An application of the sine-Gordon model to the two component plasma

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Outline

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1. The system and how to solve it
2. Mapping into a sine-Gordon field theory
3. Debye-Hückel Limit

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The system
The two component plasma

The two component plasma (TCP) is a system of mobile pointlike particles composed of two kinds of charges $\{q_\alpha\}$ where $\alpha = \pm$ and

$$q_+ = q = -q_-$$

Throughout this talk, we'll consider the case where the system is 2 dimensional.

1. Potential and charge density

$$\Delta v(\mathbf{r}) = -2\pi q \delta(\mathbf{r})$$

$$\rho(\mathbf{r}) = q \left(\sum_{j=1}^{N_+} \delta(\mathbf{r} - \mathbf{R}_j) - \sum_{k=1}^{N_-} \delta(\mathbf{r} - \mathbf{X}_k) \right)$$

2. Energy of the system

$$E = \frac{1}{2} \int_{\Lambda^2} \mathrm{d}^2\mathbf{r}\mathrm{d}^2\mathbf{r}' \rho(\mathbf{r}') v(|\mathbf{r} - \mathbf{r}'|) \rho(\mathbf{r}) - \frac{1}{2} N v(0)$$

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Preliminaries

1. The Hubbard-Stratonovich transform

$$\frac{\int_{\mathbb{R}^n} \exp(-\frac{1}{2}(\mathbf{X}^t A \mathbf{X}) + \mathbf{b}^t \mathbf{X}) \mathrm{d}\mathbf{X}}{\int_{\mathbb{R}^n} \exp(-\frac{1}{2}\mathbf{X}^t A \mathbf{X}) \mathrm{d}\mathbf{X}} = \exp\left(\frac{1}{2}\mathbf{b}^t A^{-1} \mathbf{b}\right)$$

2. Wick's theorem

$$\langle x_{k_1} x_{k_2} \cdots x_{k_n} \rangle = \sum_{\text{pairs of k's}} (A^{-1})_{k_{p_1} k_{p_2}} \cdots (A^{-1})_{k_{p_{n-1}} k_{p_n}}$$

TCP

From TCP to sG [2, 3]

1. Partition Function

$$\Xi[z] = \sum_{N_-,N_+=0}^{\infty} \frac{1}{N_-!N_+!} \int_\Lambda \prod_{i=1}^{N} \left[\mathrm{d}^2\mathbf{r}_i z_{q_i}(\mathbf{r}_i)\right] \exp\left(-\beta E(\mathbf{r}_i, \mathbf{r}_j)\right)$$

$$\exp(-\beta E) = \int \frac{\mathcal{D}\phi}{D} \exp\left[\int \mathrm{d}^2\mathbf{r} \left(\frac{1}{2}\phi\Delta\phi + i\sqrt{2\pi\beta}\hat{\rho}\phi\right)\right]$$

2. For equilibrium, if $\bar{z} = ze^{\beta v(0)/2}$, we end up with

$$\Xi[z] = \frac{\int \mathcal{D}\phi e^{-S[z]}}{\int \mathcal{D}\phi e^{-S[0]}},$$

where

$$S[z] = \int \mathrm{d}^2\mathbf{r} \left(\frac{1}{2}|\nabla\phi(\mathbf{r})|^2 - 2\bar{z}\cos(\sqrt{2\pi q^2\beta}\phi(\mathbf{r}))\right)$$

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TCP

Correlations [1]

1. To give meaning to $\bar{z}$, one has to fix the normalization of z:

$$n_{+-}(\mathbf{r}, \mathbf{r}') \sim z^2 |\mathbf{r} - \mathbf{r}'|^{-q^2\beta}$$

valid for $|\mathbf{r} - \mathbf{r}'| \to 0$.

2. Using the TBA one gets in the thermodynamic limit, with $\gamma = \frac{q^2\beta}{4-q^2\beta}$

$$\omega := -\frac{\ln \Xi}{V} = -\frac{m_1^2}{8\sin(\pi\gamma)} = -\frac{M^2}{4}\tan\left(\frac{\pi\gamma}{2}\right)$$

3. The fugacity z is, then, for the normalization chosen:

$$z = \frac{\Gamma(q^2\beta)}{\pi\Gamma(1-q^2\beta)}\left[M\frac{\sqrt{\pi}\Gamma(1/2+\gamma/2)}{2\Gamma(\gamma/2)}\right]^{2-q^2\beta/2}$$

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Some other thermodynamical quantities

1. Particle density ($\Gamma = q^2\beta$)

$$\frac{n^{1-\Gamma/4}}{2z} = \left(\frac{\pi\Gamma}{2}\right)^{\Gamma/4} + O(\Gamma^2) \qquad ;\Gamma \to 0$$

2. Specific heat

$$\frac{c_v}{k_B} = \frac{\Gamma}{4} + \frac{4}{4-\Gamma} + \frac{\Gamma^2}{16}\left[\psi'\left(1-\frac{\Gamma}{4}\right) - \psi'\left(1+\frac{\Gamma}{4}\right)\right] - \frac{2\Gamma^2}{(4-\Gamma^3)}\left[\psi'\left(\frac{2}{4-\Gamma}\right) - \psi'\left(\frac{8-\Gamma}{8-2\Gamma}\right)\right] - \frac{4\pi^2\Gamma^2}{(4-\Gamma)^3}\frac{\cos(\pi\Gamma/(4-\Gamma))}{\sin^2(\pi\Gamma/(4-\Gamma))} \qquad 0 < \Gamma < 2$$

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What we knew before s-G

Debye-Hückel theory is based on a mean field approximation where the density of charges differs from a constant by the Boltzmann factor, that is,

$$\begin{aligned}\Delta\Phi(\mathbf{r}) &= -2q\pi\left[\delta(\mathbf{r}) + n_+ - n_-\right] \\ &= -2q\pi\left\{\delta(\mathbf{r}) + n\left[e^{-q\beta\Phi(\mathbf{r})} - e^{q\beta\Phi(\mathbf{r})}\right]\right\}\end{aligned}$$

As it is, we still cannot solve it. However, for large temperatures ($\beta \to 0$)

$$\Phi(\mathbf{r}) = -2q\pi\delta(\mathbf{r}) + \kappa^2\Phi(\mathbf{r})$$

where $\kappa^2 = 4n\pi\Gamma$.

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This corresponds to an action of the form

$$S = \int \mathrm{d}^2\mathbf{r} \left[\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}\kappa^2\phi^2 \right]$$

And we see this agrees with the original sine Gordon action for the case where $T \to \infty$!.

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