Término de Frontera en Teorías de Gravedad Modificada $f(R)$: Ecuaciones de campo en el Formalismo Métrico

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Índice

1. Relatividad General: Acción de Einstein-Hilbert + Termino de Gibbons-York-Hawking

Sea un par $(\mathcal{M}, \mathbf{g})$ con $\mathcal{M}$ una variedad cuadri-dimensional, y $\mathbf{g}$ una métrica sobre $\mathcal{M}$. La Relatividad General esta basada en postulados que incluyen el de las ecuaciones de campo de Einstein (en unidades de $c = 1$ y con $\kappa = 8\pi G$), las cuales determinan la forma de $\mathbf{g}$ dada una forma de momento-energía:

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \kappa T_{\alpha\beta} \tag{1}$$

Podemos obtener las ecuaciones de campo de un principio variacional, a partir de la acción de **Einstein-Hilbert**, más una acción asociada a los campos de materia:

$$S = \frac{1}{2\kappa} \underbrace{\int_{\mathcal{V}} d^4x \sqrt{-g} R}_{S_{EH}} + \underbrace{\int_{\mathcal{V}} d^4x \sqrt{-g} \mathcal{L}_M[g_{\alpha\beta}, \psi]}_{S_M} \tag{2}$$

La variación se realiza con respecto a $g^{\alpha\beta}$ fijando $\delta g^{\alpha\beta}|_{\partial\mathcal{V}} = 0$.

Usando

$$\delta R = \delta g^{\alpha\beta} R_{\alpha\beta} + \nabla_\sigma \big(g^{\alpha\beta} (\delta \Gamma^\sigma_{\beta\alpha}) - g^{\alpha\sigma} (\delta \Gamma^\gamma_{\alpha\gamma}) \big), \qquad \delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\delta g^{\alpha\beta}$$

tendremos para la variación del termino geométrico:

$$\begin{aligned}
\delta S_{EH} &= \int_{\mathcal{V}} d^4x \, \left(R\delta\sqrt{-g} + \sqrt{-g}\, \delta R \right) \\
&= \int_{\mathcal{V}} d^4x \left(R_{\alpha\beta}\sqrt{-g}\delta g^{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}\sqrt{-g}\, \delta g^{\alpha\beta} + \sqrt{-g}\nabla_\sigma \big(g^{\alpha\beta}(\delta\Gamma^\sigma_{\beta\alpha}) - g^{\alpha\sigma}(\delta\Gamma^\gamma_{\alpha\gamma}) \big) \right) \\
&= \int_{\mathcal{V}} d^4x \, \sqrt{-g} \Big(\underbrace{R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}}_{G_{\alpha\beta}} \Big) \delta g^{\alpha\beta} + \int_{\mathcal{V}} d^4x \sqrt{-g}\nabla_\sigma \big(g^{\alpha\beta}(\delta\Gamma^\sigma_{\beta\alpha}) - g^{\alpha\sigma}(\delta\Gamma^\gamma_{\alpha\gamma}) \big)
\end{aligned} \tag{3}$$

Definamos la siguiente cantidad

$$V^\sigma = g^{\alpha\beta}(\delta\Gamma^\sigma_{\beta\alpha}) - g^{\alpha\sigma}(\delta\Gamma^\gamma_{\alpha\gamma}) \tag{4}$$

de modo que la integral adicional se escribe como

$$\int_{\mathcal{V}} d^4x \sqrt{-g}\nabla_\sigma \big(g^{\alpha\beta}(\delta\Gamma^\sigma_{\beta\alpha}) - g^{\alpha\sigma}(\delta\Gamma^\gamma_{\alpha\gamma}) \big) = \int_{\mathcal{V}} d^4x \sqrt{-g}\nabla_\sigma V^\sigma \tag{5}$$

Usamos ahora el teorema de Gauss-Stokes

$$\boxed{\int_{\mathcal{V}} d^n x\,\sqrt{|g|}\nabla_\mu A^\mu = \oint_{\partial\mathcal{V}} d^{n-1}y\,\varepsilon\sqrt{|h|}n_\mu A^\mu} \tag{6}$$

la integral adicional queda escrita como

$$\int_{\mathcal{V}} d^4 x\sqrt{-g}\nabla_\sigma V^\sigma = \oint_{\partial\mathcal{V}} d^3 y\,\varepsilon\sqrt{|h|}n_\sigma V^\sigma \tag{7}$$

Para la evaluación de $\delta\Gamma^\sigma_{\beta\alpha}$ usamos que $\Gamma^\sigma_{\beta\alpha}$ es el simbolo de Christoffel $\left\{{\sigma \atop \beta\alpha}\right\}$:

$$\begin{aligned}\delta\Gamma^\sigma_{\beta\alpha} &= \delta\left(\frac{1}{2}g^{\sigma\gamma}\left[\partial_\beta g_{\gamma\alpha} + \partial_\alpha g_{\gamma\beta} - \partial_\gamma g_{\beta\alpha}\right]\right)\\ &= \frac{1}{2}\delta g^{\sigma\gamma}\left[\partial_\beta g_{\gamma\alpha} + \partial_\alpha g_{\gamma\beta} - \partial_\gamma g_{\beta\alpha}\right] + \frac{1}{2}g^{\sigma\gamma}\left[\partial_\beta(\delta g_{\gamma\alpha}) + \partial_\alpha(\delta g_{\gamma\beta}) - \partial_\gamma(\delta g_{\beta\alpha})\right]\end{aligned} \tag{8}$$

De las condiciones en la frontera $\delta g_{\alpha\beta} = \delta g^{\alpha\beta} = 0$, tenemos

$$\delta\Gamma^\sigma_{\beta\alpha}\Big|_{\partial\mathcal{V}} = \frac{1}{2}g^{\sigma\gamma}\left[\partial_\beta(\delta g_{\gamma\alpha}) + \partial_\alpha(\delta g_{\gamma\beta}) - \partial_\gamma(\delta g_{\beta\alpha})\right] \tag{9}$$

y

$$V_\sigma\Big|_{\partial\mathcal{V}} = g_{\sigma\mu}V^\mu\Big|_{\partial\mathcal{V}} = g^{\alpha\beta}\left[\partial_\beta(\delta g_{\sigma\alpha}) - \partial_\sigma(\delta g_{\beta\alpha})\right] \tag{10}$$

Evaluamos el termino $n^\sigma V_\sigma\big|_{\partial\mathcal{V}}$ usando que

$$g^{\alpha\beta} = h^{\alpha\beta} + \varepsilon n^\alpha n^\beta \tag{11}$$

de modo que

$$\begin{aligned} n^\sigma V_\sigma\Big|_{\partial\mathcal{V}} &= n^\sigma (h^{\alpha\beta} + \varepsilon n^\alpha n^\beta)[\partial_\beta(\delta g_{\sigma\alpha}) - \partial_\sigma(\delta g_{\beta\alpha})] \\ &= n^\sigma h^{\alpha\beta}[\partial_\beta(\delta g_{\sigma\alpha}) - \partial_\sigma(\delta g_{\beta\alpha})] \\ &= -n^\sigma h^{\alpha\beta}\partial_\sigma(\delta g_{\beta\alpha}) \end{aligned} \tag{12}$$

donde usamos la parte antisimétrica de $\varepsilon n^\alpha n^\beta$ con $\varepsilon = n^\mu n_\mu = \pm 1$, y también que la derivada tangencial $h^{\alpha\beta}\partial_\beta(\delta g_{\sigma\alpha}) = 0$ pues $\delta g_{\alpha\beta} = 0$ en la frontera. Finalmente la variación del termino geométrico es

$$\delta S_{EH} = \int_{\mathcal{V}} d^4x\,\sqrt{-g}\left(R_{\alpha\beta} - \frac{1}{2}R g_{\alpha\beta}\right)\delta g^{\alpha\beta} - \oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}h^{\alpha\beta}\partial_\sigma(\delta g_{\beta\alpha})n^\sigma \tag{13}$$

Para evitar terminos en la frontera NO basta con fijar $\delta g^{\alpha\beta}|_{\partial\mathcal{V}} = 0$, es necesario también tener $\partial_\sigma(\delta g_{\beta\alpha})|_{\partial\mathcal{V}} = 0$.

1.1. Termino de frontera de Gibbons-York-Hawking

Para evitar fijar ambas condiciones, introducimos un termino de frontera (Gibbons-York-Hawking):

$$S_{GYH} = 2\oint_{\partial \mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}K \qquad \Longrightarrow \qquad \delta S_{GYH} = 2\oint_{\partial \mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}\delta K \tag{14}$$

usamos la definición de curvatura extrínseca

$$\begin{aligned} K &= \nabla_\alpha n^\alpha \\ &= g^{\alpha\beta}\nabla_\beta n_\alpha \\ &= h^{\alpha\beta}(\partial_\beta n_\alpha - \Gamma^\gamma_{\beta\alpha}n_\gamma) \end{aligned} \tag{15}$$

la variación es

$$\begin{aligned} \delta K &= -h^{\alpha\beta}\delta\Gamma^\gamma_{\beta\alpha}n_\gamma \\ &= -\frac{1}{2}h^{\alpha\beta}\left[\partial_\beta(\delta g_{\sigma\alpha}) + \partial_\alpha(\delta g_{\sigma\beta}) - \partial_\sigma(\delta g_{\beta\alpha})\right]n^\sigma \\ &= \frac{1}{2}h^{\alpha\beta}\partial_\sigma(\delta g_{\beta\alpha})n^\sigma \end{aligned} \tag{16}$$

Con esta variación tendremos

$$\delta S_{GYH} = \oint_{\partial\mathcal{V}} d^3y\, \varepsilon \sqrt{|h|} h^{\alpha\beta} \partial_\sigma(\delta g_{\beta\alpha}) n^\sigma \tag{17}$$

de modo que

$$\delta S_{EH} + \delta S_{GYH} = \int_{\mathcal{V}} d^4x\, \sqrt{-g} \left(R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} \right) \delta g^{\alpha\beta} \tag{18}$$

Si introducimos un termino de frontera, la parte goemétrica de las ecuaciones de Campo de Einstein se obtiene de un principio variacional fijando tan solo $\boldsymbol{\delta g^{\alpha\beta}|_{\partial\mathcal{V}} = 0}$.

Para la acción asociada a los campos de materia tendremos

$$\begin{aligned} \delta S_M &= \int_{\mathcal{V}} d^4x\, \delta(\sqrt{-g}\mathfrak{L}_M) \\ &= \int_{\mathcal{V}} d^4x\, \sqrt{-g} \left(\frac{\partial \mathfrak{L}_M}{\partial g^{\alpha\beta}} - \frac{1}{2} \mathfrak{L}_M g_{\alpha\beta} \right) \delta g^{\alpha\beta} \\ &= -\frac{1}{2} \int_{\mathcal{V}} d^4x\, \sqrt{-g} T_{\alpha\beta} \delta g^{\alpha\beta} \end{aligned} \tag{19}$$

con

$$T_{\alpha\beta} \equiv -2 \frac{\partial \mathfrak{L}_M}{\partial g^{\alpha\beta}} + \mathfrak{L}_M g_{\alpha\beta} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\alpha\beta}} \tag{20}$$

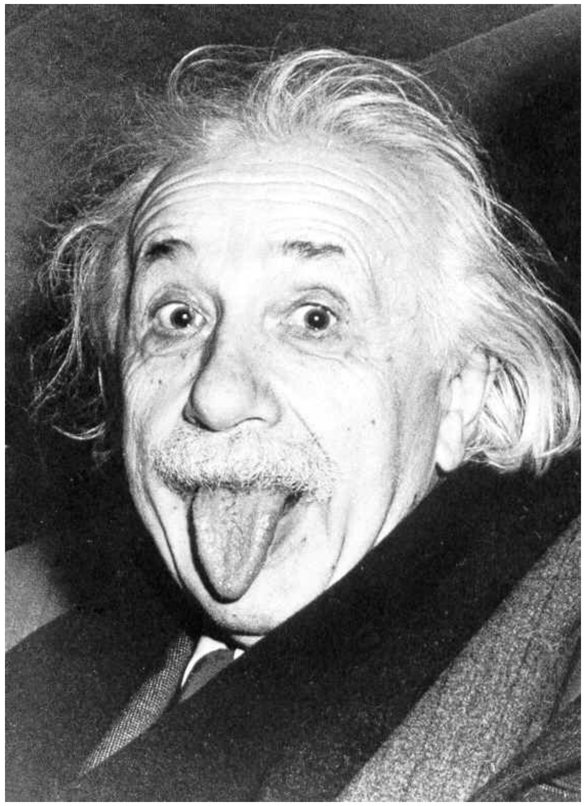
En Conclusión:

$$S = \frac{1}{2\kappa}\left(\int_{\mathcal{V}} d^4x\,\sqrt{-g}R + 2\oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}K\right) + \int_{\mathcal{V}} d^4x\,\sqrt{-g}\mathcal{L}_M[g_{\alpha\beta},\psi]$$

$$+$$

$$\frac{\delta S}{\delta g^{\alpha\beta}} = 0, \qquad \delta g^{\alpha\beta}|_{\partial\mathcal{V}} = 0$$

$$\Downarrow$$



2. Teorías de gravedad modificada $f(R)$

2.1. Introducción

Problemas fundamentales de la cosmología estándar

- Expansión acelerada del universo
- Contenido de materia-energía en el universo

Posible solución: Constante Cosmológica Λ : Expansión acelerada del universo

$$\frac{\ddot{a}}{a} = \frac{\kappa}{3}\rho_\Lambda, \qquad \rho_\Lambda = \frac{\Lambda}{\kappa}$$

Contenido de Materia-energía en el universo: ρ_Λ implica una ecuación de estado de la forma

$$P = -\rho \qquad \Longrightarrow \qquad \textbf{DARK ENERGY}$$

Parámetros cosmológicos:

$$\Omega_m\left(=\frac{8\pi G\rho}{3H^2}\right) + \Omega_\Lambda\left(=\frac{\Lambda}{3H^2}\right) + \Omega_k\left(=-\frac{k}{a^2H^2}\right) = 1$$

Hoy en dia tenemos $\Omega_m = 0.3$ y $\Omega_\Lambda = 0.7$

Pero Λ tiene problemas:

- *Problema de la constante cosmológica*
- *Problema de coincidencia*

Una posible solución:

Ver el efecto de la expansión acelerada del universo como producto de términos puramente geométricos en las ecuaciones de campo.

$$\Downarrow$$

Teorías de Gravedad Modificada $f(R)$

$$\Downarrow$$

$$\boxed{\int_{\mathcal{V}} d^4x\,\sqrt{-g}R \qquad \Longrightarrow \qquad \int_{\mathcal{V}} d^4x\,\sqrt{-g}f(R)}$$

$$\Downarrow$$

DARK GRAVITY

2.2. Ecuaciones de Campo en el Formalismo Métrico

Partimos de la acción:

$$S' = \frac{1}{2\kappa}\Big[\underbrace{\int_{\mathcal{V}} d^4x\,\sqrt{-g}\,f(R)}_{S_{met}} + \underbrace{2\oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}f'(R)K}_{S'_{GYH}}\Big] + \int_{\mathcal{V}} d^4x\,\sqrt{-g}\mathcal{L}_M[g_{\alpha\beta},\psi]$$

con $f'(R) = df(R)/dR$. De nuevo la variación se realiza con respecto a $g^{\alpha\beta}$ fijando $\delta g^{\alpha\beta}|_{\partial\mathcal{V}} = 0$. La variación del primer termino es:

$$\delta S_{met} = \int_{\mathcal{V}} d^4x\,\left(f(R)\delta\sqrt{-g} + \sqrt{-g}\,\delta f(R)\right) \tag{21}$$

la derivada funcional de $f(R)$ se puede escribir como

$$\delta f(R) = f'(R)\delta R \tag{22}$$

y la variación del escalar de Ricci como:

$$\begin{aligned}\delta R &= \delta g^{\alpha\beta}R_{\alpha\beta} + \nabla_\sigma\left(g^{\alpha\beta}(\delta\Gamma^\sigma_{\beta\alpha}) - g^{\alpha\sigma}(\delta\Gamma^\gamma_{\alpha\gamma})\right)\\ &= \delta g^{\alpha\beta}R_{\alpha\beta} + g_{\mu\nu}\nabla_\sigma\nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\sigma\nabla_\gamma(\delta g^{\sigma\gamma})\\ &= \delta g^{\alpha\beta}R_{\alpha\beta} + g_{\alpha\beta}\Box(\delta g^{\alpha\beta}) - \nabla_\alpha\nabla_\beta(\delta g^{\alpha\beta})\end{aligned} \tag{23}$$

Con $\Box \equiv \nabla_\sigma \nabla^\sigma$. Así tendremos:

$$
\begin{aligned}
\delta S_{met} &= \int_{\mathcal{V}} d^4x \left(f(R)\delta\sqrt{-g} + \sqrt{-g}\, f'(R)\delta R \right) \\
&= \int_{\mathcal{V}} d^4x \left(-f(R)\frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\delta g^{\alpha\beta} + f'(R)\sqrt{-g}\Big(\delta g^{\alpha\beta}R_{\alpha\beta} + g_{\alpha\beta}\Box(\delta g^{\alpha\beta}) - \nabla_\alpha\nabla_\beta(\delta g^{\alpha\beta})\Big)\right) \\
&= \int_{\mathcal{V}} d^4x \sqrt{-g}\left(f'(R)\Big(\delta g^{\alpha\beta}R_{\alpha\beta} + g_{\alpha\beta}\Box(\delta g^{\alpha\beta}) - \nabla_\alpha\nabla_\beta(\delta g^{\alpha\beta})\Big) - \frac{1}{2}f(R)\, g_{\alpha\beta}\delta g^{\alpha\beta}\right) \qquad (24)
\end{aligned}
$$

Consideramos ahora las siguientes integrales:

$$\int_{\mathcal{V}} d^4x \sqrt{-g} f'(R) g_{\alpha\beta}\Box(\delta g^{\alpha\beta}), \qquad \int_{\mathcal{V}} d^4x \sqrt{-g} f'(R)\nabla_\alpha\nabla_\beta(\delta g^{\alpha\beta}) \tag{25}$$

para expresarlas de forma distinta definimos:

$$M_\tau = f'(R)g_{\alpha\beta}\nabla_\tau(\delta g^{\alpha\beta}) - \delta g^{\alpha\beta}g_{\alpha\beta}\nabla_\tau(f'(R)) \tag{26}$$

y

$$N^\sigma = f'(R)\nabla_\gamma(\delta g^{\sigma\gamma}) - \delta g^{\sigma\gamma}\nabla_\gamma(f'(R)) \tag{27}$$

La combinación $g^{\sigma\tau}M_\tau + N^\sigma$ en el caso particular $f(R) = R$, se reduce a nuestra cantidad V^σ.

Tomando la derivada covariante en M_τ:

$$\begin{aligned}\nabla^\tau M_\tau &= \nabla^\tau\big(f'(R)g_{\alpha\beta}\nabla_\tau(\delta g^{\alpha\beta})\big) - \nabla^\tau\big(\delta g^{\alpha\beta}g_{\alpha\beta}\nabla_\tau(f'(R))\big)\\ &= f'(R)g_{\alpha\beta}\Box(\delta g^{\alpha\beta}) - \delta g^{\alpha\beta}g_{\alpha\beta}\Box(f'(R))\end{aligned}$$

integrando

$$\int_{\mathcal{V}} d^4x\,\sqrt{-g}\nabla^\tau M_\tau = \int_{\mathcal{V}} d^4x\,\sqrt{-g}f'(R)g_{\alpha\beta}\Box(\delta g^{\alpha\beta}) - \int_{\mathcal{V}} d^4x\,\sqrt{-g}\delta g^{\alpha\beta}g_{\alpha\beta}\Box(f'(R)) \tag{28}$$

usando el teorema de Gauss-Stokes podemos escribir:

$$\int_{\mathcal{V}} d^4x\,\sqrt{-g}f'(R)g_{\alpha\beta}\Box(\delta g^{\alpha\beta}) = \int_{\mathcal{V}} d^4x\,\sqrt{-g}\delta g^{\alpha\beta}g_{\alpha\beta}\Box(f'(R)) + \oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}n^\tau M_\tau \tag{29}$$

de forma similar, tomando la derivada covariante en N^σ:

$$\begin{aligned}\nabla_\sigma N^\sigma &= \nabla_\sigma\big(f'(R)\nabla_\gamma(\delta g^{\sigma\gamma})\big) - \nabla_\sigma\big(\delta g^{\sigma\gamma}\nabla_\gamma(f'(R))\big)\\ &= f'(R)\nabla_\sigma\nabla_\beta(\delta g^{\sigma\beta}) - \delta g^{\sigma\beta}\nabla_\sigma\nabla_\beta(f'(R))\end{aligned}$$

integrando y usando de nuevo el teorema de Gauss-Stokes tendremos

$$\int_{\mathcal{V}} d^4x\,\sqrt{-g}f'(R)\nabla_\sigma\nabla_\beta(\delta g^{\sigma\beta}) = \int_{\mathcal{V}} d^4x\,\sqrt{-g}\delta g^{\sigma\beta}\nabla_\sigma\nabla_\beta(f'(R)) + \oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}n_\sigma N^\sigma \tag{30}$$

De modo que

$$\delta S_{met} = \int_{\mathcal{V}} d^4x\,\sqrt{-g}\left(f'(R)R_{\alpha\beta}+g_{\alpha\beta}\Box f'(R)-\nabla_\alpha\nabla_\beta f'(R)-f(R)\frac{1}{2}\,g_{\alpha\beta}\right)\delta g^{\alpha\beta} + \oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}n^\tau M_\tau + \oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}n_\sigma N^\sigma \quad (31)$$

Expresamos las cantidades M_τ y N^σ evaluadas en la frontera $\partial\mathcal{V}$. Es conveniente primero escribir estas cantidades en función de las variaciones $\delta g_{\alpha\beta}$, ($\delta g_{\alpha\beta} = -g_{\alpha\mu}g_{\beta\nu}\delta g^{\mu\nu}$):

$$M_\tau = -f'(R)g^{\alpha\beta}\nabla_\tau(\delta g_{\alpha\beta}) + g^{\alpha\beta}\delta g_{\alpha\beta}\nabla_\tau(f'(R)) \quad (32)$$

$$N^\sigma = -f'(R)g^{\sigma\mu}g^{\gamma\nu}\nabla_\gamma(\delta g_{\mu\nu}) + g^{\sigma\mu}g^{\gamma\nu}\delta g_{\mu\nu}\nabla_\gamma(f'(R)) \quad (33)$$

con lo cual

$$M_\tau\Big|_{\partial\mathcal{V}} = -f'(R)g^{\alpha\beta}\partial_\tau(\delta g_{\alpha\beta}), \qquad N^\sigma\Big|_{\partial\mathcal{V}} = -f'(R)g^{\sigma\mu}g^{\gamma\nu}\partial_\gamma(\delta g_{\mu\nu}) \quad (34)$$

calculamos ahora $n^\tau M_\tau|_{\partial\mathcal{V}}$ y $n_\sigma N^\sigma|_{\partial\mathcal{V}}$ que son los terminos que aparecen en las integrales sobre la frontera

$$n^\tau M_\tau\Big|_{\partial\mathcal{V}} = -f'(R)n^\tau(\varepsilon n^\alpha n^\beta + h^{\alpha\beta})\partial_\tau(\delta g_{\alpha\beta})$$
$$= -f'(R)n^\sigma h^{\alpha\beta}\partial_\sigma(\delta g_{\alpha\beta}) \quad (35)$$

$$n_\sigma N^\sigma\Big|_{\partial\mathcal{V}} = -f'(R)n_\sigma(h^{\sigma\mu} + \varepsilon n^\sigma n^\mu)(h^{\gamma\nu} + \varepsilon n^\gamma n^\nu)\partial_\gamma(\delta g_{\mu\nu})$$
$$= -f'(R)n^\mu(h^{\gamma\nu} + \varepsilon n^\gamma n^\nu)\partial_\gamma(\delta g_{\mu\nu})$$
$$= -f'(R)n^\mu h^{\gamma\nu}\partial_\gamma(\delta g_{\mu\nu})$$
$$= 0 \quad (36)$$

donde usamos que $n_\sigma h^{\sigma\mu} = 0$, $\varepsilon^2 = 1$ y el hecho de que la derivada tangencial $h^{\gamma\nu}\partial_\gamma(\delta g_{\mu\nu})$ es nula. Con estos resultados la variación de la acción S_{met} es:

$$\delta S_{met} = \int_{\mathcal{V}} d^4x\,\sqrt{-g}\Big(f'(R)R_{\alpha\beta}+g_{\alpha\beta}\Box f'(R)-\nabla_\alpha\nabla_\beta f'(R)-f(R)\frac{1}{2}\,g_{\alpha\beta}\Big)\delta g^{\alpha\beta}$$
$$-\oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}f'(R)n^\sigma h^{\alpha\beta}\partial_\sigma(\delta g_{\alpha\beta}) \quad (37)$$

2.2.1. Termino de frontera en gravedad $f(R)$

Calculamos ahora la variación del termino S'_{GYH} en la acción total:

$$\begin{aligned}\delta S'_{GYH} &= 2\oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}\big(\delta f'(R)K + f'(R)\delta K\big)\\ &= 2\oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}\big(f''(R)\delta R\,K + f'(R)\delta K\big)\end{aligned} \tag{38}$$

usando la expresión para la variación de K, podemos escribir

$$\begin{aligned}\delta S'_{GYH} &= 2\oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}\left(f''(R)\delta R\,K + \frac{1}{2}f'(R)h^{\alpha\beta}\partial_\sigma(\delta g_{\beta\alpha})n^\sigma\right)\\ &= 2\oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}f''(R)\delta R\,K + \oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}f'(R)h^{\alpha\beta}\partial_\sigma(\delta g_{\beta\alpha})n^\sigma\end{aligned}$$

Puesto que la función $f(R)$ es arbitraria, para evitar nuevos grados de libertad, imponemos $\delta R = 0$ en la frontera.

$$\begin{aligned}\delta S' = \frac{1}{2\kappa}\int_{\mathcal{V}} d^4x\,\sqrt{-g}\left(f'(R)R_{\alpha\beta} + g_{\alpha\beta}\Box f'(R) - \nabla_\alpha\nabla_\beta f'(R) - \frac{1}{2}f(R)\,g_{\alpha\beta}\right)\delta g^{\alpha\beta}\\ - \frac{1}{2}\int_{\mathcal{V}} d^4x\,\sqrt{-g}T_{\alpha\beta}\delta g^{\alpha\beta}\end{aligned} \tag{39}$$

$$\frac{1}{\sqrt{-g}}\frac{\delta S_{mod}}{\delta g^{\alpha\beta}} = 0 \Longrightarrow f'(R)R_{\alpha\beta} + g_{\alpha\beta}\Box f'(R) - \nabla_\alpha\nabla_\beta f'(R) - \frac{1}{2}f(R)\,g_{\alpha\beta} = \kappa T_{\alpha\beta}$$

$$\boxed{f'(R)R_{\alpha\beta} + g_{\alpha\beta}\Box f'(R) - \nabla_\alpha\nabla_\beta f'(R) - \frac{1}{2}f(R)\,g_{\alpha\beta} = \kappa T_{\alpha\beta}} \tag{40}$$

Podemos escribir

$$G_{\alpha\beta} = \frac{\kappa T_{\alpha\beta}}{f'(R)} + g_{\alpha\beta}\frac{[f(R) - Rf'(R)]}{2f'(R)} + \frac{[\nabla_\alpha\nabla_\beta f'(R) - g_{\alpha\beta}\Box f'(R)]}{f'(R)}$$

o

$$G_{\alpha\beta} = \frac{\kappa}{f'(R)}\left(T_{\alpha\beta} + T^{eff}_{\alpha\beta}\right)$$

donde

$$T^{eff}_{\alpha\beta} \equiv \frac{1}{\kappa}\left[\frac{[f(R) - Rf'(R)]}{2}g_{\alpha\beta} + \nabla_\alpha\nabla_\beta f'(R) - g_{\alpha\beta}\Box f'(R)]\right] \tag{41}$$

$$\Downarrow$$

DARK GRAVITY

En Conclusión:

$$S = \frac{1}{2\kappa}\left(\int_{\mathcal{V}} d^4x\,\sqrt{-g}\,f(R) + 2\oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}\,f'(R)K\right) + \int_{\mathcal{V}} d^4x\,\sqrt{-g}\,\mathcal{L}_M[g_{\alpha\beta},\psi]$$

$$+$$

$$\frac{\delta S}{\delta g^{\alpha\beta}} = 0, \qquad \delta g^{\alpha\beta}|_{\partial\mathcal{V}} = 0, \quad \delta R|_{\partial\mathcal{V}} = 0$$

$$\Downarrow$$

3. Bibliografía

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GRUPO DE GRAVITACION Y COSMOLOGIA

Muchas Gracias