WEYL ALTERNATIVE, SEPARATION OF VARIABLES AND SELFADJOINTNESS OF THE DIRAC OPERATOR

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Part I

Mathematical tools

Adjointable, Symmetric and Self Adjoint operators

$\mathcal{H}$: complex separable Hilbert space, with sesquilinear product $(|)$;
$A \equiv (A, D_A)$ linear operator in $O(\mathcal{H})$ if

- D_A is a linear subset of $\mathcal{H}$,
- $A : D_A \to \mathcal{H}$ is a linear map

A is adjointable if densely defined: $\bar{D}_A = \mathcal{H}$.
Set $D_{A^\dagger} = \{x \in \mathcal{H} | D_A \to \mathbb{C}, z \mapsto (x|Az) \text{ is continuous } \}$.
Riesz theorem: $\exists! y \; s.t. \; (y|z) = (x|Az)$. Then $A^\dagger$ on $D_{A^\dagger}$ is defined by $y =: A^\dagger x$.
A is symmetric if $A \subseteq A^\dagger$ ($D_A \subseteq D_{A^\dagger}$ and $A^\dagger x = Ax \; \forall x \in D_A$).
A is selfadjoint if $A = A^\dagger$.

CLOSABLE OPERATORS AND CLOSURE

The graphic of $A \in O(\mathcal{H})$ is the set $\mathcal{G}_A = \{(x, y) \in \mathcal{H} \times \mathcal{H} | x \in D_A, y = Ax\}$.

EXERCISE: $\mathcal{G}$ is the graphic of an operator iff it is a linear subset of $\mathcal{H} \times \mathcal{H}$ and $(0, y) \in \mathcal{G} \Leftrightarrow y = 0$.

A is closed if G_A is closed. A is closable if $\bar{G}_A$ is a graphic; then $\bar{G}_A =: G_{\bar{A}}$ defines $\bar{A}$, the closure of A.

EXERCISE: A is closable iff for any sequence $\{x_n\} \subset D_A$, $x_n \to 0 \ \Rightarrow \ Ax_n \to 0$.

A is essentially selfadjoint if A is closable and $\bar{A} = \bar{A}^\dagger$

EXERCISE: If A is closed then ImA is closed.

MINIMAL REDUCED DIRAC OPERATORS

On $\mathcal{H} := \mathcal{L}_2((a,b);w)^2$, where $-\infty \leq a < b \leq +\infty$, w is a positive function on (a,b), and

$$(f|g) = \int_a^b w(x)(f(x),g(x))dx.$$

Here $(,)$ denotes the usual scalar product in $\mathbb{C}^2$.
Let us define the minimal reduced Dirac operator A_{V0}

$$A_{V0} : D_0 \longrightarrow \mathcal{H},\ f \mapsto \frac{1}{w}(\Omega\frac{df}{dx} + Vf) \equiv \hat{D}f,$$
$$D_0 = \{f \in \mathcal{AC}_c((a,b),\mathbb{C}^2)|A_{V0}(f) \in \mathcal{H}\}.$$

Here $\mathcal{AC}_c$ means absolutely continuous with compact support, whereas V is a 2×2 Hermitian valued function on (a,b), and

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Further assumptions:

1. w and V are measurable on (a, b)
2. w is positive a.e. in (a, b)
3. |w| and |V| are locally integrable

We will say that A_{V0} is regular in a (in b) if $a > -\infty$ ($b < \infty$) and such conditions are satisfied in $[a, b)$ (in $(a, b]$). A_{V0} is regular if it is regular both in a and b.
We say that f is locally $\mathcal{L}_2((a, b); w)^2$ (write $\mathcal{L}_2((a, b); w)^2_{loc}$) if $f|_{(x,y)} \in \mathcal{L}_2((x, y); w)^2$ for any $[x, y] \subset (a, b)$.

GREEN'S FORMULA: for $f, g \in \mathcal{L}_2((a, b); w)^2_{loc}$ set

$$[f, g]_x^y := {}^t\overline{f}(x)\Omega g(x) - {}^t\overline{f}(y)\Omega g(y).$$

Then the following formula holds

$$\int_x^y [(\hat{D}f, g)(s) - (f, \hat{D}g)(s)]w(s)ds = [f, g]_x^y.$$

Maximal reduced Dirac operators

Set

$$D := \{f \in \mathcal{H} | f \in \mathcal{AC}, \hat{D}f \in \mathcal{H}\}.$$

Then, we define on D the maximal reduced Dirac operator A_V as

$$A_V : D \longrightarrow \mathcal{H},\ f \mapsto A_V f := \hat{D}f.$$

Prop. (W1, Th. 3.1)

If $f \in D_0$ and $g \in D$ then $(A_{V0}f|g) = (f|A_V g)$.

Proof: if $f \in D_0$ let be $\mathrm{supp}(f) \in [x, y] \subset (a, b)$. Then we can apply the Green's formula. □

This means that A_{V0} is Hermitian but not yet that it is symmetric. To this end we need to show that D_0 is dense in $\mathcal{H}$. This will be easily done for regular operators. Strategy: if $[x, y] \subset (a, b)$, $\hat{D}|_{[x,y]}$ is regular.

Regular reduced Dirac operator

Let us consider the case $\hat{D}$ is regular.
Define $D_1 = \{d \in D | f(a) = f(b) = 0\}$. This define the operator $A_{V1} = A_v|_{D_1}$:

$$A_{V0} \subseteq A_{V1} \subseteq A_V.$$

Prop. (W1, Th. 3.3)

If $\hat{D}$ is regular and $f \in D_1$, $g \in D$ then $(A_{V1} f | g) = (f | A_V g)$.

Again, this means that A_{V1} is Hermitian.

We now need some technical results.

PROP. (W1, TH. 3.4)

If $\hat{D}$ is regular and $\lambda \in \mathbb{C}$ then:
a) $\mathrm{Im}(A_{V1} - \lambda) = \mathrm{Ker}(A_V - \bar{\lambda})^{\perp}$;
b) $\mathrm{Im}(A_{V1} - \lambda)^{\perp} = \mathrm{Ker}(A_V - \bar{\lambda})$.

PROOF: a) Assume $f \in \mathrm{Im}(A_{V1} - \lambda)$ and let h be the unique solution of the Cauchy problem $(\hat{D} - \lambda)h = f$, $h(a) = 0$. Moreover, let g_1, g_2 be the solution of $(\hat{D} - \bar{\lambda})g = 0$ with Cauchy condition $g_1(b) = \binom{1}{0}$ and $g_2(b) = -\binom{0}{1}$ respectively. Then

$$(f|g_i) = ((\hat{D} - \lambda)h|g_i) = ((\hat{D} - \lambda)h|g_i) - (h|(\hat{D} - \bar{\lambda})g_i) = [h, g_i]_a^b = \bar{h}_{(i)}(b),$$

where $h_{(i)}$, $i = 1, 2$ is the i-th component of the $\mathbb{C}$ vector. Then $h \in D_1 \Leftrightarrow f \perp g_i$. This proves a).
b) as $\dim \mathrm{Ker}(A_V - \bar{\lambda}) = 2 < \infty$ we have a) $\Rightarrow$ b). $\square$

PROP. (W1, TH. 3.5)

Let $\hat{D}$ be regular and $\alpha, \beta \in \mathbb{C}^2$. Then $\exists g \in D$ (not unique) s.t. $g(a) = \alpha$ and $g(b) = \beta$.

PROOF: Take g_1, g_2 solutions of the Cauchy problems $\hat{D}g_i = 0$, $i = 1, 2$ and $g_1(b) = -\binom{0}{1}$, $g_2(b) = \binom{1}{0}$ respectively. Then $g_i \in D$ and any $f \in \mathrm{Ker} A_V$ takes the form $f = \mu g_1 + \nu g_2$. In particular we can choose f s.t. $(f|g_i) = \bar{\beta}_{(i)}$. Take h as the unique solution of $\hat{D}h = f$, $h(a) = 0$. Then:

$$\bar{\beta}_{(i)} = (f|g_i) = (\hat{D}h|g_i) = (\hat{D}h|g_i) - (h|\hat{D}g_i) = [h, g_i]_a^b = \bar{h}(b)_{(i)}$$

so that $h(b) = \beta$.
In the same way construct $k \in D$ such that $k(a) = \alpha$, $k(b) = 0$.
Then $g = h + k$. □

PROP. (W1, TH. 3.6)

Let $\hat{D}$ be regular. Then a) A_{V1} is symmetric, b) $A_V = A_{V1}^\dagger$, c) $A_{V1} = A_V^\dagger$. In particular A_{V1} is closed.

PROOF: a) First we have to prove that D_1 is dense in $\mathcal{H}$. Let $f \perp D_1$ and g a solution of $\hat{D}g = f$, so that $f \in D$. $\forall h \in D_1$ we have

$$(g|A_{V1}h) = (A_V g|h) = (f|h) = 0. \qquad (1)$$

Then $g \perp \mathrm{Im}(A_{V1}) \Rightarrow g \in \mathrm{Ker}(A_V)$ and then $f = 0$.

b) Obviously $A_V \subseteq A_{V1}^\dagger$. Viceversa take $f \in D(A_{V1}^\dagger)$. Set $h = A_{V1}^\dagger f$ and solve $\hat{D}g = h$. Then $g \in D(T)$. If $k \in D_1$ then

$$(f - g|A_{V1}k) = (A_V(f - g)|k) = 0$$

so that $f - g \in (\mathrm{Im}(A_{V1}))^\perp = \mathrm{Ker}(A_V) \subset D$. Then $f \in D$.

c) From b) we have ${A_{V1}^{\dagger}}^{\dagger} = A_V^{\dagger}$. By construction $A_{V1} \subseteq {A_{V1}^{\dagger}}^{\dagger}$, so that $A_{V1} \subseteq A_V^{\dagger}$.
Now, $A_{V1} \subset A_V \Rightarrow A_V^{\dagger} \subset A_{V1}^{\dagger} = A_V$. Take $f \in D(A_V^{\dagger})$. The $A_V^{\dagger} f = A_V f$ and

$$(A_V f|g) = (A_V^{\dagger} f|g) = (f|A_V g) \qquad \forall\, g \in D.$$

Then

$$0 = (A_V f|g) - (f|A_V g) = [f, g]_a^b \qquad \forall\, g \in D.$$

In particular, choosing $g_i \in D$ s. t, $g_i(a) = 0$ and $g_1(b) = -\binom{0}{1}$, $g_2(b) = \binom{1}{0}$, we get $f(b) = 0$. In a similar way $f(a) = 0$. Then $f \in D_1$ and $A_{V1} = A_V^{\dagger}$.
The closure of A_{V1} follows from the next exercise. $\square$

THE GENERAL CASE

EXERCISE: Shew that a densely defined operator A is closable iff $A^\dagger$ is adjointable and then $\bar{A} = A^{\dagger\dagger}$.

PROP. (W1, TH. 3.7)

Let $\hat{D}$ arbitrary. Then A_{V0} is symmetric and $A_{V0}^\dagger \subseteq A_V$.

PROOF: For any $I \equiv [x, y] \subset (a, b)$ define $A_{V1,I}$ as $\hat{D}$ on the domain $D_{1,I} = \{f \in D | f(x) = 0 \; \forall \, x \in (a, x] \cup [y, b)\}$. Then $\bar{D}_{1,I} = \mathcal{L}_2(I; w)^2$. The density of D_0 follows from $D_0 = \cup_I D_{1,I}$.
Assume $f \in D(A_{V0}^\dagger)$ and $g \in D_{1,I}$. Then $(A_{V0}^\dagger f | g) = (f | A_{V0} g) = (f | A_{V1,I} g)$ so that

$$f|_I \in D(A_{V1,I}^\dagger)|_I = D_I := \{f \in \mathcal{L}_2(I; w)^2 | f \in \mathcal{AC}, \; \hat{D}f \in \mathcal{L}_2(I; w)^2\}, \; \Rightarrow$$

$$(A_{V0}^\dagger f)_I = (\hat{D}f)|_I \; \Rightarrow \hat{D}f = A_{V0}^\dagger f \in \mathcal{H}.$$

□

EXERCISE: Shew that if $\hat{D}$ is regular then $A_{V1} = \bar{A}_{V0} = {A_{V0}^{\dagger}}^{\dagger}$.

In the general case $A_{V1} := \bar{A}_{V0} = {A_{V0}^{\dagger}}^{\dagger}$.

PROP. (W1, TH. 3.9)

$A_{V0}^{\dagger} = A_{V1}^{\dagger} = A_V$.

PROOF: The first identity is obvious. Also we know that $A_{V0}^{\dagger} \subseteq A_V$. On the other hand from $(A_{V0}f|g) = (f|A_V g)\ \forall f \in D_0$, $g \in D$ and the density of D_0 it follows $g \in D(A_{V0}^{\dagger})$. □

Technical tools

The Wronskian of two solutions f,h of $\hat{D}g = \lambda g$ is

$$W(f,h;x) := \det \begin{pmatrix} f_{(1)}(x) & h_{(1)}(x) \\ f_{(2)}(x) & h_{(2)}(x) \end{pmatrix} = [f,gx.$$

NOTE: f and h determine a fundamental system for $\hat{D}g = \lambda g$ iff $W(f,h;x) \neq 0$ for some $x \in (a,b)$.

PROP. (W1, TH. 5.2)

Assume h,k fundamental system of $\hat{D}g = \lambda g$ and assume $|wf|$ is locally integrable in (a,b). Then all solutions of $(\hat{D} - \lambda)g = f$ have the form

$$\begin{aligned} g(x) &= a(x)h(x) + b(x)k(x), \\ a(x) &= a_0 - \int_c^x W(h,k;s)^{-1}(\bar{k}(s),f(s))w(s)ds \\ b(x) &= b_0 + \int_c^x W(h,k;s)^{-1}(\bar{h}(s),f(s))w(s)ds, \qquad c \in (a,b). \end{aligned}$$

PROOF. Variation of constant:
$g(x) = a(x)h(x) + b(x)k(x)$ and $(\hat{D} - \lambda)g = f$ imply

$$\begin{pmatrix} h_1 & k_1 \\ h_2 & k_2 \end{pmatrix} \begin{pmatrix} da/dx \\ db/dx \end{pmatrix} = -w\Omega f,$$

so that

$$\begin{pmatrix} da/dx \\ db/dx \end{pmatrix} = \frac{w(x)}{W(h,k;x)} \begin{pmatrix} -k_1 & -k_2 \\ h_1 & h_2 \end{pmatrix} f.$$

□

PROP.

If h and k are a fundamental system for $(\hat{D}-\lambda)g=0$, *then* $y(x)=\bar{h}(x)/\bar{W}(h,k;x)$ *and* $\chi(x)=\bar{k}(x)/\bar{W}(h,k;x)$ *solve* $(\hat{D}-\bar{\lambda})g=0$.

PROOF: $[\Omega\partial_x+V(x)-w(x)\lambda]h(x)=0 \Rightarrow [\Omega\partial_x+\bar{V}(x)-w(x)\bar{\lambda}]\bar{h}(x)=0$.
Then $dW(h,k;x)/dx=-(\bar{k}(x),(V(x)-\bar{V}(x))h(x))$ gives

$$\begin{aligned} w(x)(\hat{D}-\bar{\lambda})\frac{\bar{h}(x)}{\bar{W}(h,k;x)} &= -\frac{\Omega}{\bar{W}(h,k;x)^2}\bar{h}(x)(k(x),(V(x)-\bar{V}(x))\bar{h}(x)) \\ &+(V(x)-\bar{V}(x))\frac{\bar{h}(x)}{\bar{W}(h,k;x)} =: z(x). \end{aligned}$$

Ω and $V(x)-\bar{V}(x)$ antisymmetric imply $(h(x),z(x))=0$. By definition of W $(k(x),z)=0$. Then $z(x)\in\{h(x),k(x)\}^{\perp}=0$. □

Prop. (W1, Th. 5.3)

Assume $\exists\ \lambda_0 \in \mathbb{C}$ s.t. all solutions of $(\hat{D} - \lambda_0)g = 0$ and $(\hat{D} - \bar{\lambda}_0)g = 0$ lie right in $\mathcal{L}_2((a,b);w)^2$. Then it holds for every $\lambda \in \mathbb{C}$.

Proof: Write $(\hat{D} - \lambda)g = 0$ as $(\hat{D} - \lambda_0)g = (\lambda - \lambda_0)g$. Choose a fundamental system h, k for $(\hat{D} - \lambda_0)g = 0$. Then

$$\begin{aligned} g(x) = a_0 h(x) + b_0 k(x) &- (\lambda - \lambda_0)k(x) \int_c^x w(s)(\chi(s), g(s))ds \\ &+ (\lambda - \lambda_0)h(x) \int_c^x w(s)(y(s), g(s))ds. \end{aligned}$$

By hyp. $y, \chi \in \mathcal{L}_2((c,b);w)^2$. Set

$$M := 2|\lambda - \lambda_0|^2 [\int_c^b (|y(s)|^2 + |\chi(s)|^2) w(s) ds], \qquad A = \text{Max}\{|a_0|, |b_0|\}.$$

Using Cauchy and some manipulations we get

$$|g(x)|^2 \leq 2A^2 (|h(x)| + |k(x)|)^2 + M(|h(x)| + |k(x)|)^2 \int_c^x |g(s)|^2 w(s) ds.$$

As $|h(x)| + |k(x)| \in \mathcal{L}_2((c,b),w)^2$ it exists $d \in (c,b)$ such that

$$\int_d^b (|h(x)| + |k(x)|)^2 w(x)dx \leq \frac{1}{2M}$$

so that

$$\int_d^z |g(x)|^2 w(x)dx \leq 2A^2 \int_d^b (|h(x)| + |k(x)|)^2 w(x)dx + \frac{1}{2} \int_c^z |g(x)|^2 w(x)dx.$$

As this is true for any $z > d \Rightarrow g \in \mathcal{L}_2((d,b);w)^2$. □

Deficiency indices

Define the deficiency indices $m^{\pm} = \dim K^{\pm}$ where

$$K^{\pm} = \mathrm{Ker}(A_V \mp i) = \mathrm{Im}(A_{V1} \pm i)^{\perp}.$$

Lemma (von Neumann I; W2, Th. 8.12)

$D = D_1 \oplus K^+ \oplus K^-$.

Proof: Assume $g \in D$. As A_{V1} is closed $\mathrm{Im}(A_{V1} + i)$ is closed and we can write $\mathrm{Im}(A_V + i) = \mathrm{Im}(A_{V1} + i) \oplus \mathrm{Im}(A_V + i) \cap (\mathrm{Im}(A_{V1} + i))^{\perp}$. Then $(A_{V1} + i)g_0 + g_1$, $g_1 \in (\mathrm{Im}(A_{V1} + i))^{\perp}$. Set $g_+ = ig/2$. Then one easily sees that $g_- := g - g_0 - g_+ \in K^-$. Then $g = g_0 + g_- + g_+$.
Finally we see that $g = 0 \Rightarrow g_0 = g_+ = g_-$. Indeed,

$$0 = g = A^{\dagger}_{V1} g_0 + ig_+ - ig_-.$$

Then $(A_{V1} - i)g_0 = 2ig_-$. But $g_- \in (\mathrm{Im}(A_{V1} - i))^{\perp}$ then $g_- = 0$. Similarly $g_+ = 0$ and then $g_0 = 0$. □

Define the right deficiency indices

$$m_b^{\pm} := \dim\{g \in \mathrm{Ker}(\hat{D} \mp i) \mid g \in \mathcal{L}_2((a,b);w)^2_{right}\}.$$

LEMMA

$m_b^+ + m_b^- \geq 2.$

PROOF: Let $\hat{D}_c$ and $\hat{D}_{c0}$ the maximal and the closed minimal operators associated to $\hat{D}$ in $\mathcal{L}_2((c,b);w)^2$, $a < c < b$. $\hat{D}$ is regular in $[c,d]$, then $\exists$ $h,k \in \hat{D}_0$ such that $h(x) = k(x) = 0$ for $x > d$ and $h(c) = \vec{v}_1$, $k(c) = \vec{v}_2$ are a basis for $\mathbb{C}^2$. Thus follows

$$D(\hat{D}_{c0}) + L(h,k)_{\mathbb{C}} \subset D(\hat{D}_c)$$

so that $\dim(D(\hat{D}_c)/D(\hat{D}_{c0})) \geq 2$. But from von Neumann I, $m_b^+ + m_b^- = \dim(D(\hat{D}_c)/D(\hat{D}_{c0}))$. □

Weyl's alternative

Prop. (Weyl's alternative)

Suppose $\hat{D}$ real ($\bar{V} = V$), and consider the equation $(\hat{D} - \lambda)g = 0$. Then either

1. *$\forall\lambda \in \mathbb{C}$ all solutions lie right in $\mathcal{L}_2((a,b);w)$ (limit circle case LCC) or*
2. *$\forall\lambda \in \mathbb{C}\backslash\mathbb{R}$ $\exists!$, up to a multiplicative constant, solution which lies right in $\mathcal{L}_2((a,b);w)$ (limit point case LPC)*

Proof: $\hat{D}$ is real $\Rightarrow$ if g is a solution for a given λ, than $\bar{g}$ is for λ. Then, as $\dim \mathrm{Ker}(\hat{D} - \lambda) = 2$ it suffices to show that if $\lambda \in \mathbb{C}\backslash\mathbb{R}$ then exists at least a a solution. As $\hat{D}$ is real then $m_b^+ = m_b^- \geq 1$. □

Obviously similar results hold true in a.

PROP.

If A is a closed symmetric operator and $\mathrm{Im}(A \pm i) = \mathcal{H}$, then A is selfadjoint.

PROOF: Take $y \in D(A^\dagger)$. By hypothesis $\exists\, x_\pm \in D(A)$ s.t. $(A^\dagger \pm i)y = (A \pm i)x_\pm$. As $A = A^\dagger|_{D(A)}$ we have $(A^\dagger \pm i)(y - x_\pm) = 0$. But $\mathrm{Ker}(A^\dagger \pm i) = (\mathrm{Im}(A^\dagger \pm i))^\perp = 0$. Then $y = x_\pm \in D(A)$. Then $A = A^\dagger$. □

PROP. (W1, TH. 5.8(I))

Let $\hat{D}$ be real and $\lambda \in \mathbb{R}$. If $\hat{D}$ is LPC at both a and b then $A_V = A_{V1}$ is the only selfadjoint extension of A_{V1}.

PROOF: Using the same methods as in the last Lemma, it is easy to show that $m^\pm = m_a^\pm + m_b^\pm - 2$. As $\hat{D}$ is real, and is LPC at both hands, then $m^+ = m^- = 0$. Then $\mathrm{Im}(A_{V1} \pm i) = \mathcal{H}$ and A_{V1} is selfadjoint. □

PROP. (W1, TH. 6.8)

Let $\hat{D}$ be real. If the non vanishing constant functions 1 do not lie right in $\mathcal{L}_1((a,b);w)$, then $\hat{D}$ is LPC at b.

PROOF: Take a fundamental system h, k for $\hat{D}g = 0$ and set $W := |W(h,k;x)|$. It is easy to check that W is constant (as $\hat{D}$ is real). Then

$$Ww(x) \leq w(x)(|h_1 k_2| + |h_2 k_1|) \leq w(x)|h(x)||k(x)| \leq \frac{1}{2} w(x)(|h(x)|^2 + |k(x)|^2).$$

Then, if w does not lie right in $\mathcal{L}_1((a,b);w)$, h and k cannot lie both right in $\mathcal{L}_1((a,b);w)$. □

Physical applications

Dirac operator on a spacetime manifold

$M, \mathbf{g}$ four dimensional manifold with Lorentzian metric
$\mathbf{g} = g_{\mu\nu} dx^\mu \otimes dx^\nu = \eta_{ab} e^a \otimes e^b$, $\eta = \mathrm{diag}\{-1, 1, 1, 1\}$.
Flat Dirac matrices: Γ^a, $a = 0, 1, 2, 3$ and $\Gamma_a = \eta_{ab}\Gamma^b$ satisfy
$\{\Gamma_a, \Gamma_b\} = -2\eta_{ab}$.
Curved Dirac matrices: $\gamma_\mu = e^a_\mu \Gamma_a \Rightarrow \{\gamma_\mu, \gamma_\nu\} = -2g_{\mu\nu}$.
$S \longrightarrow M$ principal spin bundle with infinitesimal Levi-Civita connection ω,
$P \longrightarrow M$ principal $U(1)$ bundle with infinitesimal connection A.
In local gauge $S \otimes P \longrightarrow M$ the Dirac equation for a field ψ having charge q and mass μ is

$$\left[-i\hbar\gamma^\mu \left(\partial_\mu + \frac{1}{4}\omega^{ab}_\mu \Gamma_a \Gamma_b + i\frac{q}{c} A_\mu\right) + \mu c^2\right] \psi = 0.$$

TIME EVOLUTION: $i\hbar\partial_t\psi = \hat{H}\psi$, w.r.t. a chosen foliation Σ_t of M. Unitary evolution requires $\hat{H}$ Hermitian operator.
If $M, \mathbf{g}$ is a stationary spacetime then $\hat{H}$ is time independent.

PROBLEM: consider selfadjointness of $\hat{H}$ in $\mathcal{H} = \mathcal{L}_2(\vec{x}, \mu)^4$ where μ induced by

$$(\psi|\phi) = \int_\Sigma \sqrt{-g}\,(\psi, \Gamma^0\gamma^t\phi)dx^3.$$

where $(,)$ is the usual sesquilinear product in $\mathbb{C}^4$.

STRATEGY: Exploit variable separation to reduce the problem to the one of a reduced Dirac operator.

DIRAC OPERATOR ON REISSNER-NORDSTRÖM ADS

Reissner-Nordström anti de Sitter black hole manifold ($c = 1,\ \hbar = 1$)

$$\mathbf{g} = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2,$$

$$f(r) = 1 + \frac{r^2}{L^2} - \frac{2M}{r} + \frac{Q^2}{r^2}; \qquad t \in \mathbb{R},\ r \in (r_+, \infty),\ \Omega \in S^2,$$

$$A_\mu = \delta^t_\mu \frac{Q}{r}.$$

r_+ is the larger root of $f(r)$.
Gamma matrices

$$\Gamma^0 = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix}, \qquad \vec{\Gamma} = \begin{pmatrix} \mathbb{O} & \vec{\sigma} \\ -\vec{\sigma} & \mathbb{O} \end{pmatrix},$$

where $\vec{\sigma}$ are the Pauli matrices. Choose the vierbein
$e^0 = \sqrt{f}\, dt,\ e^1 = r\, d\theta,\ e^2 = dr/\sqrt{f},\ e^3 = r \sin\theta d\phi.$

Set $D_\mu := \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\Gamma_a\Gamma_b$. Then

$$\begin{aligned} \hat{H} &= -i\frac{1}{4}\omega_t^{ab}\Gamma_a\Gamma_b + \frac{qQ}{r} - if(r)\Gamma^0\Gamma^2 D_r + \frac{\sqrt{f(r)}}{r}\Gamma^0\Gamma^1\hat{L} \\ &\quad + \Gamma^0\sqrt{f(r)}\mu c^2, \\ \hat{L} &:= -i(D_\theta + i\Gamma^2 \sin\theta D_\phi). \end{aligned}$$

The Hilbert space is
$\mathcal{H} = \mathcal{L}_2((r_+,\infty)\times(0,\pi)\times(0,2\pi); \sin^2\theta r^2 f(r)^{-\frac{1}{2}} dr\, d\theta\, d\phi)$.

EXERCISE: Let compute the spin connection coefficients ω_μ^{ab}. Recall

$$de^a + \omega^{ab} \wedge \eta_{ac}e^c = 0, \qquad \omega^{ab} + \omega^{ba} = 0.$$

Separation of variables

We want to look at essential selfadjointness of $\hat{H}$ on the domain $D_c = \mathcal{C}_c^\infty((r_+, \infty) \times (0, \pi) \times (0, 2\pi))$. Here one easily verify that both $\hat{H}$ and $\hat{L}$ are symmetric (D_c is dense in $\mathcal{H}$). This can be achieved by setting

$$\psi(r, \theta, \phi) = \begin{pmatrix} R_1(r)S_1(\theta, \phi) \\ R_2(r)S_2(\theta, \phi) \\ R_2(r)S_2(\theta, \phi) \\ R_1(r)S_1(\theta, \phi) \end{pmatrix}.$$

EXERCISE: Shew that the angular operator $\hat{L}$ then reduces to the operator

$$\hat{L}_{red} = -i(D_\theta + \sin\theta\sigma_3 D_\phi),$$

on $C_c^\infty((0, \pi) \times (0, 2\pi))$, which is dense in $\mathcal{H}_\theta = \mathcal{L}_2((0, \pi) \times (0, 2\pi); \sin^2\theta d\theta d\phi)$.

It is easy to see that the spherical spinors $\Omega_{j,l,m}, j \in \mathbb{Z}+\frac{1}{2}$ are a complete set for $\mathcal{H}_\theta$ and diagonalize $\hat{L}_{red}$, with eigenvalues of the form k, $k \in \mathbb{Z}+\frac{1}{2}$.
Then $(\mathrm{Im}(\hat{L}_{red} \pm i))^\perp = 0$, and then $\hat{L}_{red}$ is essentially selfadjoint.
This means that we can reduce to consider the reduced Hamiltonians

$$\hat{H}^k_{red} := \begin{pmatrix} \sqrt{f}\mu + \frac{qQ}{r} & -f\partial_r + k\frac{\sqrt{f}}{r} \\ f\partial_r + k\frac{\sqrt{f}}{r} & -\sqrt{f}\mu + \frac{qQ}{r} \end{pmatrix}$$

with domain $C_c^\infty((r_+,\infty))^2$ dense in $\mathcal{L}_2((r_+,\infty); f(r)^{-1}dr)^2$, where we have defined

$$\begin{pmatrix} R_1(r) \\ R_2(r) \end{pmatrix} = \frac{1}{rf^{\frac{1}{4}}}\psi_{red}(r).$$

This is a reduced Dirac operator with $w = 1/f$ and V of class $\mathcal{C}^\infty$ in (r_+,∞).

To look at $r \sim r_+$ introduce tortoise coordinate y so that

$$\frac{dy}{dr} = -\frac{1}{f(r)}, \qquad y \in (0, \infty),$$

where $r = r_+$ correspond to $y = \infty$. In this coordinate the measure function becomes $w(y) = 1$ and

$$\hat{H}^k_{red} := \begin{pmatrix} \sqrt{f}\mu + \frac{qQ}{r(y)} & \partial_y + k\frac{\sqrt{f}}{r(y)} \\ -\partial_y + k\frac{\sqrt{f}}{r(y)} & -\sqrt{f}\mu + \frac{qQ}{r(y)} \end{pmatrix}.$$

As $w = 1$, at $y \sim \infty$ constant functions are not in $\mathcal{L}_1$ right and we are in LPC.

Near $r \sim \infty$ $w(r) \sim L^2/r^2$ so we cannot use the same argument.
EXERCISE: Shew that near $r \sim \infty$ the eigenvalue equation $(\hat{H}^k_{red} - \lambda)\psi_{red}$ takes the form

$$\frac{d\psi_r ed}{dr} = \frac{\mu L}{r}\sigma_2\psi_{red} + O(\frac{1}{r^2}).$$

From this it follows that a fundamental set of solutions is

$$\psi^{\pm}_{red}(r) = \frac{1}{r^{\pm\mu L}}(1 + O(\frac{1}{r}))$$

if $2\mu L \notin \mathbb{N}$, and

$$\psi^{+}_{red}(r) = \frac{1}{r^{\mu L}}(1 + O(\frac{1}{r})),$$
$$\psi^{-}_{red}(r) = r^{\mu L}(1 + O(\frac{1}{r})) + c\psi^{+}_{red}(r)\log(r/L),$$

if $2\mu L$ is integer. Thus, ψ^{+}_{red} lies right in $\mathcal{L}_2$, whereas ψ^{-}_{red} does not iff $2\mu L \geq 1$.
Thus, $\hat{H}$ is essentially selfadjoint iff $\mu L \geq \frac{1}{2}$.

Dirac operator on Kerr-Newman AdS

Metric

$$\mathbf{g} = -\frac{\Delta_r}{\rho^2}\left[dt - \frac{a\sin^2\theta}{\Xi}d\phi\right]^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \Delta_\theta\frac{\sin^2\theta}{\rho^2}\left[a\,dt - \frac{r^2+a^2}{\Xi}d\phi\right]^2 ,$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta\,, \qquad \Xi = 1 - \frac{a^2}{l^2}\,,$$

$$\Delta_r = (r^2+a^2)\left(1+\frac{r^2}{l^2}\right) - 2mr + z^2\,,$$

$$\Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta\,, \qquad z^2 = q_e^2 + q_m^2\,.$$

Electromagnetic potential

$$A = -\frac{q_e r}{\rho\sqrt{\Delta_r}} e^0 - \frac{q_m \cos\theta}{\rho\sqrt{\Delta_\theta}\sin\theta} e^1 \ ,$$

where we introduced the vierbein

$$\begin{aligned} e^0 &= \frac{\sqrt{\Delta_r}}{\rho}\left(dt - \frac{a\sin^2\theta}{\Xi} d\phi\right) \ , \\ e^1 &= \frac{\sqrt{\Delta_\theta}\sin\theta}{\rho}\left(a dt - \frac{r^2+a^2}{\Xi} d\phi\right) \ , \\ e^2 &= \frac{\rho}{\sqrt{\Delta_r}} dr \ , \\ e^3 &= \frac{\rho}{\sqrt{\Delta_\theta}} d\theta \ . \end{aligned}$$

Mass, angular momentum, electric and magnetic charge are

$$M = \frac{m}{\Xi^2} \ , \quad J = \frac{am}{\Xi^2} \ , \quad Q_e = \frac{q_e}{\Xi} \ , \quad Q_m = \frac{q_m}{\Xi} \ .$$

Choose gamma matrices

$$\Gamma^0 = \begin{pmatrix} \mathbb{O} & -\mathbb{I} \\ -\mathbb{I} & \mathbb{O} \end{pmatrix} , \qquad \vec{\Gamma} = \begin{pmatrix} \mathbb{O} & -\vec{\sigma} \\ \vec{\sigma} & \mathbb{O} \end{pmatrix} ,$$

and a Newman-Penrose frame

$$\begin{aligned}
\theta^1 &= \frac{1}{\sqrt{2}} |Z(r,\theta)|^{\frac{1}{2}} \left[\frac{W(r)}{Z(r,\theta)} \left(dt + \frac{a \sin^2\theta}{\Xi} d\phi \right) + \frac{dr}{W(r)} \right] , \\
\theta^2 &= \frac{1}{\sqrt{2}} |Z(r,\theta)|^{\frac{1}{2}} \left[\frac{W(r)}{Z(r,\theta)} \left(dt + \frac{a \sin^2\theta}{\Xi} d\phi \right) - \frac{dr}{W(r)} \right] , \\
\theta^3 &= \frac{1}{\sqrt{2}} |Z(r,\theta)|^{\frac{1}{2}} \left[\frac{X(\theta)}{Z(r,\theta)} \left(adt - \frac{r^2 + a^2}{\Xi} d\phi \right) + i \frac{\sin\theta d\theta}{X(\theta)} \right] , \\
\theta^4 &= \bar{\theta}^3 ,
\end{aligned}$$

with

$$Z(r,\theta) = \frac{r^2 + a^2\cos^2\theta}{\Xi}\,, \quad W(r) = \frac{\sqrt{\Delta_r}}{\Xi^{\frac{1}{2}}}\,, \quad X(\theta) = \frac{\sqrt{\Delta_\theta}\sin\theta}{\Xi^{\frac{1}{2}}}\,.$$

Then

$$ds^2 = -2(\theta^1\theta^2 - \theta^3\theta^4)\,,$$

and

$$A = -\frac{1}{\sqrt{2|Z(r,\theta)|}}\left[\frac{H(r)}{W(r)}(\theta^1+\theta^2) + \frac{G(\theta)}{X(\theta)}(\theta^3+\theta^4)\right]\,,$$

where

$$H(r) = Q_e r\,, \quad G(\theta) = Q_m\cos\theta\,.$$

EXERCISE: Set

$$\mathcal{B}(r,\theta) = \frac{i}{4}\log\frac{r - ia\cos\theta}{r + ia\cos\theta}\ ,$$

Perform the transformation $\psi \mapsto S^{-1}\psi$, with

$$S = Z^{-\frac{1}{4}}\operatorname{diag}(e^{i\mathcal{B}}, e^{i\mathcal{B}}, e^{-i\mathcal{B}}, e^{-i\mathcal{B}})\ ,$$

and introduce the new wave function

$$\tilde{\psi} = (\Delta_\theta\Delta_r)^{\frac{1}{4}} S^{-1}\psi\ .$$

Shew that then the Dirac equation takes the form

$$(\mathcal{R}(r) + \mathcal{A}(\theta))\tilde{\psi} = 0\ ,$$

where

$$\mathcal{R} = \begin{pmatrix} i\mu r & 0 & -\sqrt{\Delta_r}\mathcal{D}_+ & 0 \\ 0 & -i\mu r & 0 & -\sqrt{\Delta_r}\mathcal{D}_- \\ -\sqrt{\Delta_r}\mathcal{D}_- & 0 & -i\mu r & 0 \\ 0 & -\sqrt{\Delta_r}\mathcal{D}_+ & 0 & i\mu r \end{pmatrix}\ ,$$

$$\mathcal{A} = \begin{pmatrix} -a\mu\cos\theta & 0 & 0 & -i\sqrt{\Delta_\theta}\mathcal{L}_- \\ 0 & a\mu\cos\theta & -i\sqrt{\Delta_\theta}\mathcal{L}_+ & 0 \\ 0 & -i\sqrt{\Delta_\theta}\mathcal{L}_- & -a\mu\cos\theta & 0 \\ -i\sqrt{\Delta_\theta}\mathcal{L}_+ & 0 & 0 & a\mu\cos\theta \end{pmatrix} ,$$

and

$$\mathcal{D}_\pm = \partial_r \pm \frac{1}{\Delta_r}\left((r^2+a^2)\partial_t - a\Xi\partial_\phi + ieq_e r\right) ,$$

$$\mathcal{L}_\pm = \partial_\theta + \frac{1}{2}\cot\theta \pm \frac{i}{\Delta_\theta \sin\theta}\left(\Xi\partial_\phi - a\sin^2\theta\partial_t + ieq_m\cos\theta\right) .$$

From the exercise it follows

$$H = \left[\left(1 - \frac{\Delta_r}{\Delta_\theta} \frac{a^2 \sin^2\theta}{(r^2+a^2)^2} \right)^{-1} \left(\mathbb{I}_4 - \frac{\sqrt{\Delta_r}}{\sqrt{\Delta_\theta}} \frac{a \sin\theta}{r^2+a^2} BC \right) \right] (\tilde{\mathcal{R}} + \tilde{\mathcal{A}}) \,,$$

where

$$\tilde{\mathcal{R}} = -\frac{\mu r \sqrt{\Delta_r}}{r^2+a^2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{E}_- & 0 & 0 & 0 \\ 0 & -\mathcal{E}_+ & 0 & 0 \\ 0 & 0 & -\mathcal{E}_+ & 0 \\ 0 & 0 & 0 & \mathcal{E}_- \end{pmatrix} \,,$$

$$\tilde{\mathcal{A}} = \frac{a\mu \cos\theta \sqrt{\Delta_r}}{r^2+a^2} \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\mathcal{M}_- & 0 & 0 \\ \mathcal{M}_+ & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{M}_- \\ 0 & 0 & -\mathcal{M}_+ & 0 \end{pmatrix} \,,$$

$$\mathcal{E}_\pm = i \frac{\Delta_r}{a^2+r^2} \left[\partial_r \mp \frac{a\Xi}{\Delta_r} \partial_\phi \pm i \frac{e q_e r}{\Delta_r} \right] \,,$$

$$\mathcal{M}_\pm = \frac{\sqrt{\Delta_r}\sqrt{\Delta_\theta}}{r^2+a^2} \left[\partial_\theta + \frac{1}{2} \cot\theta \pm \frac{i\Xi}{\Delta_\theta \sin\theta} \partial_\phi \mp \frac{e q_m \cot\theta}{\Delta_\theta} \right] \,,$$

$$B = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \,, \qquad C = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} .$$

satisfy $[B, C] = 0$, $B^2 = C^2 = \mathbb{I}_4$.

The Hilbert space is $\mathcal{H}_{\langle\rangle} = \mathcal{L}^2 := (L^2((r_+, \infty) \times S^2; d\mu))^4$ with measure

$$d\mu = \frac{r^2 + a^2}{\Delta_r} \frac{\sin\theta}{\sqrt{\Delta_\theta}} dr d\theta d\phi,$$

and scalar product

$$\langle\tilde{\psi}|\tilde{\chi}\rangle = \int_{r_+}^{\infty} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \frac{r^2 + a^2}{\Delta_r} \frac{\sin\theta}{\sqrt{\Delta_\theta}} \, {}^t\tilde{\psi}^* \left(\mathbb{I}_4 + \frac{\sqrt{\Delta_r}}{\sqrt{\Delta_\theta}} \frac{a \sin\theta}{r^2 + a^2} BC \right) \tilde{\chi} \, .$$

PROBLEM: Complete separability is made difficult by the presence of the matrix

$$\Xi^2(r, \theta) := \mathbb{I}_4 + \frac{\sqrt{\Delta_r}}{\sqrt{\Delta_\theta}} \frac{a \sin\theta}{r^2 + a^2} BC.$$

To solve it, let us introduce a second Hilbert space
$\mathcal{H}_{()} = \mathcal{L}^2 := (L^2((r_+, \infty) \times S^2; d\mu))^4$, with scalar product

$$(\psi|\chi) = \int_{r_+}^{\infty} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \frac{r^2 + a^2}{\Delta_r} \frac{\sin\theta}{\sqrt{\Delta_\theta}} \, {}^t\psi^* \chi \; = \int d\mu \, {}^t\psi^* \chi \, .$$

EXERCISE: Shew that Ξ^2 is indeed an Hermitian bounded positive matrix function with bounded inverse, so that its positive square root Ξ is well defined as well as Ξ^{-1}. Next, shew that the map

$$V_\Xi : \mathcal{H}_{\langle\rangle} \mapsto \mathcal{H}_{()},$$

defined by $(V_\Xi\psi)(r,\theta,\phi)$ is an isomorphism of Hilbert spaces.

From such an isomorphism, it follows that essential selfadjointness of $\hat{H}$ on a domain $D \subset \mathcal{H}_{\langle\rangle}$ is equivalent to essential selfadjointness of $V_\Xi \hat{H} V_\Xi^{-1}$ on $V_\Xi D \subset \mathcal{H}_{()}$.
Now, set

$$\hat{H}_0 := \Xi^2 \hat{H} = \tilde{\mathcal{R}} + \tilde{\mathcal{A}}.$$

Prop.

$\hat{H}$ is essentially selfadjoint on $D \subset \mathcal{H}_{\langle\rangle}$ if and only if $\hat{H}_0$ is essentially selfadjoint on $D \subset \mathcal{H}_{()}$.

Proof: The isomorphism V_Ξ implies that essential selfadjointness of $\hat{H}$ on D is equivalent to essential selfadjointness of $V_\Xi \hat{H} V_\Xi^{-1} = \hat{\Xi}^{-1}\hat{H}_0\hat{\Xi}^{-1}$ on $V_\Xi D$, where $\hat{\Xi}^{-1}$ is the multiplication operator by Ξ^{-1} in over $\mathcal{H}_{()}$. As Ξ is real, bounded with bounded inverse, then $\hat{\Xi}^{-1}$ is selfadjoint over $\mathcal{H}_{()}$. Then

$$(\hat{\Xi}^{-1}\hat{H}_0\hat{\Xi}^{-1})^\dagger = \hat{\Xi}^{-1}\hat{H}_0^\dagger\hat{\Xi}^{-1}.$$

□

It is convenient to introduce the unitary map $V : \mathcal{H}_{()} \to \mathcal{H}_{()}$:

$$V = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 & i \\ i & 0 & -i & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix},$$

and then to consider the operator $V\hat{H}_0V^\dagger$ on VD.

EXERCISE: Shew that

$$VH_0V^* = \begin{pmatrix} \frac{1}{r^2+a^2}(ia\Xi\partial_\phi + eq_e r + \mu r\sqrt{\Delta_r})\mathbb{I} & \frac{\Delta_r}{r^2+a^2}\partial_r\mathbb{I} + \frac{\sqrt{\Delta_r}}{r^2+a^2}\mathbb{U} \\ -\frac{\Delta_r}{r^2+a^2}\partial_r\mathbb{I} + \frac{\sqrt{\Delta_r}}{r^2+a^2}\mathbb{U} & \frac{1}{r^2+a^2}(ia\Xi\partial_\phi + eq_e r - \mu r\sqrt{\Delta_r})\mathbb{I} \end{pmatrix},$$

$$\mathbb{U} = \begin{pmatrix} -\mu a\cos(\theta) & i\sqrt{\Delta_\theta}(\partial_\theta + \frac{1}{2}\cot(\theta) + g) \\ i\sqrt{\Delta_\theta}(\partial_\theta + \frac{1}{2}\cot(\theta) - g)) & \mu a\cos(\theta) \end{pmatrix}.$$

Separation of variable is now obtained by looking at solutions of the eigenvalue problem of the form $\psi(r,\theta,\phi) = V\chi(r,\theta,\phi)$, with

$$\chi(r,\theta,\phi) = \varepsilon(\phi)\begin{pmatrix} R_1(r)S_2(\theta) \\ R_2(r)S_1(\theta) \\ R_2(r)S_2(\theta) \\ R_1(r)S_1(\theta) \end{pmatrix},$$

where $\varepsilon(\phi) \in C_c^\infty(0,2\pi)$, $R(r) := \begin{pmatrix} R_1(r) \\ R_2(r) \end{pmatrix} \in C_c^\infty(r_+,\infty)^2$ and $S(\theta) := \begin{pmatrix} S_1(\theta) \\ S_2(\theta) \end{pmatrix} \in C_c^\infty(0,\pi)^2$.

We now look at essential selfadjointness of $V\hat{H}_0V^*$ on the domain $D = \mathcal{C}_c^\infty((r_+, \infty) \times S^2)^4 \subset \mathcal{H}_{()}$.
The first reduction arises by looking at the operator $i\partial_\phi$ on $\mathcal{C}_c^\infty((0, 2\pi))$, with anti-periodic boundary conditions at 0 and at 2π. It is obviously essentially selfadjoint and the subspace L_k spanned by the eigenfunctions $e^{-ik\phi}$, $k \in \mathbb{Z} + \frac{1}{2}$ is such that $L^2((r_+, \infty), \frac{r^2+a^2}{\Delta_r} dr)^2 \otimes L^2((0, \pi), \frac{\sin(\theta)}{\sqrt{\Delta_\theta}} d\theta)^2 \otimes L_k$ is a reducing subspace for $V\hat{H}_0V^*$.

The restriction $\hat{\mathbb{U}}_k \otimes I_k$ of $\mathbb{U}$ to $C_0^\infty(0, \pi)^2 \otimes L_k$ (I_k is the identity operator on L_k) is

$$\mathbb{U}_k = \begin{pmatrix} -\mu a \cos(\theta) & i\sqrt{\Delta_\theta}(\partial_\theta + \frac{1}{2}\cot(\theta) + b_k(\theta)) \\ i\sqrt{\Delta_\theta}(\partial_\theta + \frac{1}{2}\cot(\theta) - b_k(\theta)) & \mu a \cos(\theta) \end{pmatrix},$$

where $b_k(\theta) := \frac{1}{\Delta_\theta \sin(\theta)} \Xi k - \frac{1}{\Delta_\theta} q_m e \cot(\theta)$.

PROP.

$\hat{\mathbb{U}}_k$ is essentially self adjoint on $C_c^\infty(0,\pi)^2$ for any $k = n + \frac{1}{2}$, $n \in \mathbb{Z}$ iff $\frac{q_m e}{\Xi} \in \mathbb{Z}$.

PROOF-EXERCISE: $\mathbb{U}_k$ has the form of a reduced Dirac operator acting on $S(\theta) := \begin{pmatrix} S_1(\theta) \\ S_2(\theta) \end{pmatrix} \in C_c^\infty(0,\pi)^2$. Consider the unitary transformations $W = \binom{0\,-i}{1\,0}$ and $R : L^2((0,\pi), \frac{\sin(\theta)}{\sqrt{\Delta_\theta}} d\theta)^2 \to L^2((0,\pi), \frac{1}{\sqrt{\Delta_\theta}} d\theta)^2$

$$(RS)(\theta) := (\sin(\theta))^{\frac{1}{2}} S(\theta) =: \Theta(\theta).$$

Shew that near $\theta_1 := 0$ and $\theta_2 := \pi$, the eigenvalue equation for $RW\mathbb{U}_k W^\dagger R^\dagger$ takes the form

$$(\theta - \theta_i)\partial_\theta \Theta = N_i \Theta + O((\theta - \theta_i)),$$

with

$$N_1 := \begin{pmatrix} -k + \frac{q_m e}{\Xi} & 0 \\ 0 & k - \frac{q_m e}{\Xi} \end{pmatrix}, \qquad N_2 := \begin{pmatrix} k + \frac{q_m e}{\Xi} & 0 \\ 0 & -k - \frac{q_m e}{\Xi} \end{pmatrix}.$$

Then, impose the LPC condition at both $\theta = 0$ and $\theta = \pi$ to complete the proof. □

Indeed, one can prove that $\mathbb{U}_k$ has a purely discrete spectrum, which is simple. Let us then introduce its (normalized) eigenfunctions $S_{k;j}(\theta) := \begin{pmatrix} S_{1\ k;j}(\theta) \\ S_{2\ k;j}(\theta) \end{pmatrix}$ with eigenvalues $\lambda_{k;j}$. Then $\mathcal{H}_{k,j} := L^2((r_+,\infty), \frac{r^2+a^2}{\Delta_r}dr)^2 \otimes M_{k,j}$, where $M_{k,j} := \{F_{k;j}(\theta,\phi)\}$, with $F_{k;j}(\theta,\phi) := S_{k;j}(\theta)\frac{e^{-ik\phi}}{\sqrt{2\pi}}$, is a reducing subspace for $V\hat{H}_0V^*$. There, it acts as

$$h_{k,j} := \begin{pmatrix} \frac{1}{r^2+a^2}(a\Xi k + eq_e r + \mu r\sqrt{\Delta_r}) & \frac{\Delta_r}{r^2+a^2}\partial_r + \frac{\sqrt{\Delta_r}}{r^2+a^2}\lambda_{k;j} \\ -\frac{\Delta_r}{r^2+a^2}\partial_r + \frac{\sqrt{\Delta_r}}{r^2+a^2}\lambda_{k;j} & \frac{1}{r^2+a^2}(a\Xi k + eq_e r - \mu r\sqrt{\Delta_r}) \end{pmatrix}$$

over $D_{k,j} := C_c^\infty(r_+,\infty)^2$

PROP.

$\hat{h}_{k,j}$ is essentially selfadjoint on $C_0^\infty(r_+,\infty)^2$ iff $\mu l \geq \frac{1}{2}$.

PROOF: Choose the tortoise coordinate $y \in (0,\infty)$ defined by

$$dy = -\frac{r^2+a^2}{\Delta_r}dr.$$

Then, $y \to \infty \Leftrightarrow r \to {r_+}^+$ and

$$h_{k,j} = \begin{pmatrix} 0 & -\partial_y \\ \partial_y & 0 \end{pmatrix} + V(r(y)).$$

As constants are not in $\mathcal{L}_1$ right the limit point case holds for $h_{k,j}$ at $y=\infty$. To look at $r \to \infty$, set $x = \frac{1}{r}$. The eigenvalue equation takes the form

$$x\partial_x X = G(x)X,$$

where the smooth matrix $G(x)$ is regular as $x \to 0^+$ and

$$\lim_{x\to 0^+} G(x) = \begin{pmatrix} 0 & \mu l \\ \mu l & 0 \end{pmatrix}.$$

This has a singularity of the first kind, with eigenvalues $w_{\pm} = \pm\mu l$. It follows that the limit point case occurs at $r = \infty$ iff

$$\int_c^\infty \frac{dr}{r^2} r^{\pm 2\mu l} = \infty.$$

For $\mu > 0$ this imply the assert. $\square$

Bibliography

[W1]
Weidmann, J.: Spectral Theory of Ordinary Differential Operators. Lecture Notes in Mathematics 1258. Berlin: Springer-Verlag, 1987.

[W2]
Weidmann, J.: Linear Operators in Hilbert Spaces. Graduate Texts in Mathematics 68. Berlin: Springer-Verlag, 1980.

[BC1]
F. Belgiorno, S.L. Cacciatori, "Quantum Effects for the Dirac Field in Reissner-Nordstrom-AdS Black Hole Background," published in Class.Quant.Grav.25:105013,2008.

[BC2]
F. Belgiorno, S. L. Cacciatori, "The Dirac Equation in Kerr-Newman-AdS Black Hole Background," J. Math. Phys. 51, 033517 (2010).