An application of differential Galois theory to the computation of exact eigenvalues and eigenfunctions of some Sturm-Liouville problems

David Blázquez Sanz

(Joint work with K. Yagasaki)

Universidad de los Andes

May 31, 2012.

Universidad Sergio Arboleda

Instituto de Matemáticas y sus Aplicaciones

1. **Statement of the problem, motivations**
2. Differential Galois approach
3. Main results
4. Applications
    - Spectrum of the asymmetric Hulthen potential
    - Bifurcation of homoclinic solutions of real coupled Ginzburg-Landau equations
    - Linear stability of front waves in Alen-Cahn (Nagumo) equation

Sturm-Liouville problems on $(-\infty, \infty)$ 1

- Consider two real analytic functions $\mu(x), \nu(x) \colon \mathbb{R} \to \mathbb{R}$ such that
$$\lim_{|x|\to\infty} \frac{\mathrm{d}^k \mu}{\mathrm{d}x^k} = \lim_{|x|\to\infty} \frac{\mathrm{d}^k \nu}{\mathrm{d}x^k} = 0, \quad \forall k \geq 1.$$
- S-L problem: Determine the complex numbers $\lambda \in \mathbb{C}$ such that the differential equation:
$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \mu(x)\frac{\mathrm{d}\psi}{\mathrm{d}x} + \nu(x)\psi = \lambda\psi \qquad \text{(SLP)}$$
has a solution $\psi_\lambda \colon \mathbb{R} \to \mathbb{C}$ satisfying
$$\int_{-\infty}^{\infty} |\psi(x)|^2 \mathrm{d}x < \infty, \quad \text{i.e.} \quad \psi_\lambda \in \mathcal{L}^2(-\infty, \infty).$$

Sturm-Liouville problems on $(-\infty, \infty)$ 2

- Our hypothesis on $\mu(x)$ and $\nu(x)$ allow us to change the convergence of the integral by the equivalent boundary conditions
$$\lim_{|x|\to\infty} \psi_\lambda(x) = 0. \tag{BC}$$
- This set $\Sigma = \{\lambda | \exists \psi_\lambda \in \mathcal{L}^2(-\infty, \infty)\} \subset \mathbb{C}$ is called the point-spectrum of the problem.
- Functional analysis provides us existence theorems and bounds for the point-spectrum Σ.
- In general terms, it is not possible to determine exactly Σ by algebraic means.
- In this work we present a methodology for some algebraically some Sturm-Lioville problems arising in dynamical systems and quantum mechanics.

S-L problems and quantum potentials

- Let us consider a quantum-mechanical potential $V(x)$, for which the position is described by a real variable x. The stationary Schrödinger equation reads,

$$E\psi(x) = \left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{dx}^2} + V(x)\right)\psi(x).$$

- We write $\lambda = -\frac{2mE}{\hbar^2}$ and get a S-L problem in the form (SLP)

$$\frac{\mathrm{d}^2\psi}{\mathrm{dx}^2} + \frac{-2mV(x)}{\hbar^2}\psi = \lambda\psi.$$

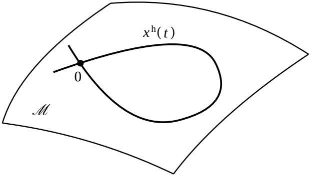
- The point-spectrum gives us the admissible values of energy for the system.

S-L problems and structural stability of Homoclinics 1

- Let us consider an analytic Hamiltonian H in $\mathbb{R}^4$, depending on a λ, $H = H(x;\lambda)$, $(x = (q_1, q_2, p_1, p_2))$,

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

- Assume that the origin is saddle-saddle point.
- Let us assume that there exist an homoclinic orbit $x^h(t;\lambda)$ do a for an open set of values of the parameter λ.



- Generic homoclinic orbits are structurally stable under Hamiltonian perturbations.

S-L problems and structural stability of Homoclinics 2

- Generic homoclinic orbits are structurally stable under Hamiltonian perturbations.
- Its structural stability can be studied by means of its linear aproximation, the first variatonal equation:

$$\dot{\xi} = \mathrm{J} \cdot \mathrm{Hess}(H(x^h(t), \lambda))\xi. \tag{VE}$$

- If the (VE) has two linearly independent bounded solutions then the homoclinic orbit is not structurally stable and it may give rise to saddle-node or pitchfork biffurcations under suitable perturbations [Yagasaki 2010].
- We reduce the (VE) to its normal part, obtaining a second order equation the reduced normal variational equation,

$$\frac{\mathrm{d}^2}{\mathrm{dt}^2}\eta + h(t, \lambda)\eta = 0 \tag{RNVE}$$

- If the (NV) has two linearly independent bounded solutions, then the (RNVE) as a solution in $\mathcal{L}^2(-\infty, \infty)$.

S-L problems and linear stability of waves

- Let us consider a non-linear evolution equation,

$$u_t = P(u_{xx}, u_x, u), \quad P \text{ analytic}$$

- Let us consider an stationary (or traveling wave) solution $v(x)$, equation along $v(x,t)$ is,

$$\xi_t = \frac{\partial P}{\partial u_{xx}}(v)\xi_{xx} + \frac{\partial P}{\partial u_x}(v)\xi_x + \frac{\partial P}{\partial u}(v)\xi = L_v(\xi)$$

- The linear stability of v is then related to the eigenvalue problem,

$$L_v(\xi) = \lambda\xi,$$

  the stationary (wave) solution is linearly stable if all elements of the point spectrum have negative real part.

1. Statement of the problem, motivations
2. **Differential Galois approach**
3. Main results
4. Applications
    - Spectrum of the asymmetric Hulthen potential
    - Bifurcation of homoclinic solutions of real coupled Ginzburg-Landau equations
    - Linear stability of front waves in Alen-Cahn (Nagumo) equation

Picard-Vessiot theory in 2 Slides, 1

- Let Γ be a Riemann surface and z a non-constant meromorphic function.
- Let K be the field of meromorphic functions on Γ.
- Then, $(K, \frac{d}{dz})$ is a differential field.
- We deal with the integrability by quadratures of differential equations,

$$L(\psi) = \frac{\mathrm{d}^2}{\mathrm{d}z^2}\psi + f\frac{\mathrm{d}}{\mathrm{d}z}\psi + g = 0, \quad f, g \in K. \qquad \text{(LE)}$$

- Let $\Gamma^{\times} \subset \Gamma$ be the Riemann surface obtained by removing the singularities of the operator L, and $\tilde{\Gamma} \to \Gamma^{\times}$ the universal covering.
- There is a $\mathbb{C}$-vector space E of holomorphic solutions in $\Gamma^{\times}$.

Picard-Vessiot theory in 2 Slides, 2

- Let ψ_1, ψ_2 be a basis of E and $P = K\langle\psi_1, \psi_2\rangle$ the differential field they span (endowed with the lift of $\frac{\mathrm{d}}{\mathrm{dz}}$). The extension $K \subset P$ is called the Picard-Vessiot extension of (LE).
- We consider field automorphisms that commute with the derivation $\mathrm{Aut}_{\frac{\mathrm{d}}{\mathrm{dz}}}(P/K) \subset \mathrm{GL}(E)$. They form a linear algebraic group that we call the Galois group.
- The analytic continuation gives a natural linear action of $\pi_1(\Gamma^\times)$ and then a group morphism $\pi_1(\Gamma^\times) \to \mathrm{Aut}_{\frac{\mathrm{d}}{\mathrm{dz}}}(P/K)$. Its image is called the monodromy group.
- Recall that a singularity is called regular if any solution approaching that singularity is bounded by some meromorphic function.
- Theorem (Schlessinger): if the singularities of (LE) are regular then the monodromy group is Zariski closed into the Galois group.

Local Fuchs-Frobenius theory 1

- Near a regular singularity $z = 0$, the (LE) has the form,

$$\frac{\mathrm{d}^2\psi}{\mathrm{dz}^2} + \frac{p(z)}{z}\frac{\mathrm{d}\psi}{\mathrm{dz}} + \frac{q(z)}{z^2}\psi = 0 \tag{RS}$$

  with p and q holomorphic at $z = 0$.
- Let α_1, α_2 be the roots of the indicial equation

$$\alpha(\alpha - 1) + p(0)\alpha + q(0).$$

  We name them in such way that $\Re(\alpha_1) \leq \Re(\alpha_2)$.
- If $\alpha_2 - \alpha_1 \notin \mathbb{Z}$ then there are two solutions,

$$\psi_1(z) = z^{\alpha_1} F_1(z), \quad \psi_2(z) = z^{\alpha_2} F_2(z), \tag{NL}$$

  with F_1, F_2 holomorphic at z.
- In this basis, the monodromy operator clockwise around $z = 0$ takes the form $\begin{pmatrix} e^{2\pi i \alpha_1} & \\ & e^{2\pi i \alpha_2} \end{pmatrix}$.

Local Fuchs-Frobenius theory 2

- If $\alpha_2 - \alpha_1 \in \mathbb{Z}$ then we may have two solutions as in (NL) or two solutions as follows,

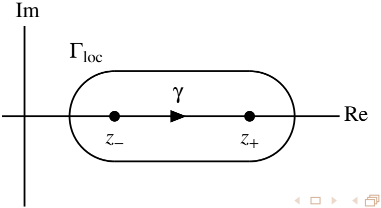
$$\psi_1(z) = z^{\alpha_1} F_1(z), \quad \psi_2(z) = z^{\alpha_2} F_2(z) + \psi_1(z) \log(z),$$

  with F_1, F_2 holomorphic at z.
- In this basis, the monodromy operator clockwise around $z = 0$ takes the form $\begin{pmatrix} e^{2\pi i \alpha_1} & 2\pi i \\ & e^{2\pi i \alpha_2} \end{pmatrix}$.
- Key lemma: Assume that $\Re(\alpha_1) < 0 < \Re(\alpha_2)$, then:
  (i) There is a non-zero solution ψ_1 which is bounded along any ray approaching the $z = 0$.
  (ii) Any other solution, independent with ψ_1, is unbounded along any ray approaching $z = 0$.
  (iii) The solution ψ_1 is an eigenvector of the monodromy around $z = 0$ of eigenvalue $e^{2\pi i \alpha_1}$.

1. Statement of the problem, motivations
2. Differential Galois approach
3. **Main results**
4. Applications
   - Spectrum of the asymmetric Hulthen potential
   - Bifurcation of homoclinic solutions of real coupled Ginzburg-Landau equations
   - Linear stability of front waves in Alen-Cahn (Nagumo) equation

Heuristics

- We present a heuristic idea for produce some Stiurm-Liouville problems that can be solved analitically.
- Let Γ real algebraic curve given by an irreducible polynomial, $P(z,w) = 0$.
- We consider in Γ a real meromorphic vector field,
$$\frac{\mathrm{dz}}{\mathrm{dx}} = f(z,w), \quad f(z,w) \in K$$
and a real solution, $\gamma \colon \mathbb{R} \to \Gamma$, $x \mapsto \gamma(x)$ connecting two zeros, namely $z_{\pm}$.
- Let Γ_{loc} be a neighbourhood of the path γ and K_{loc} its field of meromorphic functions.



- Let g, h be meromorphic functions in Γ and holomorphic in Γ_{loc}.
- We consider Sturm-Liouville problems with,

$$\mu(x) = g(\gamma(x)), \quad \nu(z) = h(\gamma(x)). \tag{CV}$$

- The Sturm-Liouville problem,

$$\frac{\mathrm{d}^2\psi}{\mathrm{dx}^2} + \mu(x)\frac{\mathrm{d}\psi}{\mathrm{dx}} + \nu(x)\psi = \lambda\psi \tag{SLP}$$

is equivalent to the following. Determine for which values of lambda, the meromorphic differential equation:

$$\frac{\mathrm{d}^2\psi}{\mathrm{dz}^2} + \frac{g + \frac{df}{dz}}{f}\frac{\mathrm{d}\psi}{\mathrm{dz}} + \frac{h-\lambda}{f^2} = 0 \tag{AE}$$

has a solution defined along γ tending to 0 at $z_{\pm}$.

- Define G_{loc} the Galois group of (AE) as a differential equation with coefficients in K_{loc}.
- Define G the Galois group of (AE) as a differential equation with coefficients in K.
- Remark: In general $G_{\mathrm{loc}} \subset G$. If (AE) has exactly three regular singularities at Γ then $G_{\mathrm{loc}} = G$.
- Define,

$$\mu_{\pm} = \lim_{x \to \pm\infty} \mu(x), \quad \nu_{\pm} = \lim_{x \to \pm\infty} \nu(x),$$

$$\alpha_0^{\pm} = \mu_- \pm \sqrt{\nu_-^2 - 4(\mu_- - \lambda)} \quad \alpha_1^{\pm} = \mu_+ \pm \sqrt{\nu_+^2 - 4(\mu_+ - \lambda)}$$

Main Result

- Theorem 1 [Yagasaki-B, 2010]: The following statements hold:
  - Assume $\Re(\alpha_0^+) > 0$, $\Re(\alpha_0^+) > 0$, $\Re(\alpha_1^+) > 0$, $\Re(\alpha_1^-) > 0$ then al solutions of (SLP) satisfy the boundary conditions (BC).
  - Assume one of the following conditions,

    (i) $\Re(\alpha_0^+) > 0$, $\Re(\alpha_0^-) > 0$, $\Re(\alpha_1^+) \cdot \Re(\alpha_1^-) < 0$.

    (ii) $\Re(\alpha_1^+) > 0$, $\Re(\alpha_1^-) > 0$, $\Re(\alpha_0^+) \cdot \Re(\alpha_0^-) < 0$.

    then one solution of (SLP) satisfy the boundary conditions.
  - Assume $\Re(\alpha_i^+) \cdot \Re(\alpha_i^+) < 0$ for $i = 0, 1$ or equivalently

$$|\Re(\sqrt{\mu_\pm^2 - 4(\nu_\pm - \lambda)}| > |\mu_\mp|,$$

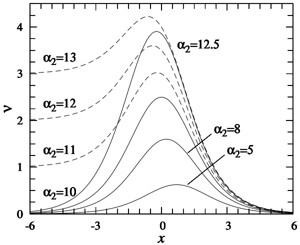
    then if there is a solution of (SLP) satisfying the boundary conditions (BC) then $G_{\rm loc}$ is a group of triangular matrices, and henceforth solvable.

1. Statement of the problem, motivations
2. Differential Galois approach
3. Main results
4. Applications
    - **Spectrum of the asymmetric Hulthen potential**
    - Bifurcation of homoclinic solutions of real coupled Ginzburg-Landau equations
    - Linear stability of front waves in Alen-Cahn (Nagumo) equation

Spectrum of the asymmetric Hulthen potential 1

- We consider the normalized Schroedinger equation for the asymmetric Hulthen potential,

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + \nu(x)\psi = \lambda\psi, \quad \nu(x) = \frac{\alpha_2}{e^x + \alpha_1} - \frac{\alpha_3}{(e^x + \alpha_1)^2} \quad \text{(AHP)}$$



Shape of the function $\nu(x)$ in (AHP) for several values of α_2 when $\alpha_1 = 10/\alpha_2$ or 1 and $\alpha_3 = 10$. Solid and dashed lines represent the cases of $\alpha_1 = 10/\alpha_2$ and 1, respectively.

Spectrum of the asymmetric Hulthen potential 2

- We know that possible eigenvalues are real numbers λ satisfying,

$$\max(\nu_-, 0) < \lambda < \frac{\alpha_2^2}{4\alpha_3}.$$

- We consider the change of variables,

$$z = \gamma(x) = \frac{e^x}{e^x + 1}$$

- It takes the Equation (AHP) to its algebraic form,

$$\frac{\mathrm{d}^2\psi}{\mathrm{dz}^2} + \frac{2z-1}{z(z-1)}\psi + \frac{(z-1)\frac{\alpha_3\beta^2}{\alpha_1^2} - \lambda(z-z_0)}{z^2(z-1)^2(z-z_0)^2}\psi = 0, \quad \text{(AAHP)}$$

where $z_0 = \frac{\alpha_1}{\alpha_1 - 1}$.

Spectrum of the asymmetric Hulthen potential 3

- Equation (AAHP) is a Riemann equation and its solutions are expressed by a Riemann P function

$$P \left\{ \begin{matrix} 0 & 1 & z_0 & \\ \rho_1^+ & \rho_2^+ & \rho_3^+ & z \\ \rho_1^- & \rho_2^- & \rho_3^- & \end{matrix} \right\},$$

- The local exponents at $z = 0$, 1 and z_0 are

$$\rho_1^\pm = \pm\sqrt{\lambda - \nu_-}, \quad \rho_2^\pm = \pm\sqrt{\lambda},$$

$$\rho_3^\pm = \frac{1}{2}\left(1 \pm \frac{1}{\alpha_1}\sqrt{\alpha_1^2 + 4\alpha_3}\right).$$

Spectrum of the asymmetric Hulthen potential 4

- Theorem [Kimura 1969]: Let us consider the Riemann function,

$$P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \rho_1^+ & \rho_2^+ & \rho_3^+ & z \\ \rho_1^- & \rho_2^- & \rho_3^- & \end{Bmatrix},$$

  and the differential field extension $\mathbb{C}(z) \subset \mathbb{C}(z)\langle P\rangle$. Its Galois group is a group of triangular matrices if and only if at least one of $\rho_1 + \rho_2 + \rho_3$, $-\rho_1 + \rho_2 + \rho_3$, $\rho_1 - \rho_2 + \rho_3$ and $\rho_1 + \rho_2 - \rho_3$ is an odd integer, where $\rho_j = \rho_j^+ - \rho_j^-$, $j = 1, 2, 3$, denote the exponent differences.
- In our particular cases it yields,

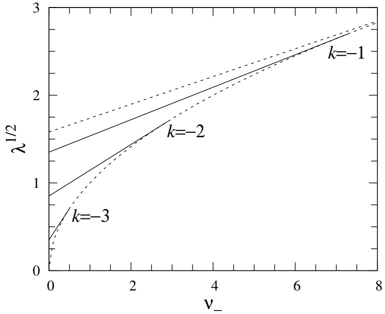
$$\lambda = \frac{((2k + 1 \pm \rho_3)^2 + 4\nu_-)^2}{16(2k + 1 \pm \rho_3)^2}, \quad k \in \mathbb{Z}$$

Spectrum of the asymmetric Hulthen potential 5

- Theorem: If for some integer $k \in (-\frac{1}{2}(\rho_3 + 1), 0)$

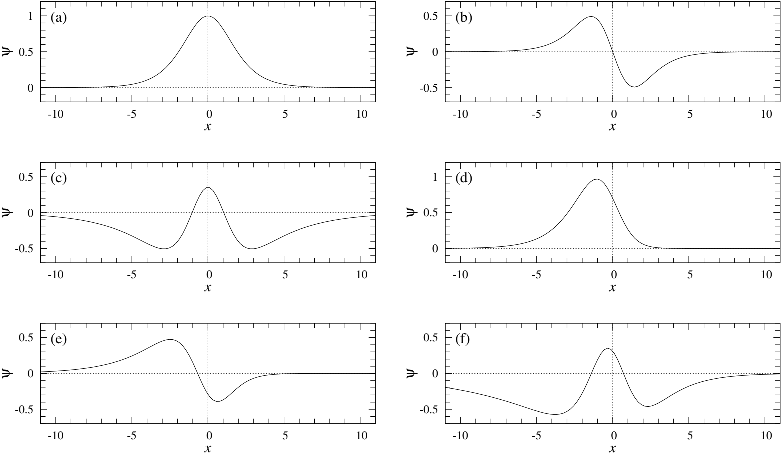
$$\lambda = \frac{((2k + 1 + \rho_3)^2 + 4\nu_-)^2}{16(2k + 1 + \rho_3)^2} \in \left(\max(\nu_-, 0), \frac{\alpha_2^2}{4\alpha_3}\right),$$

then λ is an eigenvalue for the (AHP).



Eigenvalues for (AHP) with $\alpha_1 = 1$ and $\alpha_3 = 10$. The dotted lines represent the upper bound $\sqrt{\lambda} = \frac{1}{2}\alpha_2/\sqrt{\alpha_3}$ and the lower bound $\sqrt{\lambda} = \sqrt{\nu_-}$.

Spectrum of the asymmetric Hulthen potential 6



Eigenfunctions for (AHP) with $\alpha_1 = 1$ and $\alpha_3 = 10$: (a) $(\nu_-, \sqrt{\lambda}) = (0, 1.35078)$; (b) $(0, 0.850781)$; (c) $(0, 0.350781)$; (d) $(3.5, 1.99855)$; (e) $(1.5, 1.29155)$; (f) $(0.25, 0.528955)$.

1. Statement of the problem, motivations
2. Differential Galois approach
3. Main results
4. Applications
    - Spectrum of the asymmetric Hulthen potential
    - **Bifurcation of homoclinic solutions of real coupled Ginzburg-Landau equations**
    - Linear stability of front waves in Alen-Cahn (Nagumo) equation

Bifurcation of Homoclinic orbits in G-L equations 1

- We consider the coupled real Ginzburg-Landau system of non-linear PDE ($s > 0$),

$$
\begin{aligned}
U_t &= U_{xx} - U + (U^2 + \beta_1 V^2)U_1 + \beta_3 V, \\
V_t &= V_{xx} - sV + (\beta_1 U^2 + \beta_2 V^2)V + \beta_3 U + \beta_4 V^2,
\end{aligned}
\qquad \text{(GL)}
$$

- Stationary solutions are given by the ODE system

$$
\begin{aligned}
\ddot{x}_1 &= x_1 - (x_1^2 + \beta_1 x_2^2)x_1 - \beta_3 x_2, \\
\ddot{x}_2 &= s x_2 - (\beta_1 x_1^2 + \beta_2 x_2^2)x_2 - \beta_3 x_1 - \beta_4 x_2^2,
\end{aligned}
\qquad \text{(SGL)}
$$

where $x_1 = U$, $x_2 = V$.

Bifurcation of Homoclinic orbits in G-L equations 2

- It is Hamiltonian system (denote $y_1 = \dot{x}_1$, $y_2 = \dot{x}_2$),

$$H = \frac{1}{2}(-x_1^2 - sx_2^2 + \beta_1 x_1^2 x_2^2 + y_1^2 + y_2^2) + \frac{1}{4}(x_1^4 + \beta_2 x_2^4) + \beta_3 x_1 x_2 + \frac{1}{3}\beta_4 x_2^3, \qquad \text{(SGLH)}$$

- For $\beta_3 = 0$ the origin $x = 0$ is saddle-saddle with eigenvalues,

$$\lambda_1 = -\sqrt{s}, \quad \lambda_2 = -1, \quad \lambda_3 = 1, \quad \lambda_4 = \sqrt{s}.$$

- There are two orbits homoclinic to the origin:

$$x_\pm^h(t) = (\pm\sqrt(2\mathrm{sech}(t), 0, \pm\sqrt{2}\mathrm{sech}(t)\tanh(t), 0).$$

Bifurcation of Homoclinic orbits in G-L equations 3

- Reduced normal variational equation:

$$\ddot{\eta} = (s - 2\beta_1 \mathrm{sech}^2(t))\eta.$$

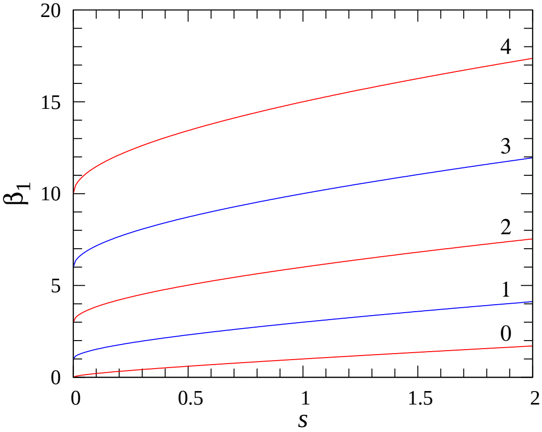
- Whic is transformed into a Riemann equation, $z = \mathrm{sech}^2(t)$, $\nu_1 = s$, $\nu_2 = 2\beta_1$,

$$\frac{d^2\eta}{dz^2} + \frac{3z-2}{2z(z-1)}\frac{d\eta}{dz} + \frac{\nu_1 - \nu_2 z}{4z^2(z-1)}\eta = 0.$$

- Applying Kimura's result we derive integrability conditions:

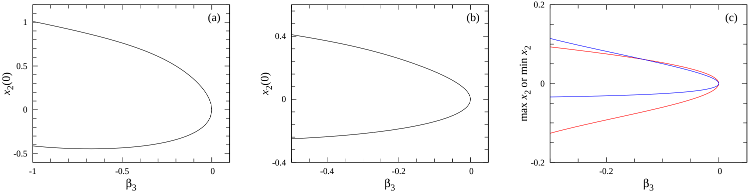
$$\beta_1 = \frac{(2\sqrt{s} + 2\ell + 1)^2 - 1}{8} \quad \ell = 0, 1, 2, \ldots \qquad \text{(IC2)}$$

Bifurcation of Homoclinic orbits in G-L equations 4



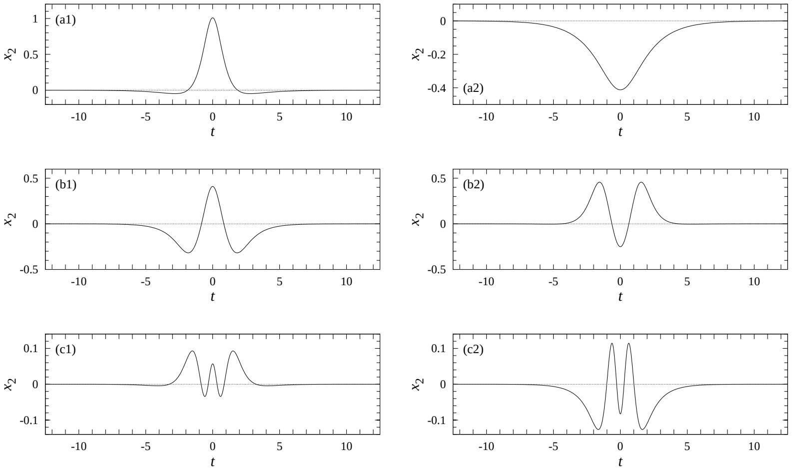
Integrability condition (IC2) for ℓ =0-4. Saddle-node bifurcations may happen along red curves but along blue curves. Pitchfork bifurcations can occur along all curves.

Bifurcation of Homoclinic orbits in G-L equations 5



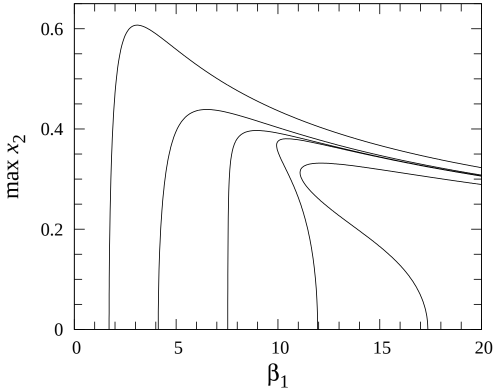
Bifurcation diagram with parameter β_3 and $s = 2$: (a) $\beta_1 = 1.7071068$, $\beta_2 = 1$ y $\beta_4 = 2$; (b) $\beta_1 = 7.5355339$, $\beta_2 = 1$ y $\beta_4 = 2$; (c) $\beta_1 = 17.36396103$, $\beta_2 = 10$ y $\beta_4 = 20$.

Bifurcation of Homoclinic orbits in G-L equations 6



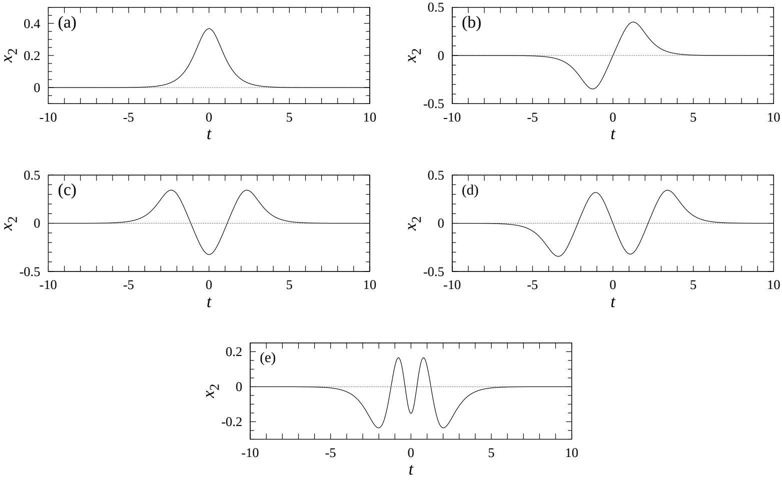
Profiles of homoclinic orbits on the branches of the previous diagrams. (a1,2) $\beta_3 = 1$; (b1,2) $\beta_3 = 0.5$; y (c1,2), $\beta_3 = 0.3$.

Bifurcation of Homoclinic orbits in G-L equations 7



Bifurcation diagram with parameter β_1, $s = 2$, $\beta_2 = 1$ y $\beta_3 = \beta_4 = 0$.

Bifurcation of Homoclinic orbits in G-L equations 8



Homoclinic orbits on the branches of ℓ =0-4 on the previous diagram with $\beta_1 = 15$: (a) $\ell = 0$; (b) $\ell = 1$; (c) $\ell = 2$; (d) $\ell = 3$ y (e) $\ell = 4$ (lower branch).

1. Statement of the problem, motivations
2. Differential Galois approach
3. Main results
4. Applications
    - Spectrum of the asymmetric Hulthen potential
    - Bifurcation of homoclinic solutions of real coupled Ginzburg-Landau equations
    - **Linear stability of front waves in Alen-Cahn (Nagumo) equation**

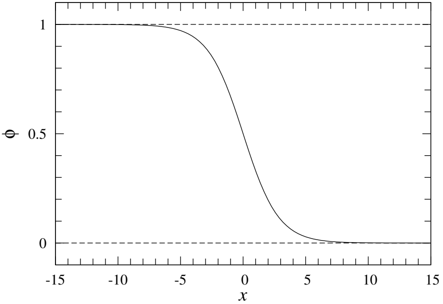
Stability of wave front solutions of Allen-Cahn equation 1

- Let us consider the Allen-Cahn (or Nagumo) equation,

$$u_t = u_{xx} + u(1-u)(u-\alpha), \quad \alpha \in (0,1).$$

- There is a traveling front solution $u(x,t) = \phi(x-ct)$ with $c = \sqrt(2)(\frac{1}{2} - \alpha)$ and

$$\phi(x) = \frac{1}{e^{\frac{x}{\sqrt{2}}+1}}$$

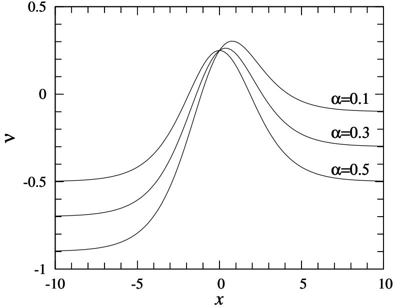


Stability of wave front solutions of Allen-Cahn equation 2

- The linear stability of the wave front solution is determined by the S-L problem,

$$\frac{\mathrm{d}^2\psi}{\mathrm{dx}^2} + \mu(x)\frac{\mathrm{d}\psi}{\mathrm{dx}} + \nu(x)\psi = \lambda\psi \qquad \text{(ACSLP)}$$

with $\mu(x) = \sqrt{2}(\frac{1}{2} - \alpha)$, $\nu(x) = -3\phi(x)^2 + 2(\alpha + 1)\phi(x) - \alpha$.



Plots of $\nu(x)$ for $\alpha = 0.1, 0.3, 0.5$.

Stability of wave front solutions of Allen-Cahn equation 2

- We consider the change of variables,

$$\frac{\mathrm{dz}}{\mathrm{dx}} = \frac{z(1-z)}{\sqrt{2}}, \quad \gamma(x) = \frac{e^{\frac{x}{\sqrt{2}}}}{e^{\frac{x}{\sqrt{2}}}+1}.$$

- The algebraic form is a Riemann equation with singularites at 0, 1 and ∞,

$$\frac{\mathrm{d}^2\psi}{\mathrm{dz}^2} + \frac{2(z+\alpha-1)}{z(z-1)}\frac{\mathrm{d}\psi}{\mathrm{dz}} + \frac{2(-3z^2+2(2-\alpha)z+\alpha-1-\lambda)}{z^2(z-1)^2}\psi = 0.$$

- The local exponents at 0, 1, ∞ are respectively

$$\rho_1^{\pm} = \frac{1}{2}(2\alpha - 1 \pm \sqrt{8\lambda + (2\alpha-3)^2}),$$
$$\rho_2^{\pm} = \frac{1}{2}(1 - 2\alpha \pm \sqrt{8\lambda + (2\alpha+1)^2}), \quad \rho_3^{+} = 3, \quad \rho_3^{-} = -2.$$

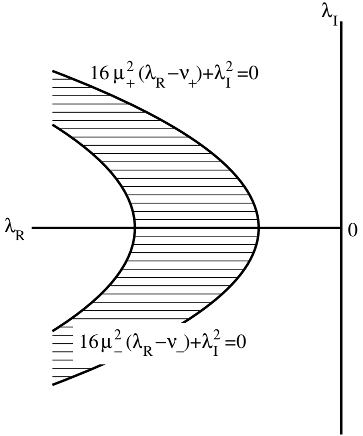
Stability of wave front solutions of Allen-Cahn equation 4

- Theorem: Assume that for $\lambda \in \mathbb{C}$ there is a solution for the (ACSL) satisfying the boundary conditions (BC). One of the following holds:
  (a) $\alpha \in (0, \frac{1}{2})$ and Theorem 1, (i) holds.
  (b) $\alpha \in (\frac{1}{2}, 1)$ and Theorem 1, (ii) holds.
  (c) λ satisfies $\max(\alpha - 1, -\alpha) < \lambda < \frac{1}{3}(\alpha^2 - \alpha + 1)$ and there is an integer k such that

$$\lambda = \frac{(k^2 - 4)(k + 1 - 2\alpha)(k - 1 + 2\alpha)}{8k^2}. \qquad \text{(IC3)}$$

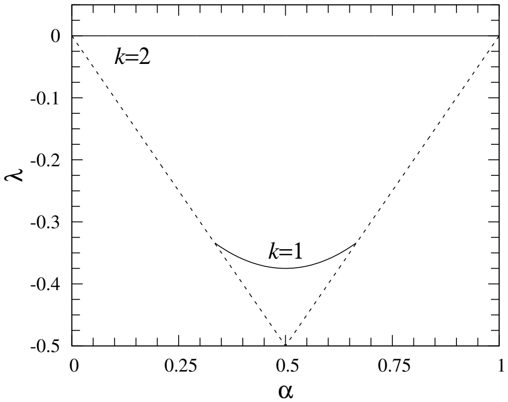
  Furthermore, conditions (a) and (b) are sufficient for the existence of solution for the boundary problem.

Stability of wave front solutions of Allen-Cahn equation 5



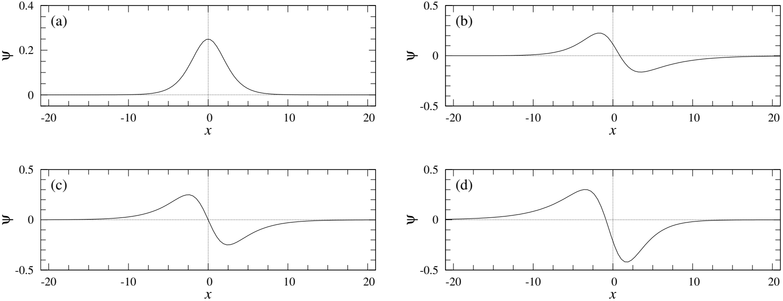
Continuous spectra (the shaded region) for (ACSL) when $\alpha \in (0, \frac{1}{2})$.

Stability of wave front solutions of Allen-Cahn equation 6



Eigenvalues given by (IC3) for $k = 1, 2$. The dotted line represents the lower bound $\lambda = \max(\alpha - 1, -\alpha)$.

Stability of wave front solutions of Allen-Cahn equation 7



Eigenfunctions for (ACSL): (a) $\lambda = 0$; (b) $\alpha = 0.35$; (c) 0.5; (d) 0.65.
Plates (b)-(c) show the functions for $\lambda = \frac{3}{2}\alpha(\alpha - 1)$.

References and related research

- P. Acosta-Humánez, J. J. Morales-Ruiz, J.-A. Weil, *Galoisian approach to integrability of Schrödinger equation*, Rep. Math. Physics, 67 (2011) 3, pp. 305–374.
- D. Blázquez-Sanz, K. Yagasaki, *Analytic and algebraic conditions for bifurcations of homoclinic orbits I: Saddle equilibria*, arXiv:1009.0977
- D. Blázquez-Sanz, K. Yagasaki, *Galoisian approach for a Sturm-Liouville problem on the infinite interval*, arXiv:1009.0979
- F. Fauvet, J.-P. Ramis, F. Richard-Jung, J. Thomann, *Stokes phenomenon for the prolate spheroidal wave equation*, Appl. Numer. Math. (2010), doi:10.1016/j.apnum.2010.05.010.

Numerical computations and figures by K. Yagasaki using AUTO97.