Logic Gates in Topological Quantum Computing

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ABSTRAC

The quantum computing based in anyons is a new type of computing characterized to be very robust with respect to environment perturbations. The proposal is based on the existence the topologic states of the matter whose quasiparticle excited are neither bosons and fermions, but rather particles called anyons. In the Topological Quantum Computation (TQC), the quantum information is stored in two-dimensional trajectories of the quasiparticle, which can be described in a space-time 2+1 dimensional. The world lines of the quasiparticle cross over each other to form the braids in that three-dimensional space. The quantum logic gates depend only from topologic of braiding. In this work, of pedagogical type, shown as finding braids leading to a universal set of quantum gates to quantum bits encoded in a kind of quasiparticle that, according literature consulted, will lead to promising implementations of quantum computing. Experiment with Fractional Quantum Hall Effect (FQHE) indicate that these elements can be created in the real world using semiconductors made of gallium arsenide near absolute cero and subject to strong magnetic field.

Topology and Quantum Computing

Topology study the properties of geometric objects that remaining unchanged for continues transformation. In our study can be said that the aim is focus in the features of the geometry that are robust to local perturbations small. This properties are invariants under a homeomorphism. Describes the TQC in terms of Temperley–Lieb Recoupling Theory (TLTR). In quantum computing, the application of topology is most interesting because the simplest non-trivial example of the Temperley–Lieb recoupling theory gives the so-called Fibonacci model. This theory allows to construct certain transfer matrices through a given algebra. This algebra is related to knot theory and the Braid group. The recoupling theory yields representations of the Artin braid group into unitary groups U (n), where n is a Fibonacci number. With the unit operation U(n) will have operators that can be used for universal quantum computing modeling in terms of representations of braid group.

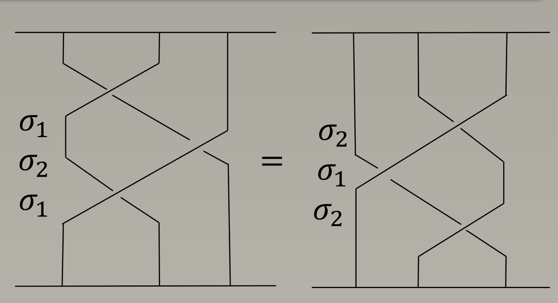
Grupo Braid B_n

B_n group can be presented as a set of generators that obey certain relations.

$\sigma_j\,\sigma_k = \sigma_k\,\sigma_j,\ |j-k| \geq 2$;

$\sigma_j\,\sigma_{j+1}\,\sigma_j = \sigma_{j+1}\,\sigma_j\,\sigma_{j+1}$, j=1,2,3,…,n-2

Yang-Baxter Relation



Labels (Charges)

We use latin letters *{a, b, c, …}* for the labels that distinguish different types of particles, and we assume that the set of possible labels is finite. The symbol a represents the value of the conserved charge carried by the particle.

-Labels: {$\mathbf{1} \rightarrow$ *Identity (trivial particle)*;

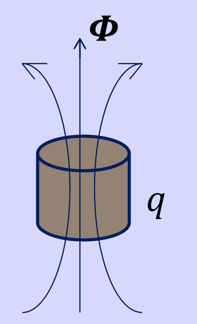
$\tau \rightarrow$ *nontrivial particle*;

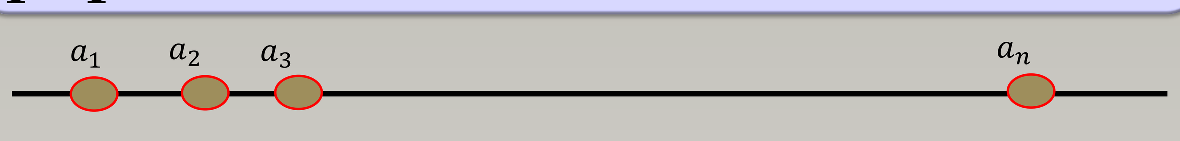
$a \rightarrow$ *generic particle (can be* $\mathbf{1}$ *o* a)}

$\mathbf{1} \times a = a$

$\exists\, \bar{a}$: $\bar{a} \times a = \mathbf{1}$; ($\bar{a} \rightarrow$ conjugate or antiparticle)

$\bar{\mathbf{1}} = \mathbf{1}$





Spin in 2D and 3D ⟶ ANYONS

Just as fermions are spin half integer (1/2, 3/2,…) particles obeying Fermi-Dirac statistics and bosons are spin integer (0, 1, 2,…) particles obeying Bose-Einstein statistics, 'anyons' (from any) are, particles with any spin obeying any statistics. The difference is that anyons can only exist in two space dimensions.

3D: The spatial components of the spin operators satisfy the commutation relation $[\hat{S}_i, \hat{S}_j] = \hbar i \epsilon_{ijk} S_k$. In quantum mechanic we have $\hat{S}^2|s\rangle = \hbar^2 s(s+1)|s\rangle$ where s can only be an integer (bosons) or half integer $(2k+1)/2$ (fermions).

2D: There is only one axis of rotation (⊥ to the plane) only $\hat{S}_z$ matter y no satisface relaciones de conmutación. and does not satisfy commutation relation. Any value of $\hat{S}_z$ is possible $\hat{S}_z|s\rangle = s\hbar|s\rangle$, $s \in \mathbb{R}$. These particles are called ANYONS.

When one particle is exchanged in a counterclockwise manner with the other, the wave function can change by an arbitrary phase $\psi \rightarrow e^{i\theta}\psi$, in a second counterclockwise exchange $\psi \rightarrow e^{i2\theta}\psi$ (not lead back to the initial state). $\theta = 0\ (bosons), \theta = \pi\ (fermions)$,other values of $\theta\ (anyons)$.

Rules for fusing and splitting

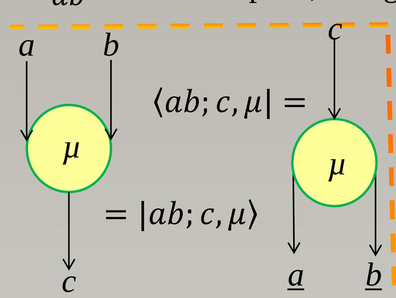
When two particles are combined together, the composite object also has a charge. The *fusion rules* of the model specify the possible values of the total charge c when the constituents have charges a y b.

$a \times b = \sum_c N_{ab}^c c$ $\quad N_{ab}^c$: Non-negative integer. The sum is over all the possible values of the charges.

If N_{ab}^c=0 c can not be obtained. Si N_{ab}^c=1 c can be obtained in a unique way. Si $N_{ab}^c > 1$ c can be obtained in N_{ab}^c distinguishable ways, each denoted by μ or ν.

The N_{ab}^c distinguishable ways that c can arise by fusing a and b can be regarded as the orthonormal basis states of a Hilbert space V_{ab}^c.

$V_{ab}^c \rightarrow$ Fusion space, being its states the fusion states $\{|ab;c,\mu\rangle, \mu = 1,2,\ldots,N_{ab}^c\}$

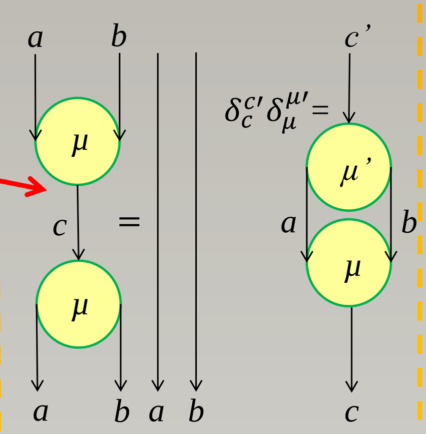


The spaces V_{ab}^c with differents c are orthogonal $\langle ab;c',\mu'|ab;c,\mu\rangle = \delta_{c}^{c'}\delta_{\mu}^{\mu'}$

Completeness relation

$\sum_{c,\mu} |ab;c,\mu\rangle\langle ab;c,\mu| = I_{ab}$

I_{ab}: Projector in the space $\oplus_c V_{ab}^c$ (full Hilbert space for the anyon pair ab.)

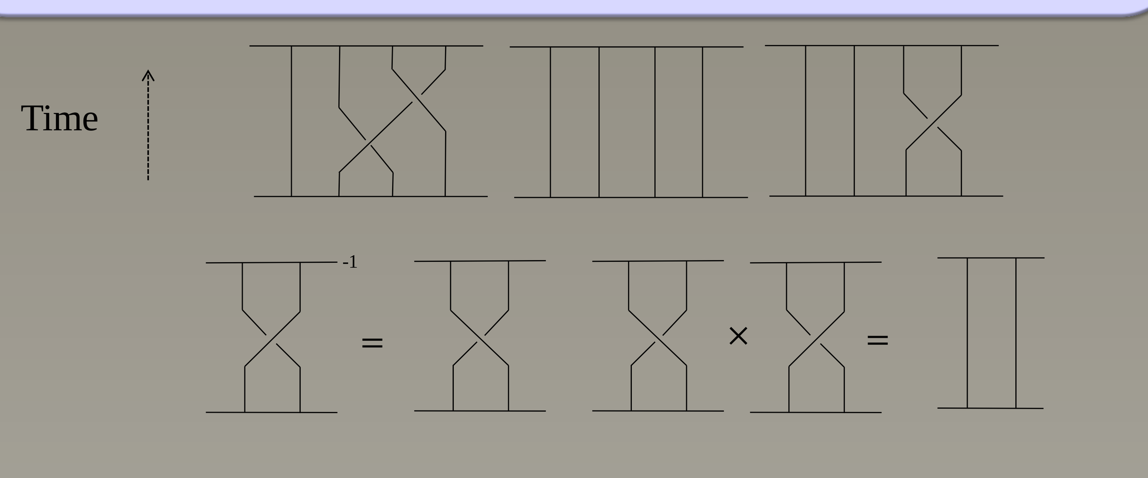


Braid Group B_n

The braid group $\boldsymbol{B_n}$ (of infinite order) of n strands generalizes the permutation group P_n (of order $n!$).

-Each element of the group is displayed as a braid of n strands. All strands are trajectories' particle as world lines in space-time 2+1 dimensional.

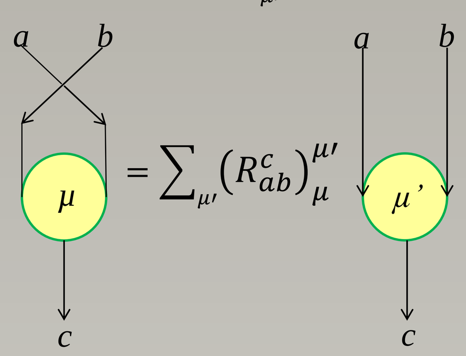
-When considering N anyons, the topological classes of trajectories which take these particles from initial positions $R_1, R_2, \ldots, R_N$ at time t to final positions $R'_1, R'_2, \ldots, R'_N$ at time t' are in one-to-one correspondence with the elements of the braid group $\boldsymbol{B_n}$ (called generators σ_i).



Rules for braiding

When two particles with labels a and b undergo a counterclockwise exchange, their total charge c is unchanged. Therefore, since the two particles swap positions on the line, the swap induces a natural isomorphism mapping the Hilbert space V_{ba}^c to V_{ab}^c; $R: V_{ba}^c \rightarrow V_{ab}^c$. If we choose the canonical basis $\{|ba;c,\mu\rangle, |ab;c,\mu'\rangle\}$ for these two spaces, R can be expressed as a unitary matrix.

$R: |ba;c,\mu\rangle \rightarrow \sum_{\mu'} |ab;c,\mu'\rangle (R_{ab}^c)_{\mu}^{\mu'}$



The fusion is associative: $(a \times b) \times c = a \times (b \times c)$ (the total charge is an intrinsic property).

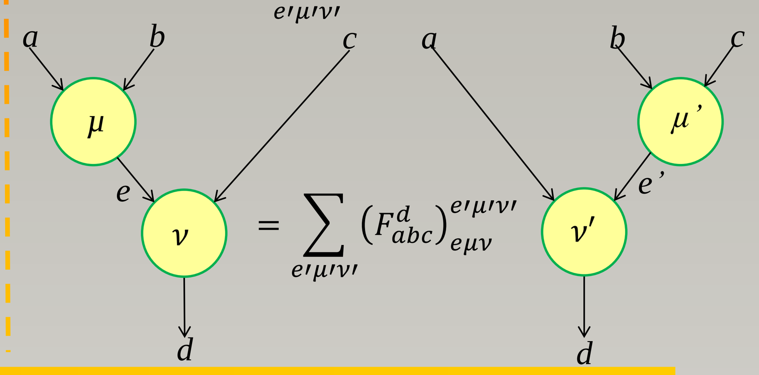
Consider orthogonal basis for V_{abc}^d

$|(ab)c;d:e,\mu\nu\rangle = |ab;e,\mu\rangle \times |ec;d,\nu\rangle$

$|a(bc);d:e',\mu'\nu'\rangle = |ae';d,\nu'\rangle \times |bc;e',\mu'\rangle$

The basis are related by unitary transformation F

$|(ab)c;d:e,\mu\nu\rangle = \sum_{e'\mu'\nu'} |a(bc);d:e',\mu'\nu'\rangle (F_{abc}^d)_{e\mu\nu}^{e'\mu'\nu'}$



QUANTUM GATES

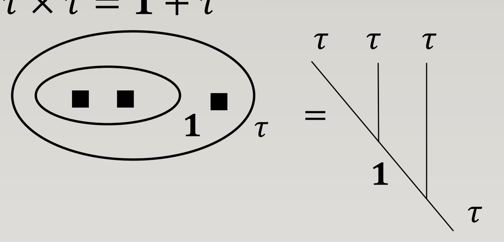
Fibonacci Anyon Models (qubits)

In the Fibonacci's model there are only two charges: the trivial $\mathbf{1}$ and the nontrivial τ that represents a non-Abelian quasiparticle.

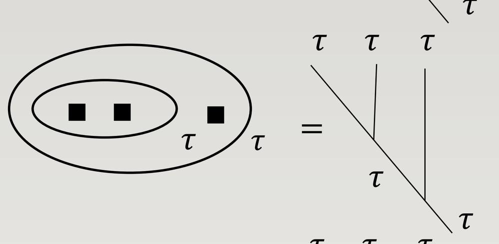
$\sum_c N_{ab}^c = 2 \rightarrow$Non abelian

$\mathbf{1} \times \tau = \tau$; $\tau \times \mathbf{1} = \tau$; $\tau \times \tau = \mathbf{1} + \tau$

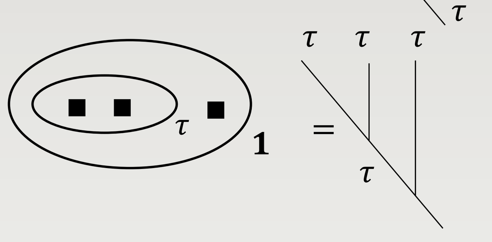
$|0\rangle = |((\blacksquare,\blacksquare)_1,\blacksquare)_\tau\rangle =$



$|1\rangle = |((\blacksquare,\blacksquare)_\tau,\blacksquare)_\tau\rangle =$



$|N\rangle = |((\blacksquare,\blacksquare)_\tau,\blacksquare)_1\rangle =$



-The dimension of the Hilbert's space of n anyons is the $(n-1)$ – th fibonacci number.

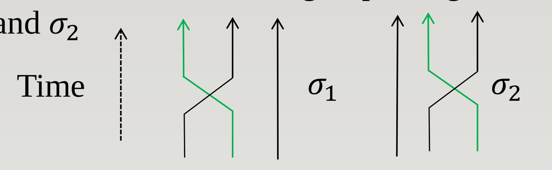
-The Fibonacci's number grows with n as $0{,}447214\phi^n$ for large n, where ϕ phi is the golden mean $\phi = (1/2)(1+\sqrt{5}) \approx 1.618$

-Three Fibonacci's anyons with total charge equals to τ are enough to encode a qubit.

To process a qubit we must find the σ operators that define the braids .

Using the R and F matrix, we can find the unitary operations that results of the realizations of braids at any number of particles.

Consider three anions in the two-dimensional representation. The Braid group it is generated by σ_1 and σ_2



Using matrix R we can determine that the unitary operation for the generator σ_1 is given by:

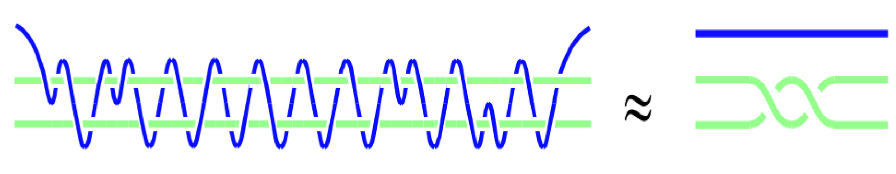
$$\sigma_1 = \begin{pmatrix} e^{-4\pi i/5} & 0 \\ 0 & e^{-2\pi i/5} \end{pmatrix}$$

To determine the unitary operation for the generator σ_2, it is necessary to use the change-of-base matrix F along with the matrix R in the form $F^{-1}RF$:

$$\sigma_2 = \begin{pmatrix} -e^{-\pi i/5}/\phi & -ie^{-\pi i/10}/\sqrt{\phi} \\ -ie^{-\pi i/10}/\sqrt{\phi} & -1/\phi \end{pmatrix}$$

σ_1 and σ_2 generates all the possibles braids over all the strands. We can use the last matrices to determine the unitary representations.

Gate on 1 qubit

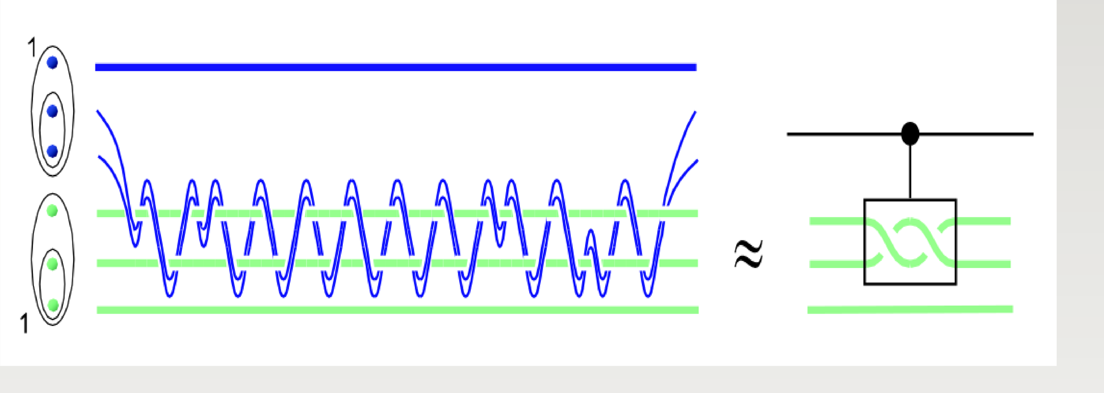


$\sigma_2^3\sigma_1^2\sigma_2^{-4}\sigma_1^2\sigma_2^2\sigma_1^{-2}\sigma_2^{-2}\sigma_1^{-2}\sigma_2^2\sigma_1^2\sigma_2^2\sigma_1^{-2}\sigma_2^2\sigma_1^{-2}\sigma_2^4\sigma_1^{-2}\sigma_2^2\sigma_1^4\sigma_2^{-2}\sigma_1^2\sigma_2^{-2}\sigma_1^2\sigma_2^{-2} \approx \sigma_1^2$

With products of powers (positive or negative) of the generators σ_1 and σ_2 can approximate any gate of a qubit with a pre-established error ϵ. In the example, the sequence shown approximates the gate $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ with $\epsilon = 8.5 \times 10^{-3}$

Controled NOT gate (CNOT)

The upper qubit represented by the three blue strands, it is the control qubit. The lower qubit, three green strands, it is the target qubit. If the state for the target qubit is $|0\rangle$, then this state it is not affected by the braiding. If the control qubit it is in the state $|1\rangle$, the target is reversed. This can be reached with a pre-established error ϵ, braiding two of the particles in the lower qubit..



CONCLUSIONS

- The charge a is a property of a localized object that can't be changed by any local physical process. Local interactions between the particle and its environment may jostle the particle, but will not alter the charge. This local conservation of charge is the essential reason that anyons are amenable to fault-tolerant quantum information processing.
- The one-dimensional representation of Braid group does not allow the construction of quantum logic gates.
- The two-dimensional representation of the Braid group allowed build simple quantum logic gates. Require at least three strands.
- The computational base is: $\{|0\rangle = |((\blacksquare,\blacksquare)_1,\blacksquare)_\tau\rangle, |1\rangle = |((\blacksquare,\blacksquare)_\tau,\blacksquare)_\tau\rangle\}$
- The states of the computational base allow the construction of a universal set of quantum logical gates (components of a quantum computer).
- Three fluxons allow physical implementation of the above computational base.

References

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