A Model of Anyons in One Dimension. Correlation Functions.

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Outline

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2. Correlation Functions
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   - Matrix Riemann-Hilbert Problem
   - Large Distance Asymptotics

Lieb-Liniger Gas of Anyons

$$H = \int dx \ \left([\partial_x \Psi_A^\dagger(x)][\partial_x \Psi_A(x)] + c\Psi_A^\dagger(x)\Psi_A^\dagger(x)\Psi_A(x)\Psi_A(x) - h\Psi_A^\dagger(x)\Psi_A(x) \right)$$

$$\Psi_A(x_1)\Psi_A^\dagger(x_2) = e^{-i\pi\kappa\epsilon(x_1-x_2)}\Psi_A^\dagger(x_2)\Psi_A(x_1) + \delta(x_1 - x_2)$$

$$\Psi_A^\dagger(x_1)\Psi_A^\dagger(x_2) = e^{i\pi\kappa\epsilon(x_1-x_2)}\Psi_A^\dagger(x_2)\Psi_A^\dagger(x_1)$$

- Statistics parameter $\kappa \in [0, 1]$ ($\kappa = 0$ bosons, $\kappa = 1$ fermions)
- $\epsilon(x) = |x|/x$ = sign of x,
- Limit $c \to \infty$ describes impenetrable case

$\Longrightarrow$ Algebraic definition of fractional statistics in one dimension

- Girardeau, Batchelor, Khundu: consistency of algebra
- Calabrese, Mintchev, Santachiara, Cabra: important results

Quantum Mechanics of 1D Anyons

Eigenstates of the Hamiltonian

$$|\Psi_N\rangle = \frac{1}{\sqrt{N!}} \int dz^N \chi_N(z_1, \cdots, z_N | \lambda_1, \cdots, \lambda_N) \Psi_A^\dagger(z_N) \cdots \Psi_A^\dagger(z_1)|0\rangle$$

$$\chi_N(\cdots, z_i, z_{i+1}, \cdots) = e^{i\pi\kappa\epsilon(z_i - z_{i+1})} \chi_N(\cdots, z_{i+1}, z_i, \cdots)$$

Self-consistent boundary conditions

$$\begin{aligned}
\chi_N(0, z_2, \cdots, z_N) &= \chi_N(L, z_2, \cdots, z_N)\,, \\
\chi_N(z_1, 0, \cdots, z_N) &= e^{i(2\pi\kappa)} \chi_N(z_1, L, \cdots, z_N)\,, \\
&\vdots \\
\chi_N(z_1, z_2, \cdots, 0) &= e^{i(2\pi(N-1)\kappa)} \chi_N(z_1, z_2 \cdots, L)\,.
\end{aligned}$$

Bethe Ansatz and Yang Thermodynamics

$$\chi_N = \frac{e^{+i\frac{\pi\kappa}{2}\sum_{j<k}\epsilon(z_j - z_k)}}{\sqrt{N!}} \prod_{j>k} \epsilon(z_j - z_k) \sum_{\pi \in S_N} (-1)^\pi e^{i\sum_{n=1}^N z_n \lambda_{\pi(n)}} .$$

Bethe equations = boundary conditions

$$e^{i\lambda_j L} = (-1)^{N-1} e^{i\overline{\eta}}, \quad \overline{\eta} = -\pi\kappa(N-1)$$

Twist $\sim N$. Momentum in the ground state.
Thermodynamic limit $N \to \infty$, $L \to \infty$, N/L is fixed.

Yang-Yang thermodynamics $\quad \vartheta(\lambda, \mathbf{h}, \mathbf{T}) = \frac{\mathbf{1}}{(\mathbf{1}+\mathbf{e}^{(\lambda^2-\mathbf{h})/\mathbf{T}})}$

$$D = \frac{1}{2\pi}\int_{-\infty}^{\infty} \vartheta(\lambda, h, T)\, d\lambda\,, \quad E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lambda^2 \vartheta(\lambda, h, T)\, d\lambda$$

Correlation Functions

$$\langle \Psi_A^\dagger(x)\Psi_A(0)\rangle_T = \frac{\mathrm{Tr}\,\{e^{-\frac{H}{T}}\Psi_A^\dagger(x)\Psi_A(0)\}}{\mathrm{Tr}\{e^{-\frac{H}{T}}\}}$$

- Method of A. Its, A. Izergin, V.Korepin and N.Slavnov for Lieb-Liniger gas: Journal of Modern Physics B. **4**, 1003 (1990).
- Represent correlation function as Fredholm determinant. Integrable integral operators.
- Differential equations.
- Riemann-Hilbert problem $\rightarrow$ large distance asymptotic.
- Textbook: QUANTUM INVERSE SCATTERING METHOD AND CORRELATION FUNCTIONS, by A. Izergin, V. Korepin, N. Bogoliubov, Cambridge University Press, 1993

Important developments: P. Calabrese, F. Colomo, V. Cheianov, F. Essler, A. Its, K. Kozlowski, G. Mussardo, N. Slavnov, M. Zvonarev .

Determinant Representations

$$\langle \Psi_A^\dagger(x)\Psi_A(0)\rangle_T = B_{++}\,\det(1-\gamma\hat{K}_T)$$

$$K_T(\lambda,\mu) = \frac{e_+(\lambda)e_-(\mu) - e_-(\lambda)e_+(\mu)}{2i(\lambda-\mu)}\,, \qquad e_\pm(\lambda) = \sqrt{\vartheta(\lambda)}e^{\pm i\lambda x}$$

$$B_{lm} \equiv \gamma\int_{-\infty}^{+\infty} e_l(\lambda)f_m(\lambda)\,d\lambda\,, \qquad l,m=\pm$$

$$f_\pm(\lambda) - \gamma\int_{-\infty}^{+\infty} K_T(\lambda,\mu)f_\pm(\mu)d\mu = e_\pm(\lambda), \qquad \gamma = \frac{(1+e^{i\pi\kappa})}{\pi}$$

The functions $f_\pm(\lambda)$ appear during inversion of the operator $\left(\hat{I}-\gamma\hat{K}_T\right)$.

$$(1-\gamma\hat{K}_T)^{-1} = 1+R, \qquad R(\lambda,\mu) = \frac{f_+(\lambda)f_-(\mu) - f_-(\lambda)f_+(\mu)}{2i(\lambda-\mu)}$$

Idea

Point of view of classical completely integrable differential equations:

$$f_\pm(\lambda) - \gamma \int_{-\infty}^{+\infty} K_T(\lambda,\mu) f_\pm(\mu) d\mu = e_\pm(\lambda), \qquad \gamma = \frac{(1+e^{i\pi\kappa})}{\pi}$$

$$e_\pm(\lambda) = \sqrt{\vartheta(\lambda)} e^{\pm i\lambda x}, \qquad \vartheta(\lambda,h,T) = \frac{1}{(1+e^{(\lambda^2-h)/T})}$$

We can consider this as Gelfand -Levitan -Marchenko equation within classical inverse scattering method.

$$\partial_x e_+ = i\lambda e_+, \qquad (\partial_h + \partial_{\lambda^2}) e_+ = 0$$

We can derive two linear differential equations for $f_\pm$. This will be a Lax representation for nonlinear differential equation for B.

Lax representation

$$\begin{aligned} \partial_x F(\lambda) &= \mathsf{L}\, F(\lambda)\,, \\ (2\lambda\partial_\beta + \partial_\lambda) F(\lambda) &= \mathsf{M}\, F(\lambda)\,, \end{aligned} \qquad (1)$$

$$F(\lambda) = \begin{pmatrix} f_+(\lambda) \\ f_-(\lambda) \end{pmatrix}. \qquad (2)$$

$$(2\lambda\partial_\beta + \partial_\lambda)\mathsf{L} - \partial_x \mathsf{M} + [\mathsf{L}, \mathsf{M}] = 0\,,$$

$$\mathsf{L} = i\lambda\sigma^z + B_{++}\sigma^x\,, \qquad (3)$$

$$\mathsf{M} = ix\sigma^z - i\partial_\beta B_{+-}\sigma^z - \partial_\beta B_{++}\sigma^y\,. \qquad (4)$$

Differential Equations for Potentials

$$\partial_\beta B_{+-} = x + \frac{1}{2}\frac{\partial_x \partial_\beta B_{++}}{B_{++}}, \qquad \partial_x B_{+-} = B_{++}^2$$

initial conditions

$$\begin{aligned} B_{++} &= \gamma d(\beta) + [\gamma d(\beta)]^2 x\,, \qquad & x \to 0\,, \\ B_{+-} &= \gamma d(\beta) + [\gamma d(\beta)]^2 x\,, \qquad & x \to 0 \end{aligned}$$

$$d(\beta) = \int_{-\infty}^{+\infty} \vartheta(\lambda) d\lambda\,, \quad \vartheta(\lambda, h, T) = \frac{1}{(1 + e^{(\lambda^2 - h)/T})}, \quad \gamma = \frac{(1 + e^{i\pi\kappa})}{\pi}$$

$$B_{++} = B_{+-} \to 0\,, \qquad \beta \to -\infty\,.$$

Differential equations for $\sigma = \ln\det(1-\gamma\hat{K}_T)$

$$\partial_x\sigma = -B_{+-}\,, \qquad \partial_x^2\sigma = -B_{++}^2\,,$$
$$\partial_\beta\sigma = -x\partial_\beta B_{+-} + \frac{1}{2}(\partial_\beta B_{+-})^2 - \frac{1}{2}(\partial_\beta B_{++})^2\,,$$

Integrable nonlinear partial differential equation for σ

$$(\partial_\beta\partial_x^2\sigma)^2 + 4(\partial_x^2\sigma)[2x\partial_\beta\partial_x\sigma + (\partial_\beta\partial_x\sigma)^2 - 2\partial_\beta\sigma] = 0$$

initial conditions depends on statistics:

$$\sigma = -\gamma d(\beta)x - [\gamma d(\beta)]^2\frac{x^2}{2} + O(x^3)\,, \qquad x\to 0$$
$$\sigma = 0\,, \qquad \beta\to-\infty\,,$$

Differential equation for σ at $T \to 0$

$\xi = xh^{1/2}$ and $\sigma_0 = \xi(\tilde{\sigma}_0)'$

$$(\xi\sigma_0'')^2 + 4(\xi\sigma_0' - \sigma_0)[4\xi\sigma_0 - 4\sigma_0 + (\sigma_0')^2] = 0$$

This Painlevé V equation was first derived by Jimbo, Miwa, Sato, Mori for impenetrable bosons. For anyons initial conditions are diferent.

$$\sigma_0 = -2\gamma\xi - 4\gamma^2\xi^2 + O(\xi^3)$$

Agrees with Santachiara and Calabrese Toeplitz determinant approach.

Matrix Riemann-Hilbert Problem

Matrix function $\chi(\lambda)$ is analytic in the upper and lower half-plane and

$$\begin{aligned} \chi_-(\lambda) &= \chi_+(\lambda) G(\lambda), \quad \chi_\pm(\lambda) = \lim_{\epsilon \to 0^+} \chi(\lambda \pm i\epsilon), \quad \lambda \in \mathbb{R} \\ \chi(\infty) &= I \end{aligned}$$

Conjugation matrix

$$G(\lambda) = \begin{pmatrix} 1 + \pi\gamma e_+(\lambda) e_-(\lambda) & -\pi\gamma e_+^2(\lambda) \\ \pi\gamma e_-^2(\lambda) & 1 - \pi\gamma e_+(\lambda) e_-(\lambda) \end{pmatrix}$$

here $e_\pm(\lambda) = \sqrt{\vartheta(\lambda)} e^{\pm i\lambda x}$

$$\lim_{\lambda \to \infty} \chi(\lambda) = I + \frac{1}{2i\lambda} \begin{pmatrix} B_{+-} & -B_{++} \\ B_{++} & -B_{+-} \end{pmatrix} + O\left(\frac{1}{\lambda^2}\right)$$

Asymptotics

$$\langle \Psi_A^\dagger(x)\Psi_A(0)\rangle_T = e^{-x\sqrt{T}C(h/T,\kappa)/2}\left(c_0 e^{ix\sqrt{T}\lambda_0} + c_1 e^{ix\sqrt{T}\lambda_1}\right),$$

$$C(\beta,\kappa) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \ln\left(\frac{e^{\lambda^2-\beta}+1}{e^{\lambda^2-\beta}-e^{i\pi\kappa}}\right) d\lambda\,,$$

$$\lambda_0 = \frac{1}{\sqrt{2}}\left(\beta + \sqrt{\beta^2+\pi^2\kappa^2}\right)^{1/2} + \frac{i}{\sqrt{2}}\left(-\beta + \sqrt{\beta^2+\pi^2\kappa^2}\right)^{1/2}, \beta = h/T$$

$$\lambda_1 = -\frac{1}{\sqrt{2}}\left(\beta + \sqrt{\beta^2+\pi^2[\kappa-2]^2}\right)^{1/2} + \frac{i}{\sqrt{2}}\left(-\beta + \sqrt{\beta^2+\pi^2[\kappa-2]^2}\right)^{1/2}$$

Conformal limit

For small temperature the expression in the exponent becomes linear function of temperature

- For $0 < \kappa < 2/3$: $\Im\lambda_1 \leq 2\Im\lambda_0$

$$\langle\Psi_A^\dagger(x)\Psi_A(0)\rangle_T \simeq c_0 e^{-x\frac{\pi T}{v_F}\left(\frac{\kappa^2}{2}+\frac{1}{2}\right)} e^{ixk_F\kappa},$$

- For $2/3 < \kappa < 1$: $\Im\lambda_1 \geq 2\Im\lambda_0$

$$\langle\Psi_A^\dagger(x)\Psi_A(0)\rangle_T \simeq c_0 e^{-x\frac{\pi T}{v_F}\left(\frac{\kappa^2}{2}+\frac{1}{2}\right)} e^{ixk_F\kappa} + c_1 e^{-x\frac{\pi T}{v_F}\left[2\left(\frac{\kappa}{2}-1\right)^2+\frac{1}{2}\right]} e^{ixk_F(\kappa-2)}$$

Agrees with harmonic fluid approach of Calabrese and Mintchev 2006.

Large time and distance asymptotic

Back to higher temperatures.
Determinant representations and Riemann-Hilbert approach also work for time and temperature dependent correlations.

Asymptotic $x \to \infty, \qquad t \to \infty, \qquad \{x/t \quad \text{is fixed}\}$

$$\langle \Psi_A(x_2,t_2)\Psi_A^\dagger(x_1,t_1)\rangle_T \sim \text{missing phase} \exp\left\{ \frac{1}{\pi}\int_{-\infty}^{+\infty} |x-2t\lambda| \ln\left|\frac{e^{\lambda^2-\beta}-e^{i\pi\kappa}}{e^{\lambda^2-\beta}+1}\right| d\lambda \right\}$$

$$x = \frac{1}{2}(x_1-x_2)\sqrt{T}, \qquad t = \frac{1}{2}(t_2-t_1)T, \qquad \beta = \frac{h}{T}$$

λ is momentum, κ is statistics parameter ($\kappa \neq 1$) and h is chemical potential.

Experimental implementation

- Bristol Centre for Quantum Photonics the group of Prof. Jeremy O'Brien created entangled state of photos which realize one dimensional abelian anyons.
- Coulomb blockage of anyons : system of antidots in fqhe can be constructed so that anyons tunnel from one antidot to another [project of experiment by Averin, Nesteroff].
- Braiding of 2D anyons in topological quantum computation.