_\xi \in L^2(\mathbb{T}^n)\}$, and obtain continuity from integrability in Fourier space.

Proposition 0.1. *Let $f \in L^2(\mathbb{T}^n)$ if $f \in H^{s+k}(\mathbb{T}^n)$ for $s > n/2$ or if $|\xi|^k\mathcal{F}f \in L^1(\mathbb{Z}^n)$ then the k-order derivatives of f exist and are continuous.*

Therefore we can show regularity if we prove certain decay rates for $\mathcal{F}u$, with u a solution to (NS).By applying $\mathcal{F}$ to our solution (NS), we obtain the following equation

$$\frac{\partial}{\partial t}\hat{u}^j_\xi = -|\xi|^2\hat{u}^j_\xi + i\sum_{\alpha\in\mathbb{Z}^n}\sum_{l,k=1}^{n}\hat{u}^k_{\xi-\alpha}\hat{u}^l_\alpha\xi_k(\frac{\xi_l\xi_j}{|\xi|^2} - \delta_{l,j}), \tag{5}$$

where $\hat{u}_\xi$ are the Fourier modes of u,

$$u^j(x,t) = \sum_{\xi\in\mathbb{Z}^n}\hat{u}^j_\xi(t)e^{i\langle x,\xi\rangle}.$$

And if we integrate with $\phi(x) = u(x,0)$, some given initial data, we obtain

$$\hat{u}^j_\xi(t) = e^{-|\xi|^2t}\hat{\phi}^j_\xi + i\int_0^t\sum_{\alpha\in\mathbb{Z}^n}\sum_{k,l=1}^{n}\hat{u}^k_\alpha(s)\hat{u}^l_{\xi-\alpha}(s)\xi_k\left(\frac{\xi_l\xi_j}{|\xi|^2} - \delta_{j,l}\right)e^{-|\xi|^2(t-s)}ds, \tag{I-NS}$$

The Main Result

Theorem 0.1. *Let u be a weak solution of (NS) whose maximal interval regularity is $(0,T)$, $T < \infty$. Then, there are absolute constants $c_s > 0, s = 3/2, 5/2$, such that there is a sequence $t_j \to T$ along which the following estimate holds*

$$\|u(\cdot,t_j)\|_{\dot{H}^s(\mathbb{T}^3)} \geq \frac{c_s}{|\log(T-t_j)|^{\frac{2s-1}{4}}(T-t_j)^{\frac{2s-1}{4}}}. \tag{6}$$

Sketch of the Proof

The idea is to assume that we have a weak solution u, whose Fourier modes $(\hat{u}_\xi)$ satisfy (I-NS) and the inverted inequality stated in the theorem ($\leq$ instead of $\geq$) call it $(6)^*$. Then we take an increasing sequence of times $t_n \to t^* < T$, which lies inside the interval of regularity.

We start with $t \geq t_1$, and by

$$\left|\hat{u}^m_\xi(t)\right| \leq e^{-|\xi|^2(t-t_0)}\left|\hat{u}^m_\xi(t_0)\right| + \left|\int_{t_0}^t e^{-|\xi|^2(t-s)}\sum_{\alpha\ j,l}\hat{u}^j_\alpha(s)\hat{u}^l_{\xi-\alpha}(s)\xi_j(\frac{\xi_l\xi_m}{|\xi|^2} - \delta_{l,m})ds\right|,$$

$$\leq e^{-|\xi|^2(t_1-t_0)}\left|\hat{u}^m_\xi(t_0)\right| + \sup_{t_0<s<t}|\sum_{\alpha\ j,l}\hat{u}^j_\alpha\hat{u}^l_{\xi-\alpha}(s)\xi_j(\frac{\xi_l\xi_m}{|\xi|^2} - \delta_{l,m})|\int_{t_0}^t e^{-|\xi|^2(t-s)}ds,$$

$$\leq e^{-|\xi|^2(t_1-t_0)}\left|\hat{u}^m_\xi(t_0)\right| + \frac{1}{|\xi|^2}\sup_{t_0<s<t}|\sum_{\alpha\ j,l}\hat{u}^j_\alpha\hat{u}^l_{\xi-\alpha}(s)\xi_j(\frac{\xi_l\xi_m}{|\xi|^2} - \delta_{l,m})|.$$

Now we to bound the non-linear term, applying $(6)^*$ to the Fourier modes inside this sum, then we get a particular bound, something alike

$$|\sum_{\alpha\in\mathbb{Z}^n}\hat{u}_{\xi-\alpha}\hat{u}_\alpha\xi| \leq K(\xi),$$

and overall

$$\left|\hat{u}^m_\xi(t)\right| \leq e^{-|\xi|^2(t-t_0)}M(\xi) + \frac{K(\xi)}{|\xi|^2} \leq D_1(\xi),\ \forall t \geq t_1,$$

where we applied $(6)^*$ again. We obtain the *base case* for an inductive process, then we assume some bound for t_n:

$$|\hat{u}_\xi(t)| \leq D_n(\xi), \quad \forall t \geq t_n,$$

and to do the inductive step, take $t \geq t_{n+1}$. It holds

$$\left|\hat{u}^m_\xi(t)\right| \leq e^{-|\xi|^2(t-t_n)}\left|\hat{u}^m_\xi(t_n)\right| + \left|\int_{t_n}^t e^{-|\xi|^2(t-s)}\sum_{\alpha\ j,l}\hat{u}^j_\alpha\hat{u}^l_{\xi-\alpha}\xi_j(\frac{\xi_l\xi_m}{|\xi|^2} - \delta_{l,m})ds\right|,$$

we use the D_n bound for the non-linear term and we obtain a better bound, for $t \in [t_{n+1}, T)$

$$|\sum_{\alpha\in\mathbb{Z}^n}\hat{u}_{\xi-\alpha}\hat{u}_\alpha\xi| \leq K_n(\xi).$$

With these considerations

$$\left|\hat{u}^m_\xi(t)\right| \leq e^{-|\xi|^2(t_{n+1}-t_n)}D_n(\xi) + \frac{K_n(\xi)}{|\xi|^2} \leq D_{n+1}(\xi),\ \forall t \geq t_{n+1},$$

therefore if $t_n \to t^* \in [T-\rho, T]$, we conclude

$$|u_\xi(t)| \leq D(\xi) = \lim_{n\to\infty}D_n(\xi) \leq \frac{Cc_s}{|\xi|^2}, \quad \forall t \in [t^*, T].$$

Given that the above limit exists and is finite. C is a universal constant, then by *Theorem 2* in [1] if c_s is small enough, u can be extended to a smooth solution in $(T-\rho, T]$, applying Leray's results with our initial data as $u(\cdot,T)$ we can extend the solution to some interval $(T-\rho, T+\eta)$, $\eta > 0$, where it is regular.

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