Teoremas de descomposición de Kolmogorov y Naimark

Arnaldo de la Barrera Correa- Universidad De Pamplona, Colombia

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RESUMEN

El objetivo de este trabajo es presentar algunos resultados conocidos respecto a la **descomposición de núcleos definidos positivos a valores operadores** . En primer lugar se presenta una versión de un resultado clásico de Kolmogorov referente a un núcleo arbitrario. A continuación se presenta una versión dada por Evans y Lewis ver ([1] Teorema 3.2), de un resultado clásico de Naimark [2] referente a la **descomposición de núcleos de Toeplitz ordinarios**.
Finalmente se presenta una versión del Teorema de Naimark para el caso de los núcleos de Toeplitz generalizados.

Positive definite kernel

Consider a family $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ of Hilbert spaces. A map K on $\mathbb{Z}\times\mathbb{Z}$ such that $K(n,m)\in L(\mathcal{H}_m,\mathcal{H}_n)$ for $n,m\in\mathbb{Z}$ is called a *kernel* on $\mathbb{Z}$.
A sequence $\{h_n\}$ in $\oplus_{n\ in\mathbb{Z}}\mathcal{H}_n$ is said to have *finite support* if $h_n=0$ except for finitely many n's. And a map K on $\mathbb{Z}\times\mathbb{Z}$ such that $K(n,m)\in L(\mathcal{H}_m,\mathcal{H}_n)$ for $n,m\in\mathbb{Z}$ is called a *positive definite kernel* on $\mathbb{Z}$ if

$$\sum_{n,m\in\mathbb{Z}}\langle K(n,m)h_m,h_n\rangle_{\mathcal{H}_n}\geq 0,$$

for all sequence $\{h_n\}$ en $\oplus_{n\in\mathbb{Z}}\mathcal{H}_n$ with finite support.

The Hilbert space induced by a positive definite kernel

Let K be a positive definite kernel. Let $\mathcal{F}$ be the linear space of the elements of $\bigoplus_{n\in\mathbb{Z}}\mathcal{H}_n$ and $\mathcal{F}_o$ the space of the elements of $\mathcal{F}$ with finite support.
Define $B_K:\mathcal{F}_o\times\mathcal{F}_o\to\mathbb{C}$ by the formula

$$B_K(f,g)=\sum_{m,n\in\mathbb{Z}}\langle K(n,m)f_m,g_n\rangle_{\mathcal{H}_n}, \tag{1}$$

for $f,g\in\mathcal{F}_o$, $f=\{f_n\}$, $g=\{g_n\}$, $f_n,g_n\in\mathcal{H}_n$.
Remark that B_K satisfies all the properties of an inner product, except for the fact that the set $\mathcal{N}_K=\{h\in\mathcal{F}_o:B_K(h,h)=0\}$ may be non trivial.
The equality $\mathcal{N}_K=\{h\in\mathcal{F}_o:B_K(h,g)=0,\ \text{for all } g\in\mathcal{F}_o\}$ is a consequence of the Cauchy-Schwarz inequality. It follows that $\mathcal{N}_K$ is a linear subspace of $\mathcal{F}_o$.
Then the quotient space $\mathcal{F}_o/\mathcal{N}_K$ is also a linear space. If $[h]$ denotes the class in $\mathcal{F}_o/\mathcal{N}_K$ of the element h then the map given by the formula

$$\langle[h],[g]\rangle_K=B_K(h,g),\qquad h,g\in\mathcal{F}_o$$

is well defined and it is easily checked that $\langle\cdot,\cdot\rangle_K$ is an inner product on $\mathcal{F}_o/\mathcal{N}_K$.
The definition of this inner product is independent of the chosen representatives.
The completion of $\mathcal{F}_o/\mathcal{N}_K$ with respect to the norm induced by this inner product is a Hilbert space denoted by $\mathcal{H}_K$ and it is known as the *Hilbert space induced by the positive definite kernel* K.

The Kolmogorov decomposition theorem

The following theorem is a version of a classical result of Kolmogorov (for reviews of the history of this result see [2]).

THEOREM Let $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ be a family of Hilbert spaces and let $K:\mathbb{Z}\times\mathbb{Z}\to L(\mathcal{H}_m,\mathcal{H}_n)$ be a positive definite kernel . Then there exits a Hilbert space $\mathcal{H}_K$ and an application V defined on $\mathbb{Z}$ such that $V(n)\in L(\mathcal{H}_n,\mathcal{H}_K)$ for each $n\in\mathbb{Z}$ and

(a) $K(n,m)=V^*(n)V(m)$, $n,m\in\mathbb{Z}$.

(b) $\mathcal{H}_K=\bigvee_{n\in\mathbb{Z}}V(n)\mathcal{H}_n$.

(c) The decomposition is unique in the following sense: if $\mathcal{H}'$ is another Hilbert space and V' defined on $\mathbb{Z}$ is an application such that $V'(n)\in L(\mathcal{H}_n,\mathcal{H}_K)$ for each $n\in\mathbb{Z}$ which satisfy (a) and (b), then there exists a unitary operator $\Phi:\mathcal{H}_K\to\mathcal{H}'$ such that $\Phi V(n)=V'(n)$ for all $n\in\mathbb{Z}$.

This theorem and its proof can be see in [1, Teorema 3.1].

Proof

The proof is as follows.
Let $\mathcal{H}_K$ be the Hilbert space induced by the positive definite kernel K.
If $h\in\mathcal{H}_n$, then the element $h_n\in\mathcal{F}_o$ is defined as follows:

$$h_n(m)=\begin{cases}h & \text{si } m=n\\ 0 & \text{si } m\neq n\end{cases} \tag{2}$$

Therefore

$$\|[h_n]\|_K^2=\langle[h_n],[h_n]\rangle_K=\langle K(n,n)h,h\rangle_{\mathcal{H}_n}.$$

The map V can be defined by the formula $V(n):\mathcal{H}_n\to\mathcal{H}_K$ by

$$V(n)h=[h_n]=[h\delta_n],\qquad h\in\mathcal{H}_n,$$

where δ_n is the Kronecker delta. Then we have

$$\|V(n)h\|_K^2=\langle K(n,n)h,h\rangle_{\mathcal{H}_n}\leq\|K(n,n)\|\,\|h\|_{\mathcal{H}_n}^2,\qquad h\in\mathcal{H}_n.$$

Which shows that $V(n)\in L(\mathcal{H}_n,\mathcal{H}_K)$.

A map V which satisfies the property (a) of the previous theorem is called a *Kolmogorov decomposition of the kernel* K or simply a *decomposition of the kernel* K (see [1]). The property (b) is referred to as the *minimality property* of the Kolmogorov decomposition. The meaning of the property (c) is that, under the minimality condition (b) the Kolmogorov decomposition is essentially unique.

Positive definite Toeplitz kernel

The kernel K is called a *positive definite Toeplitz kernel* if the family reduces to a single Hilbert space, i.e. $\mathcal{H}_n=\mathcal{H}$ for all $n\in\mathbb{Z}$, and the positive definite kernel $K:\mathbb{Z}\times\mathbb{Z}\to L(\mathcal{H})$ has the property

$$K(n,m)=W(m-n),$$

for all $n,m\in\mathbb{Z}$, for a certain map $W:\mathbb{Z}\to L(\mathcal{H})$.
Proposition. Let $\mathcal{H},\mathcal{H}_1$ be two Hilbert spaces, given $Q\in L(\mathcal{H},\mathcal{H}_1)$ and $S\in L(\mathcal{H}_1)$ a unitary operator, let $K:\mathbb{Z}\times\mathbb{Z}\to L(\mathcal{H})$ be a kernel given by

$$G(n,m)=Q^*S^{m-n}Q.$$

Then G is a positive definite hermitian Toeplitz kernel.
proof Clearly G is a Toeplitz kernel. G is Hermitian, since,

$$(G(n,m))^*=Q^*(S^{m-n})^*Q=Q^*(S^{m-n})^{-1}Q=Q^*S^{n-m}Q=G(m,n).$$

G is positive definite. Indeed, let $h:\mathbb{Z}\to\mathcal{H}$ be a sequence with finite support, then

$$\sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}\langle G(n,m)h_m,h_n\rangle_{\mathcal{H}}=\sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}\langle Q^*S^{m-n}Qh_m,h_n\rangle_{\mathcal{H}}=\sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}\langle S^mQh_m,S^nQh_n\rangle_{\mathcal{H}_1}=\left\|\sum_{m\in\mathbb{Z}}S^mQh_m\right\|_{\mathcal{H}_1}^2\geq 0.$$

Naimark Theorem

THEOREM Let K be a positive definite Toeplitz kernel defined on $\mathbb{Z}$. Then there exist a Hilbert space $\mathcal{H}_K$, a unitary operator S in $L(\mathcal{H}_K)$ and an operator Q in $L(\mathcal{H},\mathcal{H}_K)$ such that

(a) $K(n,m)=Q^*S^{m-n}Q$, $n,m\in\mathbb{Z}$.

(b) $\mathcal{H}_K=\bigvee_{n\in\mathbb{Z}}S^nQ\mathcal{H}$.

(c) If there exist another Hilbert space $\mathcal{H}'$, a unitary operator S' in $L(\mathcal{H}')$ and an operator Q' in $L(\mathcal{H},\mathcal{H}')$ such that (a), (b) hold, then there exists a unitary operator $\Phi:\mathcal{H}_K\to\mathcal{H}'$ such that $\Phi Qh=Q'h$ for all $h\in\mathcal{H}$ and $S'\Phi=\Phi S$.

Also $Q=V(0)$ where $V:\mathbb{Z}\to L(\mathcal{H},\mathcal{H}_K)$ is the Kolmogorov decomposition of the kernel K.

Proof

By Theorem Kolmogorov, there exist a Hilbert space $\mathcal{H}_K$ and a map $V:\mathbb{Z}\to L(\mathcal{H},\mathcal{H}_K)$ such that the following assertions hold

(a') $K(n,m)=V^*(n)V(m)$, for all $n,m\in\mathbb{Z}$

(b') $\mathcal{H}_K=\bigvee_{n\in\mathbb{Z}}V(n)\mathcal{H}$.

Using (b'), one defines the operator $S:\mathcal{H}_K\to\mathcal{H}_K$ as follows

$$S\left(\sum_{n\in\mathbb{Z}}V(n)h_n\right)=\sum_{n\in\mathbb{Z}}V(n+1)h_n,$$

where $\{h_n\}_{n\in\mathbb{Z}}$ is a sequence in $\mathcal{H}$ with finite support.
Using (a'), we obtain

$$\left\|\sum_{n\in\mathbb{Z}}V(n)h_n\right\|_K^2=\sum_{n,m\in\mathbb{Z}}\langle V(m)h_m,V(n)h_n\rangle_K=\sum_{n,m\in\mathbb{Z}}\langle V^*(n)V(m)h_m,h_n\rangle_{\mathcal{H}}=\sum_{n,m\in\mathbb{Z}}\langle K(n,m)h_m,h_n\rangle_{\mathcal{H}}.$$

and

$$\left\|S\left(\sum_{n\in\mathbb{Z}}V(n)h_n\right)\right\|_K^2=\left\|\sum_{n\in\mathbb{Z}}V(n+1)h_n\right\|_K^2=\sum_{n,m\in\mathbb{Z}}\langle V(m+1)h_m,V(n+1)h_n\rangle_K=\sum_{n,m\in\mathbb{Z}}\langle K(n+1,m+1)h_m,h_n\rangle_{\mathcal{H}}.$$

Since K is a Toeplitz kernel, we have that $K(n,m)=K(n+1,m+1)$ for $n,m\in\mathbb{Z}$. Therefore,

$$\left\|S\left(\sum_{n\in\mathbb{Z}}V(n)h_n\right)\right\|_K^2=\left\|\sum_{n\in\mathbb{Z}}V(n)h_n\right\|_K^2.$$

By the definition of S, it follows that

$$SV(n)=V(n+1),\qquad \text{for all}\quad n\in\mathbb{Z}.$$

Hence, for $n,m\in\mathbb{Z}$ and $m>n$, the equality

$$S^{m-n}V(0)=V(m-n)$$

holds and then

$$K(n,m)=K(0,m-n)=V^*(0)V(m-n)=V^*(0)S^{m-n}V(0).$$

Taking $Q=V(0)$, we finish the proof of (a).
Part (b) follows immediately from (b') and the previous results.
Part (c) is obtained as a repetition of the corresponding part in Theorem Kolmogorov. The operator S will be referred to as the *Naimark dilation* or *shift* in $\mathcal{H}_K$.

Núcleos de Toeplitz generalizados

Consideraremos los conjuntos
$\mathbb{Z}_1=\{n\in\mathbb{Z}:n\geq 0\}$, $\mathbb{Z}_2=\{n\in\mathbb{Z}:n<0\}$, $\mathbb{Z}^+=\{n\in\mathbb{Z}:n>0\}$ y $\mathbb{Z}_\alpha-\mathbb{Z}_\beta=\{m-n:m\in\mathbb{Z}_\alpha,n\in\mathbb{Z}_\beta\}$ con $\alpha,\beta=1,2$.
Es posible probar lo siguiente:

$$\mathbb{Z}_1-\mathbb{Z}_1=\mathbb{Z},\qquad \mathbb{Z}_2-\mathbb{Z}_2=\mathbb{Z},\qquad \mathbb{Z}_1-\mathbb{Z}_2=\mathbb{Z}^+\qquad \mathbb{Z}_2-\mathbb{Z}_1=-\mathbb{Z}^+.$$

Definicion1
Sea $K:\mathbb{Z}\times\mathbb{Z}\to L(\mathcal{H})$ un núcleo. Se dice que K es un núcleo de Toeplitz generalizado si existen cuatro funciones $k_{\alpha\beta}:\mathbb{Z}_\alpha-\mathbb{Z}_\beta\to L(\mathcal{H})$ tal que

$$K(m,n)=k_{\alpha\beta}(m-n),\qquad \text{para todo } (m,n)\in\mathbb{Z}_\alpha\times\mathbb{Z}_\beta.$$

Definicion2 Un núcleo de Toeplitz generalizado sobre $\mathbb{Z}$ consiste en un conjunto $K=\{(k_{\alpha\beta}),\alpha,\beta=1,2;\mathcal{H}_1,\mathcal{H}_2\}$ formado por dos espacios de Hilbert $\mathcal{H}_1$ y $\mathcal{H}_2$ y cuatro funciones

$$k_{\alpha\beta}:\mathbb{Z}_\alpha-\mathbb{Z}_\beta\to L(\mathcal{H}_\alpha,\mathcal{H}_\beta),\qquad \alpha,\beta=1,2,$$

tal que $k_{21}(n)=k_{21}^*(-n)$ es cierto para todo $n\in\mathbb{Z}_2$.
Se dice que el núcleo de Toeplitz generalizado K, es definido positivo si

$$\sum_{\alpha,\beta=1,2}\ \sum_{(i,j)\in\mathbb{Z}_\alpha\times\mathbb{Z}_\beta}\langle k_{\alpha\beta}(i-j)h_\alpha(i),h_\beta(j)\rangle_{H_\beta}\geq 0 \tag{3}$$

con $h_\alpha:\mathbb{Z}_\alpha\to\mathcal{H}_\alpha$ una sucesión con soporte finito y $\alpha=1,2$.

Teorema de Naimark para núcleos de Toeplitz generalizados

THEOREM Sea $K=\{(k_{\alpha\beta}),\alpha,\beta=1,2;\mathcal{H}_1,\mathcal{H}_2\}$ un núcleo de Toeplitz generalizado definido positivo sobre $\mathbb{Z}$. Entonces existen un espacio de Hilbert $\mathcal{H}_K$, un operador unitario S en $L(\mathcal{H}_K)$, operadores Q_1 en $L(\mathcal{H}_1,\mathcal{H}_K)$ y Q_2 en $L(\mathcal{H}_2,\mathcal{H}_K)$ tales que
(a)$K(i,j)=Q_1^*S^{j-i}Q_1$ si $i,j\in\mathbb{Z}_1$ (b) $K(i,j)=Q_2^*S^{j-i}Q_2$ si $i,j\in\mathbb{Z}_2$
(c)$K(i,j)=Q_2^*S^{j-i}Q_1$ si $i\in\mathbb{Z}_1$ y $j\in\mathbb{Z}_2$ (d) $K(i,j)=Q_1^*S^{j-i}Q_2$ si $i\in\mathbb{Z}_2$ y $j\in\mathbb{Z}_1$
(e) $\mathcal{H}_K=\bigvee_{\alpha=1,2}\bigvee_{n\in\mathbb{Z}_\alpha}S^nQ_\alpha\mathcal{H}_\alpha$.
(f) Si existe otro espacio de Hilbert $\mathcal{H}'$, un operador unitario S' en $L(\mathcal{H}')$ y operadores Q'_α con $\alpha=1,2$ que satisfacen (1),(2), entonces existe un operador unitario $\Phi:\mathcal{H}_K\to\mathcal{H}'$ tal que $\Phi S^nQ_\alpha=(S')^nQ'_\alpha$, para todo $h\in\mathcal{H}_\alpha$ y además $S'\Phi=\Phi S$.

Referencias

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