Expanded Vandermonde powers: applications to the two-dimensional one-component plasma

Gabriel Téllez

Universidad de los Andes

Departamento de Física
Bogotá, Colombia
gtellez@uniandes.edu.co

In collaboration with Peter J. Forrester
The University of Melbourne, Australia

Escuela de Física Matemática, 28 mayo – 1 junio 2012, Bogotá, Colombia

Contents

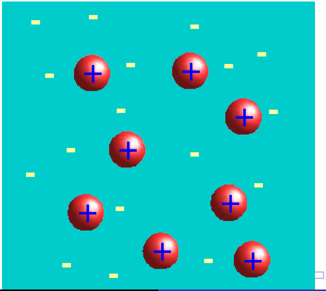
Contents

The two-dimensional one-component plasma

- System of N particles with charge q in a plane, interacting with the 2D Coulomb potential:

$$v(r) = -\ln\frac{r}{L} \qquad (1)$$

- Neutralizing background with charge density $-qn_b$.
- Coulomb coupling $\Gamma = q^2/(k_B T)$ = ratio between the electrostatic energy and the thermal energy.



Hamiltonian

$$H = -q^2 \sum_{i<j} \ln|z_i - z_j| + q^2 \sum_i v_b(\mathbf{r}_i) + V_0 \qquad (2)$$

- Particle-background potential interaction $v_b(\mathbf{r}) = \pi n_b r^2/2 +$ cst (plasma in a disk).
- V_0 background-background interaction (constant).

The Boltzmann factor as a Vandermonde determinant

$$e^{-\beta H} = \prod_{k<j} |z_k - z_j|^\Gamma \prod_i e^{-\Gamma v_b(\mathbf{r}_i)} e^{-\beta V_0} \quad (3)$$

$$= |\det(z_j^{k-1})|^\Gamma \prod_i e^{-\Gamma v_b(\mathbf{r}_i)} e^{-\beta V_0} \quad (4)$$

$$= |\det(\psi_{k-1}(\mathbf{r}_j))|^\Gamma e^{-\beta V_0} \quad (5)$$

with

$$\psi_k(\mathbf{r}) = z^k e^{-v_b(\mathbf{r})} \quad (6)$$

Analogy with the quantum Hall effect

The functions

$$\psi_l(\mathbf{r}) = z^l e^{-v_b(\mathbf{r})} \tag{7}$$

are orthogonal between them. They also are the wave function of an electron in the plane Oxy with a magnetic field in the perpendicular direction z. The angular momentum L_z of this electron is l. The Slater determinant

$$\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \det(\psi_{k-1}(\mathbf{r}_j)) \tag{8}$$

is the wave function of N independent electrons in the lowest energy level.

Partition function $\Gamma = 2$

If $\Gamma = 2$ then

$$e^{-\beta H} = e^{-\beta V_0} |\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N)|^2 . \tag{9}$$

To compute the partition function is the same as normalizing the wave function Ψ

$$Z = e^{-\beta V_0} N! \prod_{k=0}^{N-1} ||\psi_k||^2 \tag{10}$$

with

$$||\psi_k||^2 = \int |\psi_k(\mathbf{r})|^2 \, d\mathbf{r} \tag{11}$$

Contents

Partitions

Let $\mu = (\mu_1, \ldots, \mu_N)$ be a partition of non negative integers such that $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_N \geq 0$. We define $|\mu| = \sum_j \mu_j$.
A partition μ also be represented by the occupation numbers $n(\mu) = (n_0, n_1, \ldots, n_l, \ldots)$, where n_l is the frequency of the integer l in the partition $(\mu_1, \ldots, \mu_N)$.

Symmetric polynomials

Let

$$m_\mu(z) = \frac{1}{\prod_i n_i!} \sum_{\sigma \in S_N} z_{\sigma(1)}^{\mu_1} \dots z_{\sigma(N)}^{\mu_N} \tag{12}$$

They are an orthogonal basis of the space of symmetric polynomials of N variables.

The wave function of an state of N free bosons occupying the Landau levels of angular momentum $\mu_1, \dots, \mu_N$ is proportional to $m_\mu(z)$.

Antisymmetric polynomials

Let

$$A_\mu(z) = \sum_{\sigma \in S_N} \epsilon(\sigma) z_{\sigma(1)}^{\mu_1} \dots z_{\sigma(N)}^{\mu_N} = \prod_{i<j} (z_j - z_i) \, s_{\mu-\delta_N}(z) \qquad (13)$$

with $\delta_N = (N-1, N-2, \dots, 0)$ and s_μ the Schur polynomials. The $\{A_\mu\}$ form an orthogonal basis of the space of antisymmetric polynomials of N variables.

The wave function of an state of N free fermions occupying the Landau levels of angular momentum $\mu_1, \dots, \mu_N$ is proportional to $m_\mu(z)$.

A partial order is defined on partitions of the same length: $\mu < \kappa$ if

$$\sum_{j=1}^{p} \mu_i \leq \sum_{j=1}^{p} \kappa_j \qquad (p = 1, \ldots, N) \tag{14}$$

Property: $\mu < \kappa$ if μ can be constructed from κ by a sequence of *squeezing* operations:

- It is an operation between two particles that moves up one particle from the orbital m_1 to m_1' and moves down the other particle from the orbital m_2 to m_2' with $m_1 < m_1 \leq m_2' < m_2$, conserving the total angular momentum $m_1 + m_2 = m_1' + m_2'$.

Example: $m_1 = 2$, $m_1' = 3$ and $m_2' = 6$, $m_2 = 7$:

occupation: $[02\underrightarrow{4}2430\underleftarrow{2}13] \to [0233431113]$

partition: $(9,9,9,8,7,7,5,5,5,4,4,4,4,3,3,2,2,2,2,1,1)$

$\to (9,9,9,8,7,6,5,5,5,4,4,4,4,3,3,3,2,2,2,1,1)$

Estrategy

Let $r = \Gamma/2$, $z = (z_1, \ldots, z_N)$, and

$$\Psi(z) = \prod_{k<j} (z_k - z_j)^r \,. \tag{15}$$

The Boltzmann factor of the 2dOCP is proportional to $|\Psi(z)|^2$. In the analogy with the quantum Hall effect, $\Psi(z_1, \ldots, z_N)$ is the Laughlin trial wave function for filling fraction $1/r$.

To compute the partition function it is convenient to expand $\Psi(z)$ in the basis of the $\{m_\mu\}$ (if r is even) or $\{A_\mu\}$ (if r is odd)

$$\Psi(z) = \prod_{k<j} (z_k - z_j)^r = \sum_\mu c_\mu m_\mu(z) \tag{16}$$

with $|\mu| = N(N-1)r$. The Boltzmann factor of the 2dOCP is

$$e^{-\beta H} = e^{-\beta V_0} \left| \sum_\mu c_\mu m_\mu(z) \prod_{k=1}^{N} e^{-r v_b(r_k)} \right|^2 \tag{17}$$

and the partition function is

$$Z = e^{-\beta V_0} \sum_\mu c_\mu^2 \prod_{k=1}^{N} ||\psi_{\mu_k}||^2 \tag{18}$$

Jack polynomials

The symmetric Jack polynomial $J_\mu^\alpha(z)$ is an eigenfunction of the operator

$$\mathcal{H} = \sum_{j=1}^{N} \left(z_j \frac{\partial}{\partial z_j} \right)^2 + \frac{2}{\alpha} \sum_{j<k} \frac{z_j + z_k}{z_j - z_k} \left(\frac{\partial}{\partial z_j} - \frac{\partial}{\partial z_k} \right) \tag{19}$$

with eigenvalue

$$\sum_{j=1}^{N} \mu_j(\mu_j - 1) + (\alpha(N-1)+1)|\mu| - 2\alpha \sum_{j=1}^{N} (j-1)\mu_j \tag{20}$$

One can show that

$$\Psi(z) = \prod_{k<j} (z_k - z_j)^r = J_\lambda^\alpha(z) \tag{21}$$

with $\alpha = -1/(r-1)$, and λ is the partition with occupation numbers

$$[1 \underbrace{0 \ldots 0}_{r-1 \text{ times}} 10^{r-1} 1 \ldots] \tag{22}$$

with $|\lambda| = rN(N-1)$.

Furthermore

$$J_\lambda^\alpha = m_\lambda + \sum_{\mu<\lambda} c_\mu m_\mu \tag{23}$$

Applying $\mathcal{H}$ to (23), one obtains a recurrence relation between the coefficients c_μ that allows its calculation:

$$c_\rho = \frac{1}{e_\lambda(\alpha) - e_\rho(\alpha)} \frac{2}{\alpha} \sum_{\rho<\mu\leq\lambda} ((\rho_i + r) - (\rho_j - r)) c_\mu(\alpha) \tag{24}$$

where

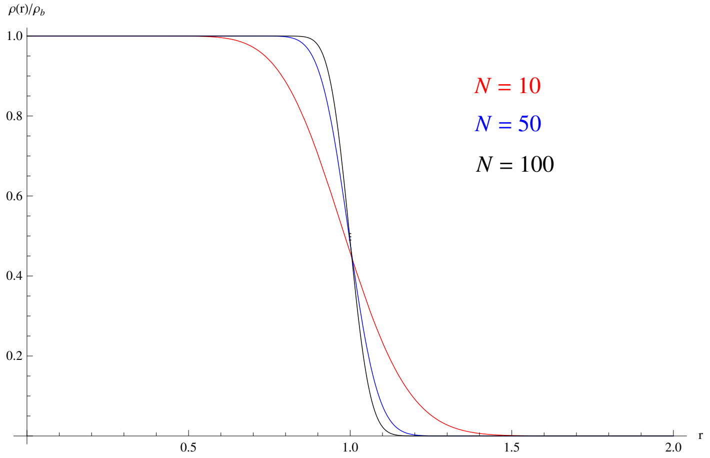
$$e_\lambda(\alpha) := \sum_{i=1}^{N} \lambda_i(\lambda_i - 1 - \frac{2}{\alpha}(i-1)).$$

Contents

Some applications to the 2dOCP

- Pair correlation function at $\Gamma = 4$, Šamaj, Percus, Kolesik, PRE 49:5623 (1994)
- Finite size corrections to the free energy, density and correlation in the disk and in the sphere for $\Gamma = 4$ and $\Gamma = 6$, Téllez, Forrester, J. Stat. Phys 97:489 (1999)
- Sum rules and fluctuation of linear statistics, Téllez, Forrester, arXiv:1204.6003 (2012)

Density profile in the soft edge disk



Density profile with $\pi\rho_b = N$ in a disk with soft edge, for $\Gamma = 2$.

Sum rules for the density: second moment

A simple scaling argument shows that

$$\iint_{\mathbb{R}^2} r^2 \rho(r)\, d^2\vec{r} = \frac{N}{2} + \frac{2}{\Gamma}\left(1 - \frac{\Gamma}{4}\right) \qquad (25)$$

The $O(1)$ term gives information about the structure of the density profile at the edge, by writting $\rho(r) = \rho_b \chi_{0<r<1} + \kappa(r)$,

$$\iint_{\mathbb{R}^2} r^2 \kappa(r)\, d^2\vec{r} = \frac{2}{\Gamma}\left(1 - \frac{\Gamma}{4}\right) \qquad (26)$$

Sum rules for the density

The above argument can be generalized

$$\iint_{\mathbb{R}^2} r^m \rho(r)\, d^2\vec{r} = \frac{2N}{m+2} + \frac{m}{\Gamma}\left(1 - \frac{\Gamma}{4}\right) + O(N^{-1/2}) \qquad (27)$$

Again, the $O(1)$ term gives more information about the structure of the density profile at the edge

$$\iint_{\mathbb{R}^2} r^m \kappa(r)\, d^2\vec{r} = \frac{m}{\Gamma}\left(1 - \frac{\Gamma}{4}\right) \qquad (28)$$

This means that, as $N \to \infty$, $\kappa(r)$ converges to the distribution

$$\kappa(r) = \frac{1}{2\pi\Gamma}\left(1 - \frac{\Gamma}{4}\right)\frac{1}{r}\,\delta'(r-1) \qquad (29)$$

Exact numerical evaluation of the density moments

The $2n$-moment of the density can be expressed as

$$\mathcal{M}_N = \frac{(N\Gamma/2)^{-n}}{Z_{\text{soft}}} \sum_{\mu} \frac{c_\mu^2}{\prod_i m_i!} \prod_{\ell=1}^{N} \mu_\ell! \sum_{k=1}^{N} \frac{(\mu_k + n)!}{\mu_k!} \,, \tag{30}$$

with

$$Z_{\text{soft}} = \sum_{\mu} \frac{c_\mu^2}{\prod_i m_i!} \prod_{\ell=1}^{N} \mu_\ell! \,, \tag{31}$$

which is the partition function of the 2dOCP in the soft disk, up to a multiplicative constant.

Numerical test of the sum rules

N	$\mathcal{M}_N$ =	aN +	b +	$c/\sqrt{N}$ +	d/N
		a	b	c	d
2	0.8125				
3	1.126262				
4	1.44563743218807				
5	1.76891109591098	0.336437	-0.0742714	0.458953	-0.221263
6	2.09454890418255	0.333419	-0.00242266	0.269645	-0.0817588
7	2.42171814295119	0.332974	0.0108746	0.230301	-0.0491322
8	2.74996856529295	0.33338	-0.003688474	0.277614	-0.0922608
9	3.07901955876735	0.333458	-0.00698637	0.289203	-0.103695
10	3.40868118671838	0.333396	-0.00397741	0.277889	-0.0917455
11	3.73882025776555	0.333353	-0.00169446	0.268779	-0.0815306
12	4.06934089384864	0.333341	-0.000983289	0.265786	-0.0779911
13	4.4001722794716	0.333341	-0.000982974	0.265784	-0.0779894
14	4.73126081887937	0.333342	-0.00105077	0.266097	-0.0783948
∞		1/3	0		

Table: Fourth moment ($n = 2$) of the density when $\Gamma = 4$.

N	$\mathcal{M}_N$ =	aN +	b +	$c/\sqrt{N}$ +	d/N
		a	b	c	d
2	0.890625				
3	1.08333333333333				
4	1.29792043399638				
5	1.52318930281443	0.249835	0.0492187	0.53598	-0.0745192
6	1.75432075208484	0.246695	0.123999	0.338947	0.0706775
7	1.98916255005449	0.248062	0.0831756	0.459736	-0.0294887
8	2.22660014721048	0.249434	0.0339754	0.619579	-0.175195
9	2.46595156197257	0.249738	0.021223	0.664392	-0.219408
10	2.70676308425724	0.249748	0.0207354	0.666226	-0.221345
11	2.94871984780432	0.249774	0.0193389	0.671799	-0.227593
12	3.19159601492947	0.249824	0.0163325	0.684451	-0.242556
13	3.43522469549061	0.249871	0.0132299	0.69815	-0.259556
14	3.67947926593368	0.249905	0.0108374	0.709186	-0.273865
∞		0.25	0		

Table: Sixth moment ($n = 3$) of the density when $\Gamma = 4$.

N	$\mathcal{M}_N$ =	aN +	b +	$c/\sqrt{N}$ +	d/N
		a	b	c	d
2	1.21875				
3	1.25420875420875				
4	1.36950440777577				
5	1.51576558282311	0.188419	0.468569	-0.269131	1.12729
6	1.67718058035561	0.189303	0.447536	-0.213712	1.08645
7	1.84742557965232	0.194189	0.301703	0.217776	0.728635
8	2.02343313425351	0.197002	0.20078	0.545661	0.42975
9	2.20346793019457	0.197864	0.164668	0.672559	0.30455
10	2.38645657506403	0.198269	0.145249	0.745579	0.227431
11	2.57169539517539	0.198622	0.126216	0.821535	0.142267
12	2.75870117826714	0.198925	0.108069	0.897903	0.0519525
13	2.9471288020509	0.199158	0.0926932	0.965792	-0.0322945
14	3.13672349360904	0.199328	0.0804934	1.02207	-0.10526
∞		0.2	0		

Table: Eighth moment ($n = 4$) of the density when $\Gamma = 4$.

N	$\mathcal{M}_N$ =	aN +	b +	$c/\sqrt{N}$ +	d/N
		a	b	c	d
3	0.829151732377539				
4	1.13999055712937				
5	1.45889183119874				
6	1.78179400313294	0.330681	-0.23698	-0.0197034	0.256393
7	2.10668148567864	0.326556	-0.113869	-0.383961	0.55846
8	2.43308749295152	0.335184	-0.423364	0.621536	-0.358107
9	2.76069703430536	0.335669	-0.443708	0.693027	-0.428642
10	3.08920070504297	0.333408	-0.335393	0.285749	0.00150004
11	3.41837572685644	0.33287	-0.306358	0.169876	0.131419
12	3.74807370935471	0.333069	-0.318295	0.220112	0.0720079
∞		$1/3$	$-1/3$		

Table: Fourth moment ($n = 2$) of the density when $\Gamma = 6$.

N	$\mathcal{M}_N$ =	aN +	b +	$c/\sqrt{N}$ +	d/N
		a	b	c	d
3	0.63878932696137				
4	0.852317437834435				
5	1.07602482170551				
6	1.30481948267947	0.238952	-0.17935	-0.0015329	0.306508
7	1.5366992098612	0.241542	-0.256653	0.227188	0.116838
8	1.77130926929194	0.254997	-0.739333	1.79534	-1.31262
9	2.00817460819162	0.253514	-0.677215	1.57705	-1.09725
10	2.24678505124783	0.249869	-0.502619	0.920542	-0.403889
11	2.48676496530133	0.249348	-0.474543	0.8085	-0.278266
12	2.72785340315076	0.249797	-0.501405	0.921547	-0.411958
∞		0.25	-0.5		

Table: Sixth moment ($n = 3$) of the density when $\Gamma = 6$.

N	$\mathcal{M}_N$ =	aN +	b +	$c/\sqrt{N}$ +	d/N
		a	b	c	d
3	0.579848665870171				
4	0.732937257370685				
5	0.897778601877637				
6	1.06814688756782	0.178851	0.00107475	-0.227029	0.519884
7	1.24220552729734	0.18911	-0.305123	0.678947	-0.231411
8	1.41969193536163	0.204817	-0.8686	2.50959	-1.90014
9	1.60002816650081	0.201673	-0.736901	2.04679	-1.44354
10	1.78261079422029	0.198035	-0.562593	1.39137	-0.751319
11	1.96700538479197	0.198245	-0.573912	1.43654	-0.801964
12	2.15290808747152	0.199242	-0.633669	1.68802	-1.09938
∞		0.2	$-2/3 \simeq -0.666667$		

Table: Eighth moment $(n = 4)$ of the density when $\Gamma = 6$.