A Galoisian approach to SUSY QM and beyond

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May 31 2012, Bogotá - Colombia

Contents

Plan

Picard-Vessiot

Galois Theory for Linear Differential Equations

Differential field: $(K, d/dt)$, field with a derivation.

Example: $K =$ field of meromorphic functions, $M(\Gamma)$, over a Riemann surface Γ (classical particular case: rational functions $\mathbf{C}(t) =$ meromorphic functions of the Riemann sphere $\mathbf{P}^1$).

Linear differential equation with coefficients in K:

$$\frac{d\xi}{dt} = A\xi, \quad A \in Mat(m, K). \tag{1}$$

Picard-Vessiot extension of (1): $L := K(u_{ij})$,
$U := (u_{ij})$: fundamental matrix associated to (1)
($K \subset L \sim$ Galois extension of a polynomial).

Differential automorphism $\sigma : L \to L$:
i) automorphism of the field L,
ii) $\sigma d/dt = d/dt\sigma$.

Galois group of the equation (1):
$G := Gal(1) = Gal(L/K) = \{\sigma : L \to L : \sigma$ differential automorphism $\sigma_K = Id\}$
($\sim$ Galois group of a polynomial).
G is the transformation group preserving all the algebraic relations of u_{ij} with coefficients in K.

Theorem

G is a linear algebraic subgroup of $GL(m, \mathbf{C})$.

(1) *integrable*: the extension L is obtained from K by a combination of algebraic extensions, quadratures and exponentials of quadratures.

Theorem

(1) is integrable $\Leftrightarrow$ G^0 is solvable.

Kovacic Algorithm

This is an effective algorithm to compute the solutions of differential equations:

$$\partial_x^2 y = ry, \quad r \in \mathbf{C}(x).$$

There are four cases in Kovacic's algorithm. Only for cases 1, 2 and 3 we can solve the differential equation, but for the case 4 the differential equation is not integrable.

Theorem

Let G be an algebraic subgroup of $\mathrm{SL}(2,\mathbb{C})$. Then, up to conjugation, one of the following cases occurs.

1. $G \subseteq \mathbb{B} :=$ *triangular unimodular group.*
2. $G \nsubseteq \mathbb{B}$, $G \subseteq \mathbb{D}_\infty :=$ *infinite dihedral group.*
3. $G \in \{A_4^{\mathrm{SL}_2}, S_4^{\mathrm{SL}_2}, A_5^{\mathrm{SL}_2}\}$.
4. $G = \mathrm{SL}(2,\mathbb{C})$.

Algebrization Procedure

Differential equations with non-rational coefficients into differential equations with rational coefficients.

A change of variable $z = z(x)$ is called hamiltonian when $(z(x), \partial_x z(x))$ is a solution curve of the hamiltonian system

$$\begin{array}{l}\partial_x z = \partial_w H \\ \partial_x w = -\partial_z H\end{array} \quad \text{with} \quad H = H(z, w) = \frac{w^2}{2} + V(z).$$

Theorem [Algebrization of Schrödinger Operator]

$H = -\partial_x^2 + V$ is algebrizable through a hamiltonian change of variable $z = z(x)$ if and only if there exists $\widehat{V}$, α such that

$$\frac{\partial_z \alpha}{\alpha}, \frac{\widehat{V}}{\alpha} \in \mathbb{C}(z), \ \textit{where}\, V(x) = \widehat{V}(z(x)), \ \alpha(z) = (\partial_x z)^2.$$

The algebrization of H is given by the operator

$$\widehat{H} = -\left(\alpha \partial_z + \frac{1}{2} \partial_z \alpha\right) \partial_z + \widehat{V}(z).$$

Plan

Non-relativistic quantum mechanics

Schrödinger equation

Consider the *one-dimensional stationary non-relativistic Schrödinger equation*

$$H\Psi = \lambda\Psi, \quad H = -\partial_x^2 + V.$$

SUSY QM. A supersymmetric quantum mechanical system is one in which there are operators Q_i and $\mathcal{H}$ satisfying

$$[Q_i, \mathcal{H}] = 0, \quad \{Q_i, Q_j\} = \delta_{ij}\mathcal{H}, \quad \{Q_i, Q_j\} = Q_iQ_j + Q_jQ_i$$

For $n = 2$, the supercharges Q_i are defined as

$$Q_\pm = \frac{\sigma_1 p \pm \sigma_2 W(x)}{2}, \quad Q_+ = Q_1,\ Q_- = Q_2, \quad p = -i\hbar\partial_x,$$

where W is the superpotential and σ_i the Pauli spin matrices.

The operator $\mathcal{H}$, satisfying $Q_i\mathcal{H} = \mathcal{H}Q_i$ and $2Q_i^2 = \mathcal{H}$, is

$$\mathcal{H} = \frac{I_2p^2 + I_2W^2(x) + \hbar\sigma_3\partial_x W(x)}{2} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \ I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Motivation: Known shape invariant potentials in Quantum Mechanics

Potential	**Name**
$\frac{1}{2}m\omega^2\left(x-\sqrt{\frac{2}{m}}\frac{b}{\omega}\right)^2$	Shifted H. O.
$\frac{1}{2}m\omega^2r^2+\frac{l(l+1)\hbar^2}{2mr^2}-\left(l+\frac{3}{2}\right)\hbar\omega$	3D H.O.
$-\frac{e^2}{r}+\frac{l(l+1)\hbar^2}{2mr^2}-\frac{me^4}{2(l+1)^2\hbar^2}$	Coulomb
$A^2+B^2e^{-2ax}-2B\left(A+\frac{a\hbar}{2\sqrt{2m}}\right)e^{-ax}$	Morse 1
$A^2+\frac{B^2-A^2-\frac{Aa\hbar}{\sqrt{2m}}}{\cosh^2 ax}+\frac{B\left(2A+\frac{a\hbar}{\sqrt{2m}}\right)\sinh ax}{\cosh^2 ax}$	Morse 2
$A^2+\frac{B^2}{A^2}+2B\tanh ax-A\frac{A+\frac{a\hbar}{\sqrt{2m}}}{\cosh^2 ax}$	Rosen-Morse 1
$A^2+\frac{B^2+A^2+\frac{Aa\hbar}{\sqrt{2m}}}{\sinh^2 ar}-\frac{B\left(2A+\frac{a\hbar}{\sqrt{2m}}\right)\cosh ar}{\sinh^2 ar}$	Rosen-Morse 2
$A^2+\frac{B^2}{A^2}-2B\coth ar+A\frac{A-\frac{a\hbar}{\sqrt{2m}}}{\sinh^2 ar}$	Eckart 1
$-A^2+\frac{B^2+A^2-\frac{Aa\hbar}{\sqrt{2m}}}{\sin^2 ax}-\frac{B\left(2A-\frac{a\hbar}{\sqrt{2m}}\right)\cos ax}{\sin^2 ax}$	Eckart 2
$-(A+B)^2+\frac{A\left(A-\frac{a\hbar}{\sqrt{2m}}\right)}{\cos^2 ax}+\frac{B\left(B-\frac{a\hbar}{\sqrt{2m}}\right)}{\sin^2 ax}$	Pöschl-Teller 1
$(A-B)^2-\frac{A\left(A+\frac{a\hbar}{\sqrt{2m}}\right)}{\cosh^2 ar}+\frac{B\left(B-\frac{a\hbar}{\sqrt{2m}}\right)}{\sinh^2 ar}$	Pöschl-Teller 2

(R. Dutt, A. Khare, U.P. Sukhatme, Am.J.Phys. 56(1988)163-168)

Plan

Notations

$\mathcal{L}_\lambda := H\Psi = \lambda\Psi, \quad H = -\partial_x^2 + V(x), \quad V \in K.$

$\Lambda \subseteq \mathbb{C}$: set of eigenvalues λ such that $\mathcal{L}_\lambda$ is integrable.

$\Lambda_+ := \{\lambda \in \Lambda \cap \mathbb{R} : \lambda \geq 0\}$, $\Lambda_- := \{\lambda \in \Lambda \cap \mathbb{R} : \lambda \leq 0\}$.

L_λ: Picard-Vessiot extension of $\mathcal{L}_\lambda$.

$\mathrm{Gal}(L_\lambda/K)$: differential Galois group of $\mathcal{L}_\lambda$.

The set Λ will be called *the algebraic spectrum* (or alternatively *the Liouvillian spectral set*) of H.

Λ can be $\emptyset$, i.e., $\mathrm{Gal}(L_\lambda/K) = \mathrm{SL}(2,\mathbb{C})\ \forall\lambda \in \mathbb{C}$.

If $\lambda_0 \in \Lambda$ then $(\mathrm{Gal}(L_{\lambda_0}/K))^0 \subseteq \mathbb{B}$.

Plan

Schrödinger Equation with Rational Potentials

Schrödinger Equation with Polynomial Potentials

Theorem (Polynomial potentials and Galois groups)

Let $V(x) \in \mathbb{C}[x]$ a polynomial of degree $k > 0$. Then

1. $\mathrm{Gal}(L_\lambda/K) = \mathrm{SL}(2, \mathbb{C})$, *or,*
2. $\mathrm{Gal}(L_\lambda/K) = \mathbb{B}$.

Corollary

Assume that $V(x)$ is an algebraically solvable polynomial potential. Then $V(x)$ is of degree 2.

Remark. When a polynomial potential is algebraically solvable or quasi-solvable, then the Galois group of the Schrödinger equation is exactly the Borel group (triangular).

Rational Potentials and Kovacic's Algorithm

Three dimensional harmonic oscillator potential

$$V(r) = r^2 + \frac{\ell(\ell+1)}{r^2} - (2\ell+3)\,, \quad \ell \in \mathbb{Z}.$$

The Schrödinger equation is

$$\partial_r^2 \Psi = \left(r^2 + \frac{\ell(\ell+1)}{r^2} - (2\ell+3) - \lambda \right) \Psi.$$

Algebraic Spectrum Applying Kovacic's algorithm we obtain $\Lambda = 2\mathbb{Z}$.

Bound states.

$$\Psi_n(r) = r^{\ell+1} P_{2n}(r) e^{\frac{-r^2}{2}}, \quad \lambda \in 4\mathbb{N}.$$

Galois groups. For $\lambda \in 4\mathbb{Z}$ we have that $\mathrm{Gal}(L_\lambda/K) = \mathbb{B}$.

Plan

Algebrized Schrödinger equations. In some cases we can algebrize $H\Psi = \lambda\Psi$ to apply Kovacic's algorithm, etc...

Theorem (Algebrization Algorithm for the exponential case)

Consider H with $V(x) = g(z_1(x), \cdots, z_n(x))$, $z_i(x) = e^{\mu_i x}$, $\mu_i \in \mathbb{C}^*$. The operator H is algebrizable if and only if

$$\frac{\mu_i}{\mu_j} \in \mathbb{Q}^*, \quad 1 \leq i \leq n, 1 \leq j \leq n, \quad g \in \mathbb{C}(z).$$

There exists $\mu \in \mathbb{C}^*$ such that $\mu_i = c_i\mu$, where $c_i \in \mathbb{Q}^*$. One Hamiltonian change of variable is $z = e^{\frac{\mu x}{q}}$, where

$$c_i = \frac{p_i}{q_i},\ p_i, q_i \in \mathbb{Z}^*,\ \gcd(p_i, q_i) = 1 \text{ and } q = \operatorname{lcm}(q_1, \cdots, q_n).$$

The algebrized Schrödinger equation of $H\Psi = \lambda\Psi$ is

$$\partial_z^2\widehat{\Psi} + \frac{1}{z}\partial_z\widehat{\Psi} - q^2\frac{g(z^{m_1}, \ldots, z^{m_n}) - \lambda}{\mu^2 z^2}\widehat{\Psi} = 0, \quad m_i = \frac{qp_i}{q_i}.$$

The Morse Potential. $V(x) = e^{-2x} - e^{-x}$.
The Schrödinger equation $H\Psi = \lambda\Psi$ is

$$\partial_x^2\Psi = \left(e^{-2x} - e^{-x} - \lambda\right)\Psi.$$

By the Hamiltonian change of variable $z = z(x) = e^{-x}$, we obtain

$$\alpha(z) = z^2, \quad \widehat{V}(z) = z^2 - z.$$

Thus, $\widehat{K} = \mathbb{C}(z)$ and $K = \mathbb{C}(e^x)$. In this way, the algebrized Schrödinger equation $\widehat{H}\widehat{\Psi} = \lambda\widehat{\Psi}$ is

$$z^2\partial_z^2\widehat{\Psi} + z\partial_z\widehat{\Psi} - (z^2 - z - \lambda)\widehat{\Psi} = 0.$$

Algebraic Spectrum and Galois Group.
$\Lambda = \{-n^2 : n \geq 0\} = \mathrm{spec}_p(H)$. $Gal(L_0/K) = \mathbb{B}$, $Gal(L_n/K) = \mathbb{G}_m$.

Plan

Recent and further research involving P-V Theory and QM

- Y. Brezhnev on finite gap potentials by means of Picard-Vessiot theory
- D. Blazquez-Sanz and K. Yagasaki on the Sturm-Liouville problems
- J.P. Ramis et al, concerning to Stokes phenomena
- T. Stachowiak on the Dirac equations
- T. Dreyfuss on the parametric Galois groups in Quantum Mechanics
- In progress there are some projects in Quantum Field Theory, Atomic and Molecular Physics, etc...

Plan

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