Entanglement Renormalisation and Boundary Conformal Field Theory

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13 May 2011

Setting

Purpose: Treat many-body quantum systems exhibiting a special kind of symmetry: scale invariance. Typically, we are interested in the ground state of some local hamiltonian:

$$H = \sum_{\langle i,j \rangle} h_{ij}, \qquad H\Psi_0 = E_0 \Psi_0, \tag{1}$$

$$\text{Example :} \qquad H = \sum_k \sigma_k^x \sigma_{k+1}^x + \sum_k \sigma^z. \tag{2}$$

Many-body (quantum) systems have too many degrees of freedom to allow a direct treatment. In the example above,

$$\dim\,(\text{Hilb}) = d^n.$$

Exact solution of (1) is generally out of question. Even storing the coefficients of Ψ_0 in some standard basis is impossible.

Renormalisation

We never want to know everything about a system. Only some very general properties are of interest: ground state energy, correlators, susceptibility, conductivity, etc.

Key observation: There are huge classes of natural systems for which only a handful of degrees of freedom are relevant when computing or measuring these quantities.

A *renormalisation group* (RG) generally refers to a process where some degrees of freedom of a system are eliminated. Hopefully, at the end of the process, our description of the system is simplified enough that a prediction can be made for the quantity we are interested in.

N.B. A renormalisation group is not a group. There is a deep reason for that.

Of course, we want an RG to be successful in

- Identifying the relevant degrees of freedom
- Estimating how irrelevant are the discarded degrees of freedom.

An example: matrix product states (White '90)

Consider again some one-dimensional local hamiltonian.

Again, let's assume we are interested in the lowest eigenvalue of H, E_0, and the corresponding eigenvector. Assuming $H \geq 0$, one possibility is to define the transfer operator $T_\epsilon(H) = exp(-\epsilon H)$ and use the identity

$$\Psi_0 = \lim_{n \to \infty} T_\epsilon(H)^n \Phi_0 / || T_\epsilon(H)^n \Phi_0 ||, \qquad \forall \Phi_0 \mid \langle \Psi_0 | \Phi_0 \rangle \neq 0, \tag{3}$$

One therefore constructs a sequence:

$$\Phi_0 \overset{T_\epsilon(H)}{\rightarrow} \Phi_1 \overset{T_\epsilon(H)}{\rightarrow} \Phi_2 \overset{T_\epsilon(H)}{\rightarrow} \dots \overset{T_\epsilon(H)}{\rightarrow} \Phi_n \overset{T_\epsilon(H)}{\rightarrow} \dots$$

Along this sequence, correlations typically grow so much that it is rapidly impossible to keep all the information about the states Φ_k. The matrix product state (MPS) RG consists in projecting successive states $T_\epsilon(H)\Phi_k$ onto a special class of states:

$$\Phi_0 \overset{T_\epsilon(H)}{\rightarrow} H\Phi_0 \overset{\text{Cutoff}}{\rightarrow} \Phi_1 \overset{T_\epsilon(H)}{\rightarrow} H\Phi_1 \overset{\text{Cutoff}}{\rightarrow} \Phi_2 \dots \overset{T_\epsilon(H)}{\rightarrow} H\Phi_{n-1} \overset{\text{Cutoff}}{\rightarrow} \Phi_n \dots$$

An example: matrix product states (White '90)

Matrix product states are states of the form

$$\Phi(x_1 \dots x_n)|x_1 \dots x_n\rangle = \mathrm{tr}\, A_1(x_1) \dots A_n(x_n)|x_1 \dots x_n\rangle. \qquad (4)$$

where, for each value $x_k \in 1 \dots d$, $A_k(x_k)$ is an $m \times m$ matrix.

Mean field methods: $\chi = 1$.

MPS have been used with great success over the past 20 years. However, their ability to describe *scale invariant* systems is limited, because they can only account for bounded correlations:

$$\langle \Psi_{\mathrm{MPS}}|X_x X'_y|\Psi_{\mathrm{MPS}}\rangle \approx exp(-\alpha|x-y|).$$

Interlude: diagrammatic notations

Entanglement renormalisation

If one is interested in scale invariance, it is of course natural to propose a renormalisation scheme where (the approximation of) the ground state is associated with a tree tensor network.

The isometries eliminate degrees of freedom.

The multiscale entanglement renormalisation ansatz (MERA) can also be associated with a tree tensor network, but with an additional ingredient: disentangling unitaries.

Motivation for the introduction of disentanglers: eliminate local degrees of freedom through local operations so that a maximal amount of information is kept when applying the isometries.

An entanglement renormalisation algorithm

Suppose we are interested in finding an approximation to the ground state of some local hamiltonian H. Then, a possibility is to proceed as follows.

1. Assume the ground state can be well described by a MERA, i.e. by a collection of tensors glued as above:

$$\bar{\tau} = \{\tau_i\}_{i=1\ldots\nu_1+\nu_2} = \{U_i\}_{i=1\ldots\nu_1} \cup \{W_i\}_{i=1\ldots\nu_2}$$

2. Initialise $\bar{\tau}$ to some fiducial value $\bar{\tau}_0$.
3. Run along the tensor network and pick a node α. Treat the tensors related to other nodes as constants. The mean value of the energy of the system can then be expressed as

$$E_0^{\mathrm{MERA}} = \langle \Psi(\tau) | H | \Psi(\tau) \rangle = \sum_i \mathrm{tr}\, A_i\, \tau_\alpha^*\, B_i\, \tau_\alpha .$$

   N.B. The sum over the index i typically only contains few terms. The quadratic form E_0^{MERA} is optimised over the tensor τ_α.
4. Repeat the previous step until convergence.

The D-quantum / $(D+1)$-classical correspondence

Consider a D-dimensional *classical* system described in terms of a set of degrees of freedom q_i, conjugate variables p_i, and some Hamiltonian H. Feynman-Hibbs quantisation (D-dimensional quantum):

$$\langle q', t'|q, t\rangle = \int_{\{\gamma\}} d\gamma \; e^{i\int_\gamma d\tau (p_i \dot{q}_i - H)}, \tag{5}$$

γ labels paths in phase space between time t and time t'.

$$\langle q', t'|\mathcal{T}\left[\hat{O}_1(t_1)\dots\hat{O}_M(t_M)\right]|q, t\rangle = \frac{\int_{\{\gamma\}} d\gamma \; \prod_{k=1}^{M} O_k(t_k)\; e^{i\int_\gamma d\tau (p_i \dot{q}_i - H)}}{\langle q', t'|q, t\rangle} \tag{6}$$

On the other hand, classical statistical mechanics tells us that:

$$Z = \sum_{\{\sigma\}} e^{-\beta H(\sigma)} \tag{7}$$

$$\langle O_1(x_1)\dots O_M(x_M)\rangle = \frac{\sum_{\{\sigma\}} O_1(x_1)\dots O_M(x_M) e^{-\beta H(\sigma)}}{Z}. \tag{8}$$

The D-quantum / $(D+1)$-classical correspondence

This *analogy* can be (formally) turned into an *identity* if we observe that the ground state mean value of $\hat{O}_1(t_1)\dots\hat{O}_M(t_M)$ can be expressed as

$$\langle\hat{O}_1(t_1)\dots\hat{O}_M(t_M)\rangle = \lim_{\tau,\tau'\to\infty}\frac{\langle q',-i\tau'|\hat{O}_1(t_1)\dots\hat{O}_M(t_M)|q,i\tau\rangle}{\langle q',-i\tau'|q,i\tau\rangle}. \tag{9}$$

Crucial to the correspondence is the analogy between the transfer operator of classical statistical mechanics and the quantum transfer operator introduced earlier:

$$e^{-\beta H^{\mathrm{classical}}(\sigma_k^{\mathrm{row}},\sigma_{k+1}^{\mathrm{row}})} \longleftrightarrow T_\epsilon(H) = exp(-\epsilon H^{\mathrm{quantum}}).$$

These considerations immediately extend to fields (at least conceptually if not mathematically).

Symmetries

We will call *symmetry* of a system a structured set of transformations of this system, together with a *representation* of this structured set spanned by a set of field $\{\Phi(x) : x \in \text{"Space"}\}$. Here, we are going to consider symmetries related to groups of space-time transformations G:

$$x \xrightarrow{g} x', \quad \Phi(x) \xrightarrow{g} \Phi'(x'). \tag{10}$$

Important examples:

	Translations	Proper rotations	Dilation
x'	$x + a$	$gx,\ gg^t = \mathbf{1}$	λx
$\Phi'(x')$	$e^{ip\cdot a}\Phi(x)$	$\pi(g)\Phi(x)$	$\lambda^{-\Delta}\Phi(x)$

We say that a (classical) system *possesses* a symmetry when its action

$$S_\gamma = \int_\gamma d^{(d+1)}x\, \mathcal{L}(\Phi, \partial_\mu\Phi) \text{ is left invariant } \forall g, \gamma.$$

Symmetries

A symmetry has consequences on the manner correlators transform. For instance,

Transformation	x'	$\langle \Phi'_1(x'_1) \dots \Phi'_n(x'_n) \rangle$
Translation	$x + a$	$e^{i(p_1 \cdot a_1 + \dots p_n \cdot a_n)} \langle \Phi_1(x_1) \dots \Phi_n(x_n) \rangle$
Rotation	gx	$\langle \Phi_1(x_1) \dots \Phi_n(x_n) \rangle$
Dilation	λx	$\lambda^{-\Delta_1} \dots \lambda^{-\Delta_n} \langle \Phi_1(x_1) \dots \Phi_n(x_n) \rangle$

Of particular importance are *conformal transformations*:

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x). \tag{11}$$

(Space-time transformations that preserve angles)

Such symmetries are important because

- Many important critical theories satisfy them. N.B. Such theories are *gapless* and are therefore delicate to tackle numerically.
- In $1 + 1$ dimensions, they yield a lot of information about correlation functions.

Global conformal transformations and quasi-primary fields

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x).$$

There are two kinds of conformal transformations: (i) global: $\Lambda(x)$ doesn't depend on x, (ii) local: the others. A field is called quasi-primary when it transforms, under a global conformal transformation as

$$\Phi(x) \to \Phi'(x') = |\frac{\partial x'}{\partial x}|^{-\Delta/d} \Phi(x), \qquad \Delta : \text{Scaling dimension.} \tag{12}$$

Importantly, if (the action of) a theory is invariant under global conformal transformations, then two- and three-point correlators of quasi-primary fields have a special form:

$$\langle \Phi_1(x_1)\Phi_2(x_2)\rangle = \begin{cases} \frac{C_{12}}{|x_1-x_2|^{2\Delta_1}} & \text{if } \Delta_1 = \Delta_2 \\ 0 & \text{if } \Delta_1 \neq \Delta_2 \end{cases} \tag{13}$$

$$\langle \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\rangle = C_{123}/x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{31}^{\Delta_3+\Delta_1-\Delta_2}. \tag{14}$$

Local conformal transformations and primary fields

Primary fields transform under *local* conformal transformations as

$$\Phi(x) \to \Phi'(x') = |\frac{\partial x'}{\partial x}|^{-\Delta/d}\Phi(x), \qquad \Delta : \text{Scaling dimension.} \tag{15}$$

Primary fields can be viewed as eigenstates of the conformal group. Primary fields play an important role in $1+1$-dimensions. Consider the identification $(x, y) \leftrightarrow z = x + iy$. Conformal transformations are associated with *holomorphic* functions:

$$z \to w(z),\ \frac{\partial w}{\partial z^*} = 0.$$

This is a rich set of symmetry. So rich that conformal field theories are essentially determined by a very little set of data.

Conformal field theory

Conformal data:

- The central charge c, governing e.g. correlations of extended parts of the system (see entanglement entropy).
- A complete list of primary fields $\{\Phi_\alpha\}$, together with their scaling dimensions $\{\Phi_\alpha\}$ [1]
- The coefficients C_{ijk} dictating the behaviour of three-point correlators of primary fields.

Other quasi-primary fields (so called descendent) are obtained by application of some operators $L_n, n \in \mathbb{Z}^+$ on primary fields:

$$\Phi_{n,\alpha} = L_{-n}\Phi_\alpha, \Phi_{n_1,n_2,\alpha} = L_{-n_2}L_{-n_1}\Phi_\alpha. \tag{16}$$

The operators L_n are some kind of moments of the energy-momentum tensor. The scaling dimension of a descendent field is determined as $\Delta(\Phi_{n_1,\ldots,n_m,\alpha}) = \Delta_\alpha + 2(n_1 + \ldots + n_m)$.

[1] To be more precise, one should rather consider their conformal dimensions $(h_\alpha, \bar{h}_\alpha)$. But this doesn't matter here.

An example: the critical Ising model

$$H^{\mathrm{quantum}} = -\frac{1}{2}\sum_{k=-\infty}^{+\infty} \sigma_k^x \sigma_{k+1}^x - \frac{\lambda}{2}\sum_{k=-\infty}^{+\infty} \sigma_k^z. \qquad (17)$$

$$H^{\mathrm{classical}} = -\frac{1}{2}\sum_{\langle i,j\rangle} \sigma_i^x \sigma_j^x. \qquad (18)$$

At criticality ($\lambda^* = 1,\ \beta(\lambda^*)$), both models are described by a CFT with central charge $c = 1/2$, three primary fields $\{\mathbf{1}, \sigma, \epsilon\}$ with respective scaling dimensions $\{0, 1/8, 1\}$.

Entanglement renormalisation and CFT

A layer of a MERA, $\mathcal{L}$, defines a lifting of operators:

$$X \to \mathcal{L}X\mathcal{L}* \equiv S(X).$$

It is of course natural to interpret this lifting as a scale transformation. Now if we consider a MERA which is a good approximation of the ground state of some critical system. Then we expect to be able to identify the eigen-operators of $\mathcal{S}$:

$$S(\Phi_\alpha) = (1/2)^{-\Delta_\alpha}\Phi_\alpha$$

with primary and quasiprimary fields of the underlying CFT.

This program has been carried out with remarkable success by Pfeifer et al.

(Quasi-)primary fields from MERA

$$S(\Phi_\alpha) = (1/2)^{-\Delta_\alpha} \Phi_\alpha, \qquad \langle \Phi_1(x_1)\Phi_2(x_2)\rangle = \delta(\Delta_1, \Delta_2)\frac{C_{12}}{|x_1 - x_2|^{2\Delta_1}}$$

Ising	Δ^{CFT}	Δ^{MERA}
σ ϵ	$1/8$ 1	0.124997 1.0001
Potts	Δ^{CFT}	Δ^{MERA}
σ_1	$2/15 = 0.133...$	0.1339
σ_2	$2/15 = 0.133...$	0.1339
ϵ	$4/5$	0.8204
Z_1	$4/3$	1.3346
Z_2	$4/3$	1.3351

First descendent fields are also found accurately.

Entanglement renormalisation and Boundary CFT

Boundary conformal field theory is an extension of CFT aimed at describing semi-infinite systems. Certainly, the physics of such systems far in the bulk is described by a CFT as we have discussed. Nevertheless the theory should be supplemented by a set of boundary scaling operators with support at finite distance from the edge of system, Φ_α^∂, that transform as primary fields under coarse graining:

$$S(\Phi_\alpha^\partial) = (1/2)^{-\Delta_\alpha^\partial}\Phi_\alpha^\partial$$

but whose two-point correlations with a bulk primary field satisfies the relation

$$\langle \Phi_\alpha^\partial(0)\Phi_\beta(x)\rangle = \frac{C_{\alpha\beta}}{x^{\Delta_\alpha^\partial+\Delta_\beta}},$$

even when $\Delta_\alpha^\partial \neq \Delta_\beta$. In particular, $\langle \Phi_\beta(x)\rangle = \frac{C_{0\beta}}{x^{\Delta_\beta}}$ in contrast to $\langle \Phi_\beta(x)\rangle_{\mathrm{bulk}} = 0$ for any bulk non-trivial scaling operator Φ_β.

Boundary MERA

A boundary MERA is an ansatz aimed at extracting information related to the edge of system. We have used this ansatz to study the critical hamiltonian

$$H_{\mathrm{Ising}} = \eta\, \sigma_0^x - \sum_{k=0}^{\infty} (\sigma_k^x \sigma_{k+1}^x - \sigma_k^z). \tag{19}$$

The constant η determines whether the system has free or fixed boundary conditions ($\eta = 0$ or $\eta = \pm 1$ respectively). We have optimised the tensors $u^{\mathrm{b}}, w^{\mathrm{b}}, w^{\partial}$ in order to get an approximation of the ground state. The resulting MERA defines a scale transformation whose eigen-operators can be identified with the (boundary) primary fields of the theory

Δ^{BCFT} (free Ising)	Δ^{MERA}	Δ^{BCFT} (fixed Ising)	Δ^{MERA}
($\mathbf{1}$) 0	0	($\mathbf{1}$) 0	0
(σ) 0.5	0.499	(σ) 2	1.992
1.5	1.503	3	2.998
2	2.001	4	4.005
2.5	2.553	4	4.062

Outlook

- MERA's bring an new perspective on old problems. The introduction of disentanglers is an interesting ingredient.
- CFT's are elegant and powerful (Think of the correlators). But how do we get there, given a concrete microscopic hamiltonian? The MERA seems to be a promising tool for this task. In particular, it seems to be accurate at identifying (quasi-)primary fields.
- MERA's also seem to be relevant to other problems. E.g. They describe *exactly* ground states of large families of topologically ordered systems.

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