An Introduction to the Spectral Theory of Dirac Operators

Osanobu Yamada *

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Abstract

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1 Essential Self-adjointness

Let us consider the Dirac operator

$$\begin{aligned} L &= L_0 + Q(x) := c(\alpha \cdot p) + mc^2\beta + Q(x) \\ &= c\sum_{j=1}^{3} \alpha_j p_j + mc^2\beta + Q(x) \qquad \left(p_j = -i\frac{\partial}{\partial x_j}\right) \end{aligned} \tag{1}$$

in the Hilbert space $\mathfrak{H} := L^2(\mathbb{R}^3)^4$, where $c > 0$ is the velocity of light, $m \geq 0$ the mass of the particle and

$$\alpha_j := \begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix}, \quad \beta := \begin{pmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & -I_2 \end{pmatrix}$$

with the 2×2 identity matrix I_2 and Pauli matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

*Department of Mathematical Sciences, Ritsumeikan University, Kusatsu, Shiga 525-8577, Japan. E-mail: yamadaos@se.ritsumei.ac.jp

Here we keep the anti-commutation relation

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I_4, \quad \beta\alpha_j + \alpha_j \beta = 0, \tag{2}$$

which implies

$$L_0^2 = c^2 \left(-\Delta + m^2 c^2\right) I_4$$

(I_4 is the unit matrix). $Q(x)$ is a 4×4 Hermitian matrix-valued function.

We often consider the case $c = m = 1$. In the present note we mention not only spectral properties but also the corresponding properties of Schrödinger operators

$$S = -\frac{1}{2m}\Delta + q(x)$$

acting on $L^2(\mathbb{R}^3)$, where $q(x)$ is a real-valued function.

Assume that each component of $Q(x)$ is locally square integrable in $\mathbb{R}^3$. Then we define a minimal operator on $\mathfrak{H}$ as

$$H_{min} u := Lu, \quad u \in D(H_{min}) = C_0^\infty(\mathbb{R}^3)^4.$$

H_{min} is a symmetric operator on $\mathfrak{H}$. It is said that H_{min} is essentially self-adjoint, if H_{min} has a unique self-adjoint extension, that is, the closure $\overline{H_{min}}$ is self-adjoint. Then we denote the self-adjoint extension by H.

If $Q(x) \equiv 0$, then H_{min} is essentially self-adjoint. Then we denote the self-adjoint extension by H_0, whose domain $D(H_0)$ coincides with the Sobolev space $W^{1,2}(\mathbb{R})^4$.

Theorem 1.1. *(Evans [Ev]) Assume a Stummel condition*

$$\int_{|x-y|\leq 1} \frac{|Q(y)|^2}{|x-y|^{1+\delta}}\, dy$$

is a locally bounded function of $x \in \mathbb{R}^3$ for some $\delta > 0$, where $|\cdot|$ denotes the matrix norm. Then, H_{min} is essentially self-adjoint.

We remark that, if $|Q(x)| \in L^3_{loc}(\mathbb{R}^3)$, then the essential self-adjointness is valid (Gross [Gr]).

It is important that the essential self-adjointness of Dirac operators does not depend on the growth order of $Q(x)$ at infinity but the local singularity (see, e.g., Jörgens [J], Thaller [Th]). One reason is that the solution $\exp(-itH)\varphi$ of the time-dependent Dirac equations $i\partial_t u = Hu$, which are symmetric hyperbolic systems, has the finite propagation property (Chernoff [C], Arai [Ar2]).

Theorem 1.2. *Assume*

$$|Q(x)| \leq \frac{e}{|x|} \qquad (x \in \mathbb{R}^3),$$

and $0 < e \leq (c/2)$, then H_{min} is essentially self-adjoint. If $0 < e < (c/2)$, then we have $D(H) = D(H_0)$.

The above theorem follows from Rellich-Kato theorem, Wüst theorem ([Wü1]) and Hardy inequality

$$\int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|^2}\, dx \le 4 \int_{\mathbb{R}^3} |\nabla u|^2\, dx \qquad (u \in C_0^\infty(\mathbb{R}^3)^4).$$

The condition $0 < e \le (c/2)$ corresponds to the atomic number $Z \le 68$.

Theorem 1.3. *Assume $Q(x) = q(x)I_4$, where $q(x)$ is a scalar function with*

$$|q(x)| \le \frac{e}{|x|} \qquad (x \in \mathbb{R}^3). \tag{3}$$

and $0 < e \le (\sqrt{3}c/2)$, then H_{min} is essentially self-adjoint. If $0 < e < (\sqrt{3}c/2)$, then we have $D(H) = D(H_0)$.

The above theorem is originally due to Schmincke [Sn1] under the condition $|Q(x)| \le e/|x|$, $0 < e < (\sqrt{3}c/2)$. For the case $e = \sqrt{3}c/2$, see, e.g., [Ym3]. The condition $0 < e \le (\sqrt{3}c/2)$ corresponds to the atomic number $Z \le 118$. It is interesting that the number of the newest atom, named in 2009 Copernisium, is 112.

It is well known that there is an distinguished self-adjoint extension of the Dirac operator for $(\sqrt{3}c/2) < e < c$ (Schmincke [Sn2], Wüst [Wü2], Arai [Ar1]).

The Schrödinger operator $S = -(1/2m)\Delta + q(x)$ on $C_0^\infty(\mathbb{R}^3)$ is essentially self-adjoint if $q(x) \in L^2_{loc}$ and

$$q(x) \ge -C|x|^2 \qquad (|x| \ge R) \tag{4}$$

for some positive constants C and R (Ikebe-Kato [IK], Kato [Kt2]).

In Kalf-Yamada [KY1] we investigate the essential self-adjointness of a type

$$L = \alpha \cdot p + m(x)\beta + q(x)I_4, \tag{5}$$

where $m(x)$ and $q(x)$ are real-valued functions with the singularities at the origin. The result can be also applied to the potentials

$$m(r) = \frac{C_1}{r^\mu}, \quad |V(x)| \le \frac{C_2}{r^\mu},$$

where $\mu > 1$ and $0 \le C_2 < |C_1|$.

It should be also remarked that Dirac operators with the anomalous magnetic moment

$$L := \alpha \cdot p + m\beta + \frac{e}{r} I_4 + \frac{\mu}{r^2} \begin{pmatrix} \mathbf{0} & i\sigma_r \\ -i\sigma_r & \mathbf{0} \end{pmatrix}$$

on $C_0^\infty(\mathbb{R}^3 \setminus \{0\})^4$ is essentially self-adjoint for every real e, if $\mu \ne 0$, where

$$\sigma_r = \sum_{j=1}^{3} \frac{x_j}{r} \sigma_j$$

(Behncke [Beh]).

2 Essential Spectrum

Let H be a self-adjoint operator on $\mathfrak{H}$ and $\sigma(H)$ the spectrum of H. We denote the set of isolated eigenvalues of H with finite multiplicity by $\sigma_d(H)$, which is called discrete spectrum of H. Then $\sigma_{ess}(H) := \sigma(H) \setminus \sigma_d(H)$ is called essential spectrum of H.

For the unperturbed operator H_0 we have

$$\sigma_{ess}(H_0) = (-\infty, -mc^2] \cup [mc^2, \infty), \qquad \sigma(H_0) \cap (-mc^2, mc^2) = \emptyset.$$

Theorem 2.1. *Let $Q(x)$ satisfy one of the assumption in Theorem 1.1 with*

$$|Q(x)| \longrightarrow 0 \qquad (|x| \to \infty).$$

or the assumption in Theorem 1.2 or Theorem 1.3. Then we have

$$\sigma_{ess}(H) = (-\infty, -mc^2] \cup [mc^2, \infty), \qquad (-mc^2, mc^2) \cap \sigma(H) \subset \sigma_d(H).$$

A similar property holds for the Schrödinger operators S, that is, if $q \in L^2_{loc}$ satisfies

$$q(x) \longrightarrow 0 \qquad (|x| \to \infty).$$

then we have

$$\sigma_{ess}(S) = [0, \infty).$$

Moreover, if $q(x) \longrightarrow +\infty$ $(|x| \to \infty)$, we have

$$\begin{aligned} &\sigma_{ess}(S) = \emptyset, \\ &\sigma(S) = \sigma_d(S) = \{\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \longrightarrow +\infty\}, \end{aligned}$$

(each λ_n is the eigenvalue with finite multiplicity) and,

if $q(x) \in C^1$ with the condition (4) and $q(x) \longrightarrow \infty$ $(|x| \to \infty)$, then

$$\sigma(S) = \sigma_{ess}(S) = (-\infty, +\infty).$$

under an additional condition on the derivatives of $q(x)$.

Let us consider

$$H = \alpha \cdot p + m(x)\beta + q(x)I_4, \tag{6}$$

where

$$m(x) \longrightarrow \infty \quad \text{or} \quad q(x) \longrightarrow \infty \quad (|x| \longrightarrow \infty).$$

Roughly speaking, we have

$$\begin{aligned} \lim_{|x|\to\infty} \frac{q(x)}{m(x)} = 0 \quad &\Rightarrow \quad \sigma_{ess}(H) = \emptyset, \qquad \sigma(H) = \sigma_d(H), \\ \lim_{|x|\to\infty} \frac{m(x)}{q(x)} = 0 \quad &\Rightarrow \quad \sigma(H) = \sigma_{ess}(H) = (-\infty, +\infty), \\ m(x) \equiv q(x) \longrightarrow +\infty \quad &\Rightarrow \quad \sigma(H) \cap (0, \infty) = \sigma_d(H), \\ &\qquad \sigma_{ess}(H) = (-\infty, 0] \end{aligned}$$

under some additional conditions (Schmidt–Yamada [SY], [Ym5]).

3 Separation of variables

Let

$$p_r := -ir^{-1}\frac{\partial}{\partial r}r = -i\left(\frac{\partial}{\partial r} + \frac{1}{r}\right),$$
$$K := 1 - \sum_{1\le j<k\le 3} i\alpha_j\alpha_k(x_jp_k - x_kp_j),$$
$$\alpha_r := \sum_{j=1}^{3}\frac{x_j}{r}\alpha_j, \quad \sigma_r := \sum_{j=1}^{3}\frac{x_j}{r}\sigma_j,$$

where K is called the spin-orbit coupling operator working on $L^2(S^2)^4$ with the discrete eigenvalues $\{\pm 1, \pm 2, \cdots, \pm k, \cdots\}$ (see, e.g., Arai [Ar1]). Then we have

$$\alpha \cdot p = \alpha_r\left(p_r + \frac{i}{r}K\right).$$

Moreover, by introducing a unitary operator U

$$U := \begin{pmatrix} I_2 & 0 \\ 0 & -i\sigma_r \end{pmatrix},$$

we obtain

$$U^{-1}\alpha_r U = \begin{pmatrix} 0 & -iI_2 \\ iI_2 & 0 \end{pmatrix}, \quad U^{-1}KU = \beta K, \quad U^{-1}\beta U = \beta,$$

which yields

$$U^{-1}(\alpha\cdot p)U = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}\frac{d}{dr} + \begin{pmatrix} 0 & (-k/r)I_2 \\ (-k/r)I_2 & 0 \end{pmatrix}$$

on the eigenspace of K with respect to the eigenvalue $k = \pm 1, \pm 2, \cdots$. Therefore, to consider the eigenvalue problem

$$Hu := (\alpha\cdot p)u + m(r)\beta u(x) + q(r)u(x) = \lambda u(x), \tag{7}$$

with $u \in \mathfrak{H}$, $\lambda \in \mathbb{R}$ and real-valued functions $m(r)$, $q(r)$, reduces to the first order differential systems

$$\begin{pmatrix} m(r) + q(r) & -(d/dr) - (k/r) \\ (d/dr) - (k/r) & -m(r) + q(r) \end{pmatrix}\begin{pmatrix} \varphi_1(r) \\ \varphi_2(r) \end{pmatrix} = \lambda\begin{pmatrix} \varphi_1(r) \\ \varphi_2(r) \end{pmatrix}. \tag{8}$$

If $q(r) = e/r$, $m(r) \equiv 0$, $\lambda = 0$, for example, the explicit solutions are well known.

Example 3.1.

$$\begin{pmatrix} e/r & -(d/dr)-(k/r) \\ (d/dr)-(k/r) & e/r \end{pmatrix} \begin{pmatrix} \varphi_1(r) \\ \varphi_2(r) \end{pmatrix} = 0.$$

If $k^2 - e^2 \neq 0$, there are two linearly independent solutions

$$\varphi_k^+(r) = r^{s_+} \begin{pmatrix} e \\ k - s_+ \end{pmatrix}, \quad \varphi_k^-(r) = r^{s_-} \begin{pmatrix} e \\ k - s_- \end{pmatrix},$$

where $s_\pm = \pm\sqrt{k^2 - e^2}$. If $k^2 - e^2 = 0$, then

$$\varphi_k^+(r) = \begin{pmatrix} e \\ k \end{pmatrix}, \quad \varphi_k^-(r) = \begin{pmatrix} 1 + k \log r \\ e \log r \end{pmatrix}$$

are linearly independent.

The above $\varphi_k^+(r)$ belongs always to $L^2(0,1)^2$ and $\varphi_k^-(r)$ belongs also to $L^2(0,1)^2$ for any $k = \pm 1, \pm 2, \cdots$ if $|e| > \sqrt{3}/2$, whereas $\varphi_k^-(r)$ does not belong to $L^2(0,1)^2$ for $k = \pm 1$ if $|e| \leq \sqrt{3}/2$. This fact suggests that the essential self-adjointness holds if and only if $|e| \leq \sqrt{3}/2$.

4 Non-existence of eigenvalues embedded in the continuous spectrum

It was believed in the beginning of Quantum Mechanics that there was no eigenvalue embedded in the continuous spectrum. However, the example by von Neumann and Wigner in 1929 was much remarkable. In fact, there is a simple example

$$-u''(x) + 2a\frac{\sin 2x}{x}u(x) = u(x),$$

which has an $L^2(0,\infty)$–eigenfunction, if $a > 1$ (see [EK], p.93). On the other hand, Kato [Kt1] prove that there is no positive eigenvalue of the Schrödinger operator $-\Delta + q(x)$ in $\mathbb{R}^n$ for *short-range* potentials

$$q(x) = o(|x|^{-1}) \quad (|x| \longrightarrow \infty).$$

The interesting result has been investigated by many authors and extended to the non-existence of $u(x) \in L^2(\mathbb{R}^n)$ such that

$$(-\Delta + q_1(x) + q_2(x))u(x) = \lambda u(x) \tag{9}$$

with $\lambda > 0$, a complex-valued short-range potential $q_1(x)$ and a real-valued *long range* potential $q_2(x)$ satisfying

$$q_2(x) = o(1), \quad \frac{\partial q_2}{\partial r} = o(|x|^{-1}) \quad (|x| \longrightarrow \infty).$$

Let us investigate the eigenvalue problem of Dirac operators

$$(\alpha \cdot p)\, u + m(x)\beta u(x) + q(x)u(x) = \lambda u(x) \quad (\lambda \in \mathbb{R}, \quad u \in \mathfrak{H}). \tag{10}$$

One way to consider the above equation is to multiply $\alpha \cdot p$ to (10). Then we have

$$-\Delta u - i\sum_{j=1}^{3}\left(\frac{\partial m}{\partial x_j}\right)\alpha_j \beta u - i\sum_{j=1}^{3}\left(\frac{\partial q}{\partial x_j}\right)\alpha_j u + m^2 u = (q-\lambda)^2 u.$$

Assuming

$$\begin{aligned}
&\lambda^2 > m_0^2 \ \ (m_0 \in \mathbb{R}), \quad q(x) = o(1), \quad m(x) - m_0 = o(1),\\
&|\nabla q| = o(|x|^{-1}), \quad |\nabla m| = o(|x|^{-1}) \ \ (|x| \longrightarrow \infty),
\end{aligned}$$

and applying a method of the result (9) and the unique continuation property, we have $u = 0$.

For the distinguished self-adjoint operator H with the Coulomb potential $q(x) = e/r$ ($|e| < 1$) and $m(x) \equiv 1$, Kalf [Kl] proves that there is no eigenvalue λ such that $|\lambda| > 1$ by showing the virial theorem.

The above method does not seem to be adequate for the diverging potentials at infinity. A non-existence theorem of eigenvalues without using any result of Schrödinger equations can be found in Kalf–Ōkaji–Yamada [KOY]. An application of our results gives the following

Proposition 4.1. *Let $u \in \mathfrak{H}$ satisfy*

$$(\alpha \cdot p)u + m(r)\beta u + q(r)u + Q(x)u = \lambda u \ \ (\lambda \in \mathbb{R}),$$

where $m(r)$, $q(r) \in C^2$ are scalar functions and $Q(x) \in C^1$ is a symmetric matrix-valued function satisfying

$$\begin{aligned}
&q(r) \longrightarrow \infty, \quad q'(r) = o\left(\frac{q^{3/2}(r)}{\sqrt{r}}\right), \quad q''(r) = o\left(\frac{q^2(r)}{r}\right)\\
&|m(r)| \le c|q(r)|, \quad |m'(r)| \le |q'(r)|, \quad m''(r) = o\left(\frac{q^2(r)}{r}\right)\\
&|Q(x)| = o\left(\frac{q^{1/2}(r)}{\sqrt{r}}\right), \quad \left|\frac{\partial Q}{\partial r}\right| = o\left(\frac{q(r)}{r}\right) \quad (r \longrightarrow \infty)
\end{aligned}$$

with some $c < 1$. Then we have $u = 0$.

5 Absolute continuity of the spectrum

Let H be a self-adjoint operator and

$$H = \int_{-\infty}^{\infty} \lambda \, dE(\lambda)$$

with the spectral spectral family $\{E(\lambda)\}$. Let $u \in \mathfrak{H}$, S a Borel set in $\mathbb{R}$, and define

$$m_u(S) = \|E(S)u\|^2,$$

which is a countably additive measure for Borel sets. If the measure m_u is absolutely continuous with respect to the Lebesgue measure (that is, $|S| = 0$ implies $E(S)u = 0$), u is absolutely continuous with respect to H. If there is a Borel set S_0 such that $|S_0| = 0$ and $m_u(\mathbb{R} \setminus S_0) = 0$, u is singular with respect to H. The set of all absolutely continuous (singular) u with respect to H is denoted by $\mathfrak{H}_{ac}$ ($\mathfrak{H}_s$) and is called absolutely continuous subspace (singular subspace). Then we have

$$\mathfrak{H} = \mathfrak{H}_{ac} \oplus \mathfrak{H}_s.$$

The spectrum of H restricted on $\mathfrak{H}_{ac}$ ($\mathfrak{H}_s$) is called the absolutely continuous (singular) spectrum of H (Kato [Kt2]).

Let $\mathfrak{H}_p$ be the closed linear hull of eigenvectors of H and define $\mathfrak{H}_c := \mathfrak{H}_p^{\perp}$. Then we have $\mathfrak{H}_{ac} \subset \mathfrak{H}_c$. The spectrum $\sigma_c(H)$ of H restricted on $\mathfrak{H}_c$ is called the continuous spectrum of H.

If $u \in \mathfrak{H}_c$, then we have RAGE theorem

$$\lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} \int_{|x|<R} |(e^{-itH}u)(x)|^2 \, dx \, dt = 0$$

for each $R > 0$. If $u \in \mathfrak{H}_{ac}$, then we have the local energy decay

$$\lim_{t \to \pm\infty} \int_{|x|<R} |(e^{-itH}u)(x)|^2 \, dx = 0$$

for each $R > 0$. (see, e.g., Weidmann [We4].)

In [Ym1] we consider the absolute continuity of the spectrum in the complement $[-1, 1]^c$

$$H = (\alpha \cdot p) + \beta + Q(x)$$

under the main assumption

$$|Q(x)| = O(|x|^{-1-\delta}) \quad (|x| \to \infty)$$

for some $\delta > 0$ (then $Q(x)$ is called a short-range potential). The absolute continuity is proved by showing that, for any $1 < a < b < \infty$ (or $-\infty < a < b < -1$) and $s > (1/2)$, there exists a positive constant C such that

$$\int_{\mathbb{R}^3} (1+|x|)^{-2s} |((H-z)^{-1}f)(x)|^2\, dx \le C \int_{\mathbb{R}^3} (1+|x|)^{2s} |f(x)|^2\, dx \qquad (11)$$

for any $a \le \operatorname{Re} z \le b$, $0 < |\operatorname{Im} z| \le 1$, which is originally due to Agmon's estimate (Agmon [Ag1], [Ag2]). The resolvent estimate (11) is often called the limiting absorption principle. The above inequality implies

$$|(E(\beta) - E(\alpha))f, f)| \le C(\beta - \alpha) \int_{\mathbb{R}^3} (1+|x|)^{2s} |f(x)|^2\, dx$$

for any $a \le \alpha \le \beta \le b$, and the absolute continuity of the spectrum in $[-1, 1]^c$. In [Ym2] we investigate the eigenfunction expansion theorem under the same condition.

For the Coulomb potential $Q = e/r$ the problem reduces to the ordinary differential problem, which is solved e.g. by Weidmann [We3].

For long-range potentials

$$|Q(x)| + |x||\nabla Q(x)| = O(|x|^{-\delta}) \quad \text{as} \quad |x| \to \infty$$

for some $\delta > 0$ there are also many extensive works on the absolute continuity in $[-1, 1]^c$ (see e.g., Gâtel-Yafaev [GY], Iwashita [Iw], Yokoyama [Yo] and their references), where The Mourre commutator method (see, e.g., [CFKS], Chapter 4) turns out to be applicable to Dirac operators as well as to Schrödinger operators. The limiting absorption method is also applicable to the long-range case (Ikebe-Saitō [IS] and Vogelsang [V2]).

Let us consider the case

$$q = q(r) \to \infty \quad (r \to \infty), \qquad \limsup_{r\to\infty} \frac{|m(r)|}{q(r)} < 1.$$

Assume that

$$\begin{array}{ll} (1) & q,\, m \in AC_{loc}([0,\infty)), \\ (2) & \liminf_{r\to\infty} m(r) > 0, \\ (3) & \dfrac{m'}{q},\ \dfrac{mq'}{q^2} \in L^1((0,\infty)) \end{array}$$

For $H = \alpha \cdot p + m(r)\beta + q(r)$ we have

$$\sigma_{ac}(H) = \mathbb{R}, \qquad \sigma_s(H) = \emptyset.$$

(Schmidt-Yamada [SY]). This proposition can be proved along the line of Gilbert-Pearson Theory by showing that every solution of the eigenvalue problem

$$\begin{pmatrix} m(r) + q(r) & -(d/dr) - (k/r) \\ (d/dr) - (k/r) & -m(r) + q(r) \end{pmatrix} \begin{pmatrix} \varphi_1(r) \\ \varphi_2(r) \end{pmatrix} = \lambda \begin{pmatrix} \varphi_1(r) \\ \varphi_2(r) \end{pmatrix}.$$

is always bounded at infinity.

6 Non-relativistic limit of Dirac operators

Let us consider the unperturbed Dirac operator

$$H_0(c) = c(\alpha \cdot p) + mc^2\beta.$$

Then we have formally

$$H_0(c) = c\sqrt{p^2 + m^2c^2}\, P_+(c) - c\sqrt{p^2 + m^2c^2}\, P_-(c),$$

where $P_\pm(c)$ are orthogonal projections such that

$$\begin{aligned} P_\pm(c) &= \frac{1}{2}\left(I_4 \pm \frac{(\alpha \cdot p) + mc\beta}{\sqrt{p^2 + m^2c^2}}\right), \\ P_\pm(c) &\to P_\pm(\infty) := \frac{1}{2}(I \pm \beta), \quad c \to \infty. \end{aligned}$$

A simple calculation

$$\begin{aligned} &(H_0(c) \mp mc^2)P_\pm(c) = \pm c(\sqrt{p^2 + m^2c^2} - mc)P_\pm(c) \\ &= \pm\frac{c^2p^2}{c\sqrt{p^2 + m^2c^2} + mc^2}P_\pm(c) \to \pm\frac{p^2}{2m}P_\pm(\infty) \ \text{ strongly as } \ c \to \infty, \end{aligned}$$

which suggests

$$e^{\mp imc^2}e^{itH_0(c)}P_\pm(c) \to e^{\pm it(-\Delta)/(2m)}P_\pm(\infty) \quad (c \to \infty)$$

for any $t \in \mathbb{R}$. Such a result can be extended to Dirac operators

$$H(c) = H_0(c) + q(x)$$

with the projection $Q_+(c)$ ($Q_-(c)$) of the positive (negative) part of $H(c)$ and

$$S_\pm = \mp\frac{1}{2m}\Delta + q(x)$$

to get

$$e^{\mp imc^2}e^{itH(c)}Q_\pm(c) \to e^{itS_\pm}P_\pm(\infty) \ \text{ strongly as } \ c \to \infty$$

for any $t \in \mathbb{R}$ (Cirincione–Chernoff [CC]). These facts imply a relation of wave operators

$$\begin{aligned} &\Omega^\pm(c) := \lim_{t\to\pm\infty} e^{itH(c)}e^{-itH_0(c)}, \\ &W_+^\pm := \lim_{t\to\pm\infty} e^{itS_+}e^{-it(-\Delta)/(2m)}, \quad W_-^\pm := \lim_{t\to\pm\infty} e^{itS_-}e^{-it(-\Delta)/(2m)} \end{aligned}$$

such that

$$\lim_{c\to\infty} \Omega^{\pm}(c) = \begin{pmatrix} W_+^{\pm} & & & \\ & W_+^{\pm} & & \\ & & W_-^{\pm} & \\ & & & W_-^{\pm} \end{pmatrix}$$

for short-range potentials $q(x)$ by Veselić-Weidmann [VW] and Yajima [Yj] (see also [Ym2] for the long-range case).

Finally, we consider the case $q(x) \to +\infty$. In this case, we have

$$\sigma(H(c)) = \sigma_{\mathrm{c}}(H(c)) = \mathbb{R},$$
$$\sigma(S_+) = \sigma_{\mathrm{d}}(S_+) = \{\lambda_1 < \lambda_2 < \cdots < \lambda_{n-1} < \lambda_n \to +\infty\},$$

under a mild condition. where

$$H(c) = c(\alpha \cdot p) + mc^2\beta + q(x), \quad S_+ = \frac{1}{2m}(-\Delta) + q(x).$$

Let

$$H(c) = \int_{-\infty}^{\infty} \lambda \, dE_c(\lambda), \quad S_+ = \int_{-\infty}^{\infty} \lambda \, dE_+(\lambda)$$

with the spectral family $\{E_c(\lambda)\}$, $\{E_+(\lambda)\}$ and an interval $I = [a, b]$ contain only λ_N in $\sigma_{\mathrm{d}}(S_+)$, and put

$$I_c := [a + mc^2, b + mc^2], \quad J_c := \left[\lambda_N + mc^2 - \frac{1}{c^\tau}, \lambda_N + mc^2 + \frac{1}{c^\tau}\right]$$

Then Veselić [Ve] shows the spectral concentration

$$E_c(I_c \setminus J_c)P_+(\infty) \to 0,$$
$$E_c(J_c)P_+(\infty) \to E_+(\{\lambda_n\})P_+(\infty)$$

strongly in $\mathfrak{H}$ as $c \to \infty$, by the phase space analysis. His condition on the Fourier transform of $q(x)$ can be weakened by [ItoY1] so that we need not assume the polynomial growth of $q(x)$ at infinity.

Amour-Brummelhuis-Nourrigat [ABN] and Ito-Yamada [ItoY2] explain the concentration property from the viewpoint of resonances of Dirac operators under the polynomial growth of $q(x)$. The conditions on $q(x)$ are similar each other. The latter contains some refined results. Although $q(x)$ is not assumed necessarily to be spherically symmetric, the readers may consider below a simple case $q(x) = |x|^\gamma$ $(\gamma > 0)$. Let

$$H(c) = c(\alpha \cdot p) + mc^2\beta + q(x),$$
$$S = -\frac{1}{2m}\Delta + q(x),$$
$$\tilde{S} = \frac{1}{2m}\Delta + q(x).$$

The <u>resonances</u> are defined as the eigenvalues of the dilated operators

$$\begin{aligned} H(c,\theta) &= ce^{-\theta}(\alpha\cdot p) + mc^2\beta + q(e^\theta x), \\ S(\theta) &= -\frac{e^{-2\theta}}{2m}\Delta + q(e^\theta x), \\ \tilde{S}(\theta) &= \frac{e^{-2\theta}}{2m}\Delta + q(e^\theta x). \end{aligned}$$

for $\theta \in \mathbb{C}$. The closed operator $H(c,\theta)$, $S(\theta)$ and $\tilde{S}(\theta)$ are purely discrete for $0 < \operatorname{Im}\theta < a$ ($\exists\, a > 0$), and the isolated eigenvalues (with finite multiplicity) are independent of θ. Such eigenvalues are called <u>resonances</u> of $H(c)$, S and $\tilde{S}$.

$S(\theta)$ can be extended to the lower half plane $-a < \operatorname{Im}\theta \le 0$, and the resonances coincide with all the real eigenvalues of S

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots .$$

Even if $\tilde{S}$ is not essentially self-adjoint, the closed operator $\tilde{S}(\theta)$ can be meaningful and the resonances consist of

$$-e^{-i\beta_0(\gamma)}\lambda_1, \quad -e^{-i\beta_0(\gamma)}\lambda_2, \quad \cdots \quad -e^{-i\beta_0(\gamma)}\lambda_n, \quad \cdots ,$$

where $\beta_0(\gamma) = 2\pi(2+\gamma)^{-1}$, that is, $\beta_0(+0) = \pi$, $\beta_0(2) = \pi/2$, $\beta_0(+\infty) = 0$, if $q(x) = |x|^\gamma$.

Roughly speaking, we can show that the resonances of $H(c) - mc^2$ concentrate into the eigenvalues λ_n of S, and the resonances of $H(c) + mc^2$ concentrate into the resonances $-e^{-i\beta_0(\gamma)}\lambda_n$ of $\tilde{S}$, including their multiplicities as $c \to \infty$ (Ito–Yamada [ItoY3]).

We can also show that, if $H(c)$ has no real eigenvalues, $H(c)$ is purely absolutely continuous.

I am also interested in other topics of Dirac operators, the strong unique continuation property and the spectral theory Dirac operators in the Kerr-Newmann metric.

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