On the Superactivation of Quantum Nonlocality

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May 30 2013

Outline

- Historical background: entanglement and nonlocality.
- Entangled-local states.
- Superactivation of quantum nonlocality [1].
- Superactivation of nonlocality and the teleportation protocol [2].
- Hidden nonlocality:
  - B-nonlocality (Buscemi Nonlocality).

[1] C. Palazuelos, 2012, PRL **109**, 190401.

[2] D. Cavalcanti et. al. 2013, PRA **87**, 042104.

Formalism; Density Matrix

Consider a bipartite system: $\mathcal{H} = \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$, $|\psi\rangle \in \mathcal{H}$,

$$\rho \in D(H) := \{\rho \in Pos(\mathcal{H}) | tr(\rho) = 1\},$$

Pure state $\quad \rho = |\psi\rangle \langle\psi|,$

Mixed State $\quad \rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|,$

$$p_i > 0, \ \forall i, \quad \sum_i p_i = 1.$$

$\rho \in D(H)$ is separable[3] if:

Separable pure state $\quad |\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle,$

$$\rho = |\psi\rangle \langle\psi|, \qquad \rho = \rho^A \otimes \rho^B,$$

Separable mixed state $\quad \rho = \sum_i q_i \rho_i^A \otimes \rho_i^B,$

$$q_i > 0, \ \forall i, \quad \sum_i q_i = 1.$$

Otherwise, we say, entangled density matrix.

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[3] R. Werner, 1989, PRA **40**, 4267

Formalism; Bipartite Correlations

A bipartite system, $x, y \in N$ Inputs, $a, b \in K$ Outputs:

- C general bipartite correlations, $P \in C$:

$$P : N^2 \otimes K^2 \to [0,1], \qquad P = \{p(a,b|x,y)\}_{a,b}^{x,y},$$
$$\sum_{a,b} P(a,b|x,y) = 1, \qquad \forall x, y.$$

- $\mathcal{L}$ local correlations, P is a Local Correlation if [4]:

$$\exists \Lambda, \mu, A, B \quad \text{s. t.}$$
$$p(a,b|x,y) = \int_\Lambda \mu(\lambda) d\lambda A(a|x,\lambda) B(b|y,\lambda),$$

(Λ, A, B, μ) is a LHVM (Local Hidden Variable Model).

- NS no-signalling bipartite correlations.

$$\mathcal{L} \subsetneq NS \subsetneq C \subsetneq [0,1]^{N^2 K^2}.$$

[4] J. S. Bell, 1964, Phys, **1**, 195.

Formalism; Quantum Correlations

$\rho \in D(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$, C_ρ correlations allowed by ρ:

$$P \in C_\rho \text{ if:} \quad \exists \{E_{x,a}\}, \{F_{y,b}\} \text{POVM's} \quad \text{s. t.}$$

$$p(a,b|x,y) = Tr\left[(E_{x,a} \otimes F_{y,b})\rho\right].$$

- Quantum correlations Q_H.

$$Q_H := \bigcup_{\rho \in D(H)} C_\rho, \qquad \mathcal{L} \subseteq Q_H \subsetneq NS \subsetneq C \subsetneq [0,1]^{N^2K^2}.$$

- Nonlocal quantum correlations respect the no signalling principle.
- There exist signalling correlations that are not allowed by quantum theory (Superquantum Correlations).

We want to know if our state allow nonlocal correlations. $\forall P \in C_\rho$ if:

$$\forall \{E_{x,a}\}, \{F_{y,b}\} \text{POVM's} \quad \exists (\Lambda, \mu, A, B) \text{LHVM} \quad \text{s. t.}$$

$$p(a,b|x,y) = Tr\left[(E_{x,a} \otimes F_{y,b})\rho\right] = \int_\Lambda \mu(\lambda) d\lambda A(a|x,\lambda) B(b|y,\lambda).$$

- Notation: LHVM General, LHVM Projective.
- Notation: ρ is local, ρ is nonlocal.

Entanglement and nonlocality relation

- If ρ Separable state $\longrightarrow \rho$ Local state.

Does the quantum mechanics allow nonlocal correlations?

- Yes!, Bell1964 [5]. $\rho = |\psi\rangle\langle\psi| \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right),$$

  Pure, entangled and allow a non local correlation.
- Multipartite nonlocality (GHZ states [6]).

Do all entangled states generate non local correlations?

- Gisin Theorem[7]: All bipartite pure entangled states are nonlocal.
- Popescu's Theorem[8]: All pure entangled states are nonlocal ones.
- Are all general mixed entangled states nonlocal?

[5] J. S. Bell, 1964, Phys, **1**, 195.

[6] D. M. Greenberger, M. A. Horne, A. Zeilinger.

[7] N. Gisin and A. Peres, 1992, Physics Letters A, **162**, 15.

[8] S. Popescu, 1992, Physics Letters A, **166**, 293.

All Known Entangled-Local states

- Werner states [9], $\rho_W^d(p) \in D(\mathbb{C}^d \otimes \mathbb{C}^d)$: (Some Region)

$$\rho_W^d(p) = \frac{p}{d(d-1)} 2P_{anti} + \frac{(1-p)}{d^2}\mathbb{1},$$

$$0 \leq p \leq 1, \qquad P_{anti} := \frac{1}{2}(\mathbb{1} - V), \qquad V := \sum_{i,j=0}^{d-1} |ij\rangle \langle ji| .$$

- Werner states generalization in $D(\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d)$: (Some Region)
- Isotropic states[10], $\rho_{iso}^d(p) \in D(\mathbb{C}^d \otimes \mathbb{C}^d)$. They are the same when $d = 2$:

$$\rho_{iso}^d(p) = p\,|\psi_d\rangle \langle\psi_d| + \frac{(1-p)}{d^2}\mathbb{1},$$

$$0 \leq p \leq 1, \qquad |\psi_d\rangle := \frac{1}{\sqrt{d}} \sum_i^d |ii\rangle .$$

- Isotropic states generalization, $\rho_{iso}^d(p) = p\rho + \frac{(1-p)}{d^2}\mathbb{1}$, $0 \leq p \leq 1$.

[9] R. Werner, 1989, PRA **40**, 4267

[10] M. Horodecki, and P. Horodecki, 1999, Phys. Rev. A, **59**, 4206.

Superactivation of Nonlocality

Superactivated entangled local states via tensor product.

- Separable states are not useful.
- Are there ρ_1 and ρ_2 entangled-local such that $\rho_1 \otimes \rho_2$ entangled and nonlocal?
- Given ρ entangled-local, Does there exist $k \in \mathbb{N}$ such that $\rho^{\otimes k}$ entangled and nonlocal?

This phenomenon is called, superactivation of nonlocality or k-copy nonlocality.

Isotropic States

A Bipartite system in the isotropic states, $\rho^d_{iso} \in D(\mathbb{C}^d \otimes \mathbb{C}^d)$:

$$\rho^d_{iso}(p) = p \left|\psi_d\right\rangle \left\langle\psi_d\right| + (1-p)\frac{\mathbb{1}}{d^2}, \qquad 0 \leq p \leq 1,$$

$$\left|\psi_d\right\rangle := \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \left|ii\right\rangle, \qquad \text{is a maximally entangled state,}$$

$$p \geq \frac{1}{d+1} \qquad \xrightarrow{\text{PPT}} \qquad \text{Entangled,}$$

$$p \leq \frac{-1 + \sum_{k=1}^{d} \frac{1}{k}}{d-1} \qquad \longrightarrow^{11} \qquad \text{LHVM Projective,}$$

$$p \leq \frac{(3d-1)(d-1)^{(d-1)}}{(d+1)d^d} \qquad \longrightarrow^{12} \qquad \text{LHVM General.}$$

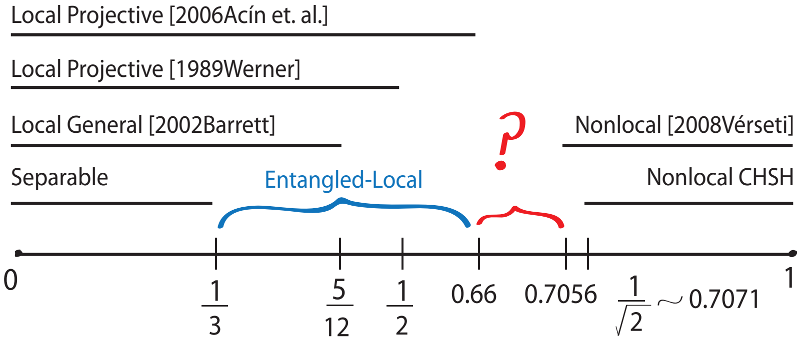
Example $d = 2$.

[11] A. Acín, et. al. 2006, PRA **73**, 062105.

[12] J. Barrett, 2002, PRA **65**, 042302.

Werner-Isotropic States $d = 2$

$$\rho^{d=2}_{W,iso}(p) = p\,|\psi_+\rangle\langle\psi_+| + (1-p)\frac{\mathbb{1}}{4}, \qquad 0 \le p \le 1.$$



How can we prove that $\rho_{iso}^{\otimes k}$ is a nolocal state?

- Violate a Bell-type inequality, (Locality $\longrightarrow$ Bell-type inequalities)
  - Quantum game theory.

Quantum nonlocal game setup

Given a quantum game G.
Given $\rho \longrightarrow M_\rho$they prepare a quantum strategy M_ρ.
They can be correlated in the game somehow by a general $P \in C_\rho$.
Then the winning probability with their strategy is:

$$|\langle M_\rho, P\rangle| := \sum_{xyab} M_{xy}^{ab} P(a,b|x,y).$$

Classical and quantum value:

$$w(M_\rho) := \mathsf{sup}\{|\langle M_\rho, L\rangle| : L \in \mathcal{L} \subseteq C_\rho\},$$
$$w^*(M_\rho) := \mathsf{sup}\{|\langle M_\rho, Q\rangle| : Q \in C_\rho\},$$

$$LV(M_\rho) := \frac{w^*(M_\rho)}{w(M_\rho)} > 1 \qquad \longrightarrow \qquad \rho \text{ is non local},$$

$$w^*(M_\rho) \leq w(M_\rho), \qquad \text{Bell-type inequality.}$$

Try to find an adecuate M_ρ such that $LV(M_\rho) > 1$. Consider a particular game, the Khot - Vishnoi Game (KV Game).

Khot - Vishnoi Game (KV Game) I [13]

Two groups. $\oplus$: addition mod 2 operation on $\{0,1\}$.

$$(\{0,1\},\oplus).$$

mod$\oplus_2$	0	1
0	0	1
1	1	0

Given $n \in \mathbb{N}$, $\{0,1\}^n$, $\oplus$ bitwise addition mod 2 operation. The group:

$$(\{0,1\}^n,\oplus_{ba}),$$

$$x \in \{0,1\}^n, \qquad x = 01110010..., \qquad \#(\{0,1\}^n) = 2^n.$$

Consider the Hadamard group H and the quotient group:

$$(\{0,1\}^n/H,\oplus_{baH}),$$

$$[x] \in \{0,1\}^n/H, \qquad \#(\{0,1\}^n/H) = 2^n/n.$$

Extra condition in the game $n = 2^l, \quad l \in \mathbb{N}$.

[13] S. Khot and N. Vishnoi, in Proceedings of the 46th IEEE Symposium on Foundations of Computer Science, Pittsburgh, 2005.

Khot - Vishnoi Game (KV Game) II

Bipartite game, Alice and Bob. The referee chooses (randomly $p \in [0, 1/2]$) two elements with probability $p \in \{0, 1/2\}$:

$$z \in \{0,1\}^n,$$
$$[x] \in \{0,1\}^n/H,$$

and the referee gives de inputs:

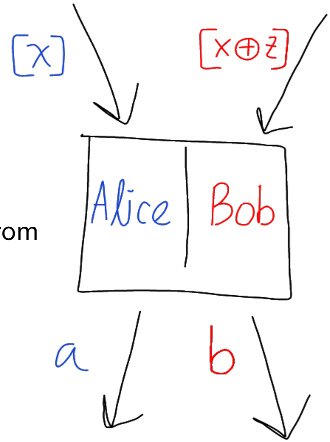
$$[x] \quad \text{to Alice,}$$
$$[x \oplus z] \quad \text{to Bob.}$$

Alice and Bob must choose an element from their own chain, play with their outputs (without comunnication):

$$a \in [x],$$
$$b \in [x \oplus z].$$

Alice and Bob win the game if:

$$a \oplus b = z.$$



Khot - Vishnoi Game (KV Game) III

There are two results[14]:

- The classical value has as upper bound $\forall M$.

$$w(M) \leq \frac{1}{n^{\frac{p}{1-p}}}.$$

- Given a maximally entangled state, they can find a quantum strategy G_{KV} such that the quantum value has a lower bound for G_{KV}: $p \in [0, 1/2]$.

$$(1-2p)^2 \leq w^*(G_{KV}).$$

If we consider the probability as: $p = \frac{1}{2} - \frac{1}{ln(n)}$,we achieve the bounds:

$$w(M) \leq C\frac{1}{n}, \qquad C'\frac{1}{(ln(n))^2} \leq w^*(G_{KV}), \qquad C, C' \text{ positive constants.}$$

To our system, consider: $n = d^k$, $k \in \mathbb{N}$.

$$w(M) \leq C\frac{1}{d^k}, \qquad C'\frac{1}{(kln(d))^2} \leq w^*(G_{KV}).$$

[14] I H. Buhrman, O. Regev, G. Scarpa, and R. de Wolf, Proceedings of the 26th IEEE Conference on Computational Complexity, 2011.

Superactivation of nonlocality on the Isotropic states I

With the KV game in mind. The Isotropic States:

$$\rho_{iso}^{d}(p) = p\left|\psi_d\right\rangle\left\langle\psi_d\right| + (1-p)\frac{\mathbb{1}}{d^2}, \qquad 0 \leq p \leq 1,$$

$$p \geq \frac{1}{d+1} \qquad \longrightarrow \qquad \text{Entangled},$$

$$F := \left\langle \psi_d \middle| \rho_{iso}^{d}(p) \middle| \psi_d \right\rangle = p + \frac{1-p}{d^2}, \qquad \text{Singlet Fraction},$$

$$\rho_{iso}^{d}(F) = F\left|\psi_d\right\rangle\left\langle\psi_d\right| + (1-F)\frac{\mathbb{1} - \left|\psi_d\right\rangle\left\langle\psi_d\right|}{d^2-1}, \qquad \frac{1}{d^2} \leq F \leq 1,$$

$$F \geq \frac{1}{d} \qquad \longrightarrow \qquad \text{Entangled},$$

$$\rho_{iso}^{\otimes k}(F) = F^k\left|\psi_d^{\otimes k}\right\rangle\left\langle\psi_d^{\otimes k}\right| + \ldots + (1-F)^k\frac{(1-\left|\psi_d\right\rangle\left\langle\psi_d\right|)^{\otimes k}}{(d^2-1)^k}.$$

Superactivation of nonlocality on the Isotropic states II [15, 16]

To prove that the new state is non local, consider the LV:

$$LV(\rho_{iso}^{\otimes k}) \geq LV(F^k \left|\psi_d^{\otimes k}\right\rangle \left\langle\psi_d^{\otimes k}\right|).$$

This is a maximally entangled state as well, $n = d^k$, then by KV game, we can find a quantum strategy G_{KV} such that:

$$LV(F^k \left|\psi_d^{\otimes k}\right\rangle \left\langle\psi_d^{\otimes k}\right|) = \frac{w^*(G_{KV})}{w(G_{KV})} \geq F^k \frac{C' \frac{1}{(kln(d))^2}}{C \frac{1}{d^k}} = F^k \frac{C' d^k}{C(kln(d))^2}.$$

When $F > 1/d$, $\epsilon > 0$:

$$LV(\rho_{iso}^{\otimes k}) \geq \left(\frac{1}{d} + \epsilon\right)^k \frac{C' d^k}{C(klnd)^2} = \left(\frac{C'}{C(lnd)^2}\right) \frac{(1+\epsilon d)^k}{k^2} > 1 \quad \square$$

All isotropic entangled states are k-copy nonlocal.

[15] C. Palazuelos, 2012, PRL **109**, 190401.

[16] D. Cavalcanti et. al. 2013, PRA **87**, 042104.

More general result[17]

Given $\rho_0 \in D(\mathbb{C}^d \otimes \mathbb{C}^d)$ a general state with $F(\rho_0) > 1/d$, we build the state, through the twirling process:

$$\rho = \sum_i p_i (U_i \otimes U_i^*) \rho_0 (U_i \otimes U_i^*)^{-1},$$

ρ is a isotropic state and also $F(\rho) = F(\rho_0) > 1/d$, then (by previous result), ρ is k-copy nonlocal.

$$\rho^{\otimes k} = \sum_{i_1 \dots i_k} p_{i_1} \dots p_{i_k} U_{i_1 \dots i_k} \rho_0^{\otimes k} U_{i_1 \dots i_k}^{-1},$$

$$U_{i \dots i_k} := (U_{i_1} \otimes \dots \otimes U_{i_k}) \otimes (U_{i_1}^* \otimes \dots \otimes U_{i_k}^*),$$

$$w^*(M) = \sum_{xyab} M_{xy}^{ab} tr[(E_{x,a} \otimes F_{y,b}) \rho^{\otimes k}] = tr(B \rho^{\otimes k}) > w(M).$$

At least one element, $tr(B' \rho_0^{\otimes k}) > w(M)$, then ρ_0 is k-copy nonlocal.

ρ entangled state with $F(\rho) > 1/d \longrightarrow \rho$ is k-copy nonlocal.

[17] D. Cavalcanti et. al. 2013, PRA **87**, 042104.

All Known entangled, local, k-copy nonlocal states

These entangled states have like a Hidden non locality.

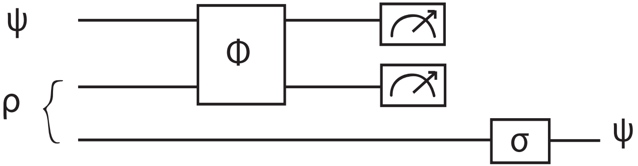
- Isotropic states (All)
- Isotropic states little generalization.(Some region)
- Werner states.(Some region)
- Multipartite Werner little generalization. (Some region)

Can all the known entangled local states be superactivated?

Superactivation importance.
Entanglement and non locality useful for different protocols.

About Teleportation [18]

Teleportation protocol general scheme.



ψ the state to teleport:

$$\psi \otimes \rho,$$

$$tr_{1,2}[\Phi(\psi \otimes \rho)],$$

$$\sigma\{tr_{1,2}[\Phi(\psi \otimes \rho)]\} =: \Lambda(\psi \otimes \rho) = \psi, \qquad \Lambda \text{ a LOCC.}$$

- Fidelity of teleportation:

$$f(\rho, \Lambda) := \int d\psi \, \langle \psi | \Lambda(\psi \otimes \rho) | \psi \rangle \, .$$

- Singlet fraction (entangled fraction):

$$F(\rho, \Lambda) := tr[\Lambda(\rho) P^+], \qquad P^+ = \left| \phi^d \right\rangle \left\langle \phi^d \right| .$$

[18] C. H. Bennett et. al. 1993, PRL **70**, 1895.

Superactivation of Nonlocality and Teleportation

- Optimal Fidelity of teleportation: $f_{max}(\rho) := max_{\Lambda \in LOCC}\{f(\rho, \Lambda)\ \}$.
- Fully entangled fraction: $F_{max}(\rho) := max_{\Lambda \in LOCC}\{F(\rho, \Lambda)\ \}$.

Main Result [19]:

$$f_{max}(\rho) = \frac{dF_{max}(\rho) + 1}{d + 1}.$$

- Result, that we need $1/d$, $f_{Classical} = \frac{2}{d+1}$ or maximal classical singlet fracion is $1/d$.

ρ is useful to teleportation $\longleftrightarrow F(\rho) > 1/d$.

With the previous result:
ρ entangled state with $F(\rho) > 1/d \longrightarrow \rho$ is k-copy nonlocal. Then:

If ρ is useful for teleportation $\longrightarrow \rho$ is k-copy nonlocal.

Teleportation uses the k-copy non locality.

ρ is useful for teleportation $\longleftarrow_{?}$ If ρ is k-copy nonlocal.

[19] M. Horodecki, et al. 1999, PRA **60**, 1888.

Hidden Nonlocality in the Entangled States

- k-copy nonlocality.
- B-Nonlocality: All the entangled states are B-Nonlocal [20].
- Other Nonlocality [21].

[20] F. Buscemi, 2012, PRL **108**, 200401.

[21] Yeong-Cherng Liang, et. al. 2012, PRA **86**, 052115.

Review Open Problems

- Can all the known entangled local states be superactivated?
- Are all k-copy nonlocal states useful to teleportation?
- Entangled local states.
- Are all entangled local states, k-copy nonlocal? relation with Hidden Nonlocality (B-nonlocality, ...).