ON AN NEW EMBEDDING THEOREM AND THE CLR-TYPE INEQUALITY FOR EUCLIDEAN AND HYPERBOLIC SPACES.

Abstract. 1/Regarding a new embedding theorem.

It is known that there is no continuous embedded theorem between the Sobolev spaces of order one $H^1(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n),\ \nabla u \in L^2(\mathbb{R}^n)\}$ and [the space of bounded functions] $L^\infty(\mathbb{R}^n)$, thus the goal of this note is to provide a new embedding theorem between the Sobolev space $H^1(\mathbb{R}^n \setminus \overline{B(0,1)})$ and the space $\mathscr{X} := L^\infty\big((1,\infty);\ r^{n-1}dr\big) \otimes \widetilde{L^2}(\mathbb{S}^{n-1}, d\sigma_{\mathbb{S}^{n-1}})$, such that $d\sigma_{\mathbb{S}^{n-1}}$ stands for the surface element on the unit sphere $\mathbb{S}^{n-1}$ and $\widetilde{L^2}(\mathbb{S}^{n-1}, d\sigma_{\mathbb{S}^{n-1}})$ is the space spanned by the orthonormal complete basis of eigenfunctions $\big(\psi_{(k+1)^2}\big)_{k\geq 0}$ associated to the operator $-\Delta_{|_{\mathbb{S}^{n-1}}}$, and $\overline{B(0,1)}$ is the closure of the unit ball. Precisely, we show that

Theorem 1. *For all* $u \in H^1(\mathbb{R}^n \setminus \overline{B(0,1)})$ *we have*

$$||u||^2_{\mathscr{X}} \leq (n-1)^{-1/2}||\nabla u||^2_{L^2(\mathbb{R}^n\setminus\overline{B(0,1)})}.$$

With

$$||u||_{\mathscr{X}} := \operatorname*{ess\,sup}_{r>1}\left(\int_{\mathbb{S}^{n-1}} |u(r,\omega)|^2 d\sigma_{\mathbb{S}^{n-1}}(\omega)\right)^{1/2}.$$

2/ Regarding the CLR-type inequality.

Since that there is no CLR inequality in terms of the $L^1(\mathbb{R}^n)$-norm of a potential, hence we stem from the Theorem 1, the CLR-type inequality for a potential belonging to a proper subspace of $L^1(\mathbb{R}^n \setminus \overline{B(0,1)})$. Precisely, we show

Theorem 2. *Let* $V \in L^1_{loc}(\mathbb{R}^n \setminus \overline{B(0,1)})$ *with* $V_+ := \max(V,0) \in \mathscr{Y}$. *Then we have*

$$N(-\Delta - V) \leq 4.47139||V_+||_{\mathscr{Y}}.$$

$N(-\Delta - V)$ is the negative eigenvalues number of the operator $-\Delta - V$.
$\mathscr{Y} := L^1((1,\infty);\ r^{n-1}dr) \otimes L^\infty(\mathbb{S}^{n-1}, d\sigma_{\mathbb{S}^{n-1}})$ and its associated norm is defined by

$$||u||_{\mathscr{Y}} = \int_1^\infty \sup_{\omega\in\mathbb{S}^{n-1}} \text{ess}|u(r,\omega)|r^{n-1}dr.$$

The version of the above theorems for an hyperbolic manifold will also provided.

(Mohammed El Aïdi) Departamento de Matemáticas. Universidad Nacional de Colombia, sede Bogotá. AK 30 número 45-03. Edificio 404. Bogotá, D.C. Colombia.
E-mail address: `melaidi@unal.edu.co`

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