PAIR PRODUCTION ON NARIAI BLACK HOLE MANIFOLD

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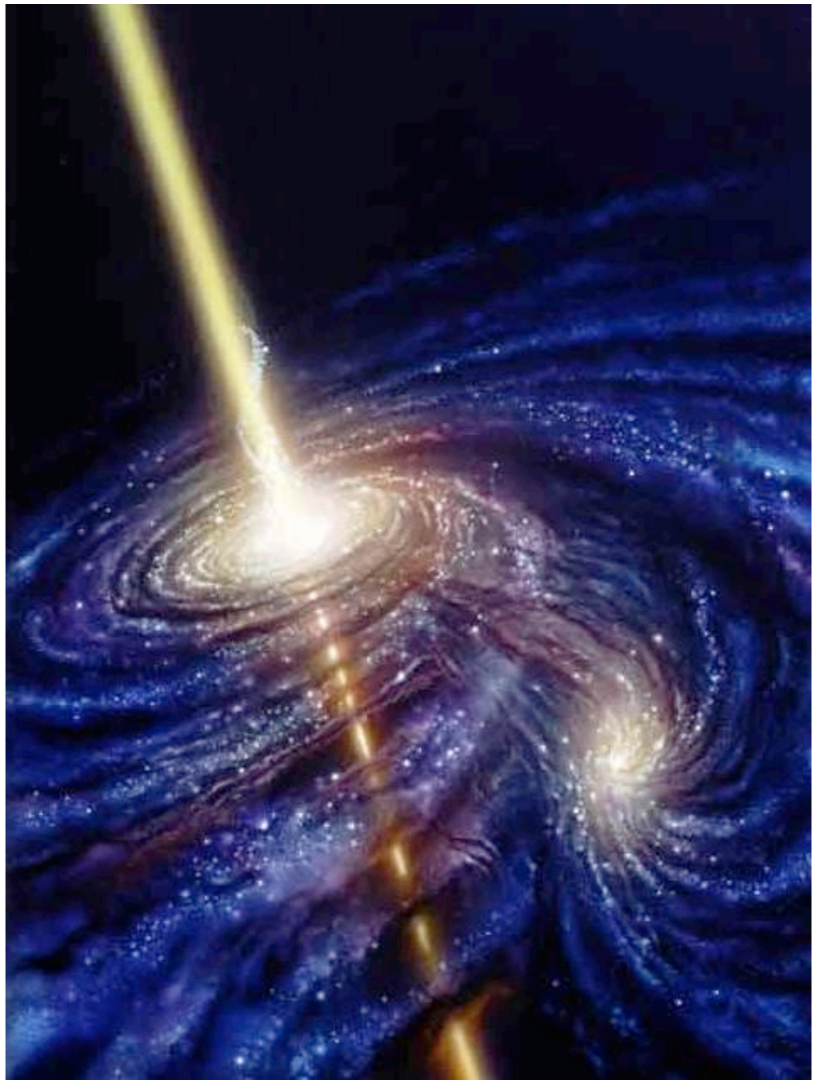
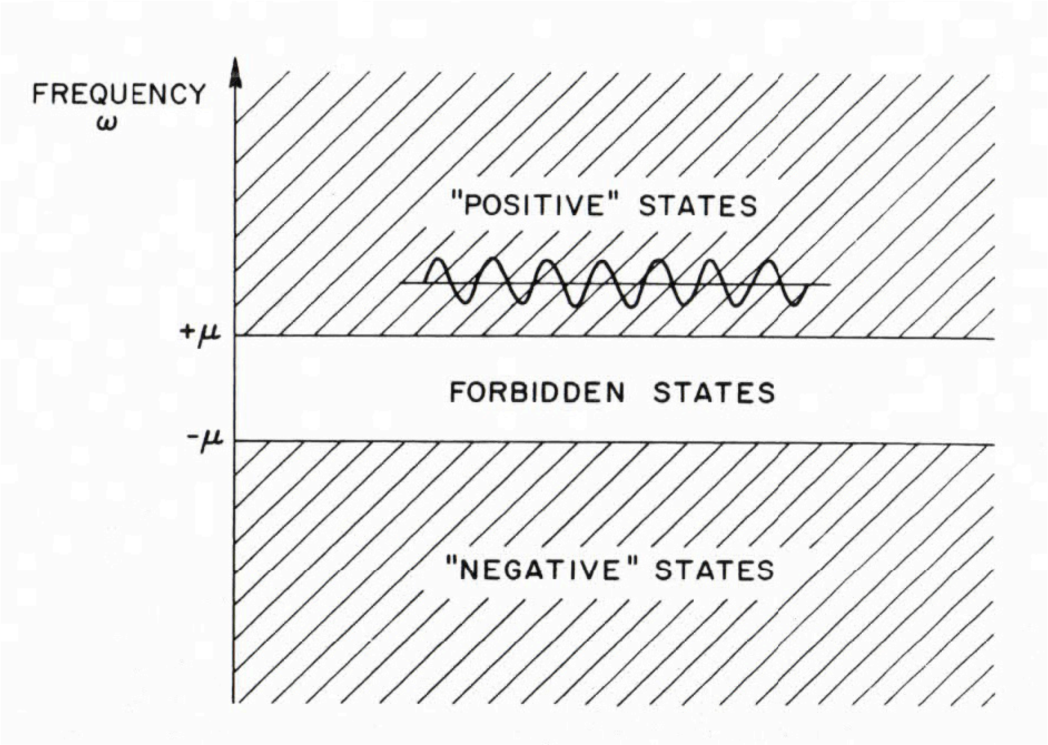
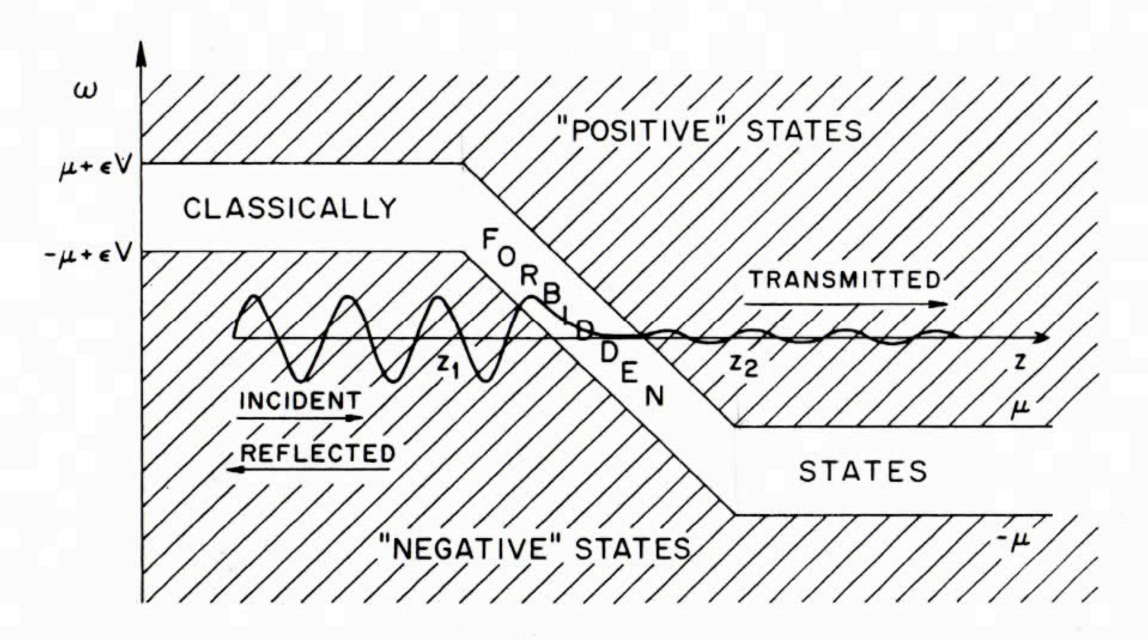


Klein Paradox

Relativistic wave equations admit symmetriclly "positive frequency" as well as "negative frequency" solutions





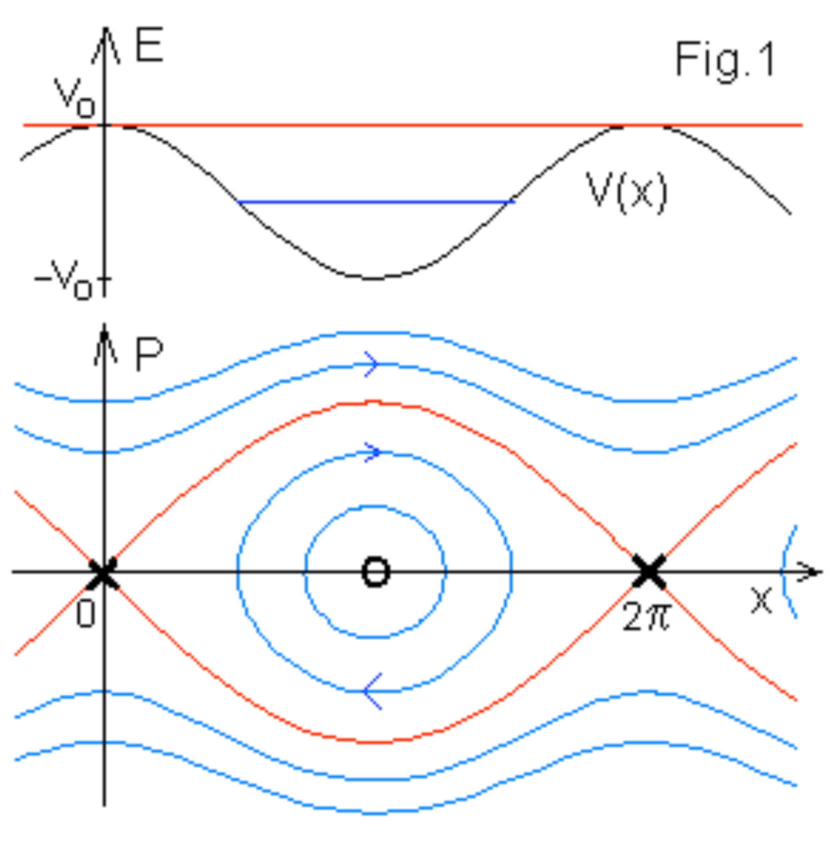
$$\omega = \pm(\mu^2 + k^2)^{1/2}$$

$$\omega = \epsilon V \pm (\mu^2 + k^2)^{1/2}$$

Positive state are stable. Ignore "negative" states

The stability of "positive" or "negative" states is lost!

Level Crossing



Study of the HJ equations

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Effective potentials $E_0^{\pm}(r)$

↓

Classical turning point for the particle motion

↓

Circular or elliptic orbits (for bound states)

$E > E_0^+(r)$	Classical orbits
$E_0^-(r) < E < E_0^+(r)$	Particles with imaginary angular momentum
$E < E_0^-(r)$	Particles with negative mass (energy)

→ Ok at quantum level

Reissner-Nordström-de Sitter metric

Charged, static, spherically symmetric solutions of Einstein equations with a cosmological constant Λ are given by the Reissner-Nordström-de Sitter metric:

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2 d\Omega^2$$

with

$$V(r) = 1 - \frac{2\mu}{r} + \frac{Q^2}{r^2} - \frac{\Lambda^2}{3}r^2$$

The function V(r) has three real positive roots:

$$r_- < r_+ < r_c$$

Inner BH horizon | Outer BH horizon | Cosmological horizon

Nariai solution

The charged Nariai solutions are the BH of maximal mass for a given charge[†]:

$$M_{max} = \frac{1}{3\sqrt{2\Lambda}}\sqrt{1+\sqrt{1-4Q^2\Lambda}}\left(2-\sqrt{1-4Q^2\Lambda}\right)$$

Geometrically this is equivalent to perform the limit $r_+ \rightarrow r_c$ and after a suitable coordinate transformation:

$$ds^2 = \frac{1}{A}(-\sin^2(\chi)d\psi^2 + d\chi^2) + \frac{1}{B}(d\theta^2 + \sin^2(\theta)d\phi^2)$$

with $\psi \in \mathbb{R}$ $\chi \in (0,\pi)$ $B = \frac{1}{2Q^2}\left(1-\sqrt{1-4Q^2\Lambda}\right)$ $A = 2\Lambda - B$

The Nariai geometry has the topology of a de Sitter space times the sphere S^2

To study the Dirac equation one needs the generalized γ matrices $\{\gamma_i,\gamma_j\} = 2g_{ij}$ for the Nariai metric:

$$\gamma_0 = \frac{\sin(\chi)}{\sqrt{A}}\tilde{\gamma}_0 \qquad \gamma^0 = -\frac{\sqrt{A}}{\sin(\chi)}\tilde{\gamma}_0$$

$$\gamma_1 = \frac{1}{\sqrt{A}}\tilde{\gamma}_1 \qquad \gamma^1 = \sqrt{A}\tilde{\gamma}_1$$

$$\gamma_2 = \frac{1}{\sqrt{B}}\tilde{\gamma}_2 \qquad \gamma^2 = \sqrt{B}\tilde{\gamma}_2$$

$$\gamma_3 = \frac{\sin(\theta)}{\sqrt{B}}\tilde{\gamma}_3 \qquad \gamma^3 = \frac{\sqrt{B}}{\sin(\theta)}\tilde{\gamma}_3,$$

[†]Belgiorno, F.; Cacciatori, S.L. “Massive Dirac particles on the background of charged de-Sitter black hole manifolds”. PRD 79 124024 (2009). ArXiv: 0810.1642.

Nariai and level crossing

Overlap between $E_0^-(r)$ and $E_0^+(r)$ ⟶ Level crossing

$$E_0^\pm(\chi) = eQ\frac{B}{A}\cos(\chi) \pm \sqrt{\frac{\mu^2}{A} + k^2\frac{B}{A}}\sin(\chi)$$

Level crossing is always present:

$$E_0^+(\pi) < E_0^-(0) \quad eQ > 0$$

$$E_0^+(0) < E_0^-(\pi) \quad eQ < 0$$

Level crossing occurs for energy ω:

$$E_0^+(\pi) \leq \omega \leq E_0^-(0) \quad eQ > 0$$

$$E_0^+(0) \leq \omega \leq E_0^-(\pi) \quad eQ < 0$$

Transmission coefficient approach I

$$|T|^2 = \frac{|\text{transmitted flux}|}{|\text{incident flux}|}$$

To investigate particle creation in presence of a given potential V:

- introduce the solutions of the wave equation;
- build localized wave packets purely ingoing into V from the past;
- do the same for outgoing states, i.e. states from which one can build localized wave packets purely outgoing from V into the future;
- we will suppose that it has been possible to define meaning fully these ingoing and outgoing states;
- it is essential that it has been possible to separate those ingoing and outgoing states in "positive" and "negative" states;

Complete basis of "positive" and "negative" modes

p_i^{in}	n_i^{in}	$\delta_{ik} = (p_i^{in}, p_k^{in}) = \pm(n_i^{in}, n_k^{in})$
p_i^{out}	n_i^{out}	$\delta_{ik} = (p_i^{out}, p_k^{out}) = \pm(n_i^{out}, n_k^{out})$

Transmission coefficient approach II

The quantized field can be expanded as:

$$\phi(x) = \sum_i a_i^{in} p_i^{in}(x) + \left(b_i^{in}\right)^\dagger n_i^{in}(x)$$

with

$$\left[a_i^{in}, \left(a_k^{in}\right)^\dagger\right]_\pm = \left[b_i^{in}, \left(b_k^{in}\right)^\dagger\right]_\pm = \delta_{ik}$$

The in-vacuum is defined as: $a_i^{in}|0^{in} >= b_i^{in}|0^{in} >= 0$

One can do the same thing with the out-states → Particle creation: the in-vacuum contains out-states

The mean number $< N_i >= \eta_i$ of out-particles described by p_i^{out} that one will find in the in-vacuum is:

$$\eta_i =< 0^{in}| \left(a_i^{out}\right)^\dagger a_i^{out}|0^{in} >$$

with a corresponding expression for antiparticles.

$$\phi = \sum_k a_k^{in} p_k^{in} + \left(b_k^{in}\right)^\dagger n_k^{in} = \sum_i a_i^{out} p_i^{out} + \left(b_i^{out}\right)^\dagger n_i^{out}$$

+ orthonormality relations between a, $a^\dagger$, b and $b^\dagger$

$$a_i^{out} = \sum_k \left(p_i^{out}, p_k^{in}\right) a_k^{in} + \left(p_i^{out}, n_k^{in}\right) \left(b_k^{in}\right)^\dagger$$

→

$$\eta_i = \sum_k | \left(p_i^{out}, n_k^{in}\right) |^2$$

Transmission coefficient approach III

To each channel for a decay $n_k^{in} \to p_i^{out}$ with a non vanishing transmission amplitude:

$$T_{ik} = (p_i^{out}, n_k^{in})$$

corresponds a mean number $|T_{ik}|^2$ of particles created in the mode p_i^{out}

The mean total number of particles created will be:

$$< N >= \sum_i \eta_i = \sum_{i,k} |T_{ik}|^2$$

In cases where it is possible to choose the in-basis and the out-basis in such a way that a n_i^{in} decays only in a p_i^{out}

↓

there is only one possible channel

↓

$$T_{ik} = T_i \delta_{ik} \quad \text{and} \quad \eta_i = |T_i|^2$$

→ The ingoing states n_i^{in} contains the outgoing part $T_i p_i^{out}$

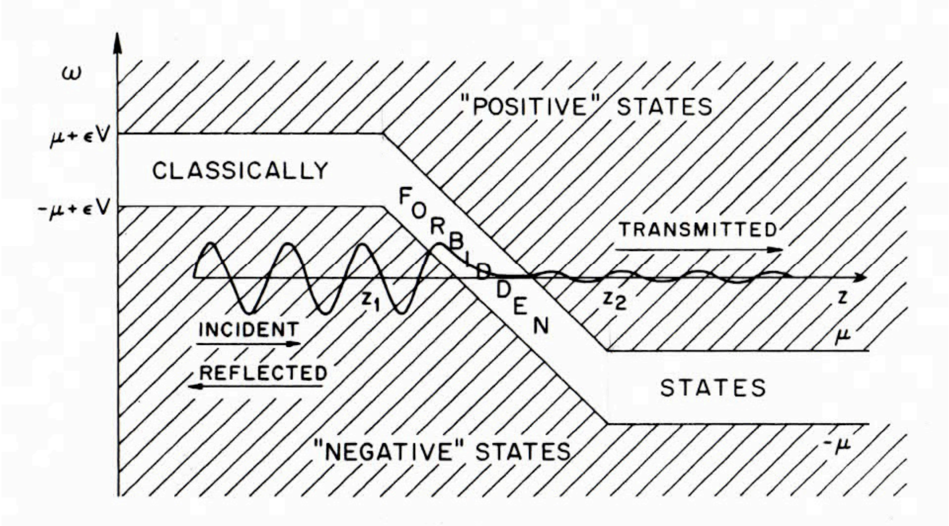
Second quantizatoin vs Dirac's sea picture I

The field $\phi(x)$ carries a charge $Q = \epsilon(\phi, \phi)$ and energy $E = (\phi, i\partial_t \phi)$ given by:

$$Q = \sum_i \epsilon (a_i^{in})^\dagger a_i^{in} + \epsilon b_i^{in} (b_i^{in})^\dagger$$
$$= \sum_i \epsilon (a_i^{in})^\dagger a_i^{in} - \epsilon (b_i^{in})^\dagger b_i^{in} + \sum_{n_i^{in}} \epsilon$$

$$E = \sum_i \omega_i^+ (a_i^{in})^\dagger a_i^{in} + \omega_i^- b_i^{in} (b_i^{in})^\dagger$$
$$= \sum_i \omega_i^+ (a_i^{in})^\dagger a_i^{in} - \omega_i^- (b_i^{in})^\dagger b_i^{in} + \sum_{n_i^{in}} \omega_i^-$$

All the negative state n_i^{in} are filled by a wave normalized to unity which bears a charge ϵ and energy ω_i^-



In the case of figure these wave will lake out of the "negative" see and appear as an outgoing positive wave

When only one channel is possible: $n_i^{in} = R_i n_i^{out} + T_i p_i^{out}$ with $|R_i|^2 = 1 - |T_i|^2$

outgoing of particles of charge +ϵ and energy $+\omega_i^-$ associated to a defect of flux (hole) over the background sea which will appear as a flux of antiparticles of charge -(+ϵ) and energy $-(+\omega_i^-)$

Second quantization vs Dirac's sea picture II

Rewriting the scattering process as:

$$n_i^{out} = R_i^{-1} n_i^{in} - R_i^{-1} T_i p_i^{out}$$

scattering of a negative mode incident from the future and which is in part refracted in the past and in part reflected in the future

The new reflection coefficient is: $|R_i^{-1} T_i|^2 = \frac{\eta_i}{1-\eta_i}$ ⟶ Relative probability for the creation of the pair n_i^{out}, p_i^{out}

The absolute probability is obtained by multiplying the relative one times the probability $p_{i,0}$ to get zero pairs in the channel i, and then the probability $p_{i,n}$ of n pair for fermions is

$$p_{i,n} = p_{i,0} \frac{\eta_i^n}{(1-\eta_i)^n}$$

The normalization condition $\sum_{n=0}^{1} p_{i,n} = 1$ leads to $p_{i,0} = 1 - \eta_i$

The persistence of the vacuum is given by:

$$P_0 = \prod_i p_{i,0} = \exp(-2\mathrm{Im}W)$$

and then

$$2\mathrm{Im}W = -\sum_i \log(1-\eta_i) = \sum_i \sum_{k=1}^{\infty} \frac{1}{k} \eta_i^k$$

Nariai in the transmission coefficient approach I

Nariai solution of Einstein equation

↓

Dirac equation $(\gamma^\mu \partial_\mu - \mu)\Psi = 0$

↓

Posing $\Psi = \frac{1}{(\sin\chi)^{1/2}} \frac{1}{(\sin\theta)^{1/2}} e^{-i\omega\psi} \eta(\chi, \eta, \varphi)$

↓

We obtain the eigenvalue equation $H_k \eta = \omega \eta$

↓

with $H_k = h_k \otimes I_2$ and h_k reduced Hamiltonian

↓

$$h_k = \begin{bmatrix} eQ\frac{B}{A}\cos(\chi) - \frac{\mu}{\sqrt{A}}\sin(\chi) & \sin(\chi)\partial_\chi + \sqrt{\frac{B}{A}}\sin(\chi)k \\ -\sin(\chi)\partial_\chi + \sqrt{\frac{B}{A}}\sin(\chi)k & eQ\frac{B}{A}\cos(\chi) + \frac{\mu}{\sqrt{A}}\sin(\chi) \end{bmatrix}$$

↓

Then the Dirac equation in hamiltonian form:

↓

$$\begin{pmatrix} eQ\frac{B}{A}\cos(\chi) - \frac{\mu}{\sqrt{A}}\sin(\chi) & \sin(\chi)\partial_\chi + \sqrt{\frac{B}{A}}\sin(\chi)k \\ -\sin(\chi)\partial_\chi + \sqrt{\frac{B}{A}}\sin(\chi)k & eQ\frac{B}{A}\cos(\chi) + \frac{\mu}{\sqrt{A}}\sin(\chi) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \omega \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Nariai in the transmission coefficient approach II

With the change of variable $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = e^{-i\frac{\pi}{4}\sigma_1} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$

The study of such equation can be reduced to the study of an hypergeometric differential equation for $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$

The asymptotic behaviors at infinities (in the coord. $x = \log\tan\frac{\chi}{2}$) for the solutions are:

$$\xi_1(x)^- \approx Ae^{i(eE-\omega)x} + e^{-i(eE-\omega)x}O(e^x)$$
$$\xi_2(x)^- \approx Be^{-i(eE-\omega)x} + e^{i(eE-\omega)x}O(e^x) \qquad x \to -\infty$$

$$\xi_1(x)^+ \approx Ce^{-i(eE+\omega)x} + e^{i(eE+\omega)x}O(e^{-x})$$
$$\xi_2(x)^+ \approx De^{i(eE+\omega)x} + e^{-i(eE+\omega)x}O(e^{-x}) \qquad x \to \infty$$

Nariai in the transmission coefficient approach III

Imposing C=0 → Incoming wave only at $x = -\infty$

$$R = \frac{A}{B} \quad \text{and} \quad T = \frac{D}{B}$$

Restoring the expression of the constant A, B, C, D given by the solution of the wave equation:

$$|T_k(\omega)|^2 = \frac{\cosh[\pi(eQ\frac{B}{A} - \omega)]\cosh[\pi(eQ\frac{B}{A} + \omega)]}{\cosh[\pi(\sqrt{\Delta} - \omega)]\cosh[\pi(\sqrt{\Delta} + \omega)]} \quad \text{and} \quad |R_k(\omega)|^2 = \frac{\sinh[\pi(\sqrt{\Delta} - eQ\frac{B}{A})]\sinh[\pi(\sqrt{\Delta} + eQ\frac{B}{A})]}{\cosh[\pi(\sqrt{\Delta} - \omega)]\cosh[\pi(\sqrt{\Delta} + \omega)]}$$

$$\Delta = \frac{\mu^2}{A} + \frac{B}{A}k^2 + (eQ\frac{B}{A})^2$$

As expected $|T|^2 < 1$, it gives the mean number of created pairs for unit time and unit volume and the property $|T_k(\omega)|^2 + |R_k(\omega)|^2 = 1$ holds.

To determine the imaginary part of the effective action we compute:

$$W_k = -\frac{1}{2}\sum_{\omega} \log(1 - |T_k(\omega)|^2)$$

We sum only over the level-crossing region, only there particle creation is expected to be present and only there an instability for the vacuum should occur:

$$-eQ\frac{B}{A} \leq \omega \leq eQ\frac{B}{A}$$

$$\mathrm{Im}W_k = -\frac{\mathcal{T}}{2\pi}eQ\frac{B}{A}\log(2\cosh[2\pi\sqrt{\Delta}] - 2\cosh[2\pi eQ\frac{B}{A}]) - \frac{\mathcal{T}}{2\pi}\frac{1}{4\pi}\Big[-\mathrm{Li}_2(-\exp[-2\pi(\sqrt{\Delta} + eQ\frac{B}{A})])$$
$$+ \mathrm{Li}_2(-\exp[2\pi(\sqrt{\Delta} + eQ\frac{B}{A})]) - \mathrm{Li}_2(-\exp[2\pi(\sqrt{\Delta} - eQ\frac{B}{A})]) + \mathrm{Li}_2(-\exp[-2\pi(\sqrt{\Delta} - eQ\frac{B}{A})])\Big]$$

$$\mathrm{Li}_2(x) = \int_2^x \frac{dt}{\ln t}$$

The ζ-function method

$$\zeta_H(s) = \sum_{n=0}^{\infty} \frac{d_n}{\lambda_n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \mathrm{Tr} e^{-Hx} dx$$

Degeneration of λ_n

Eigenvalues of H

With kernel $K_H(x) = \mathrm{Tr} \exp^{-Hx}$

$$-\log \det H = \frac{d}{ds} \zeta_H(0)$$ defines the Euclidean effective action

Turning back to the Lorentzian signature → The instability is measured by the imaginary part of the effective action

$$W = \frac{1}{2} \zeta'_{\mu^2 - \not{p}^2}(0)$$

The eigenvalues

1. Compute the eigenvalues of $-\not{D}^2$ and add the mass square μ^2

2. Exploit the Kaluza-Klein reduction, for the 4D Dirac operator we have: $\not{D} = \not{E} + \not{F}$

3. The part $\not{E}$ depends only on variables of the first 2D factor of the metric, and $\not{F}$ only on the spherical variables of the 2-sphere factor

4. For the square of the operator we obtain: $-D^2 = -E^2 - F^2$

5. The eigenvalue λ^2 of D^2 is the sum of the eigenvalue ω^2 of $-E^2$ and of the eigenvalue b^2k^2 of $-F^2$: $\lambda^2 = \omega^2 + b^2k^2$

6. b is related to radius of the 2-sphere, in the Nariai case $b^2 = B$ and k is the eigenvalue for the angular momentum operator K.

7. Eigenfunctions for $-D^2 + \mu^2$ are tensor product of eigenfunction of $-E^2$ and of $-F^2$

Application to the Nariai manifold

The operator $\not{E}$ on the first part of the manifolds is:

$$\not{E} = \frac{\sqrt{A}}{\sin\chi}\tilde{\gamma}_0\left(\partial_\psi + ieE\cos\chi\right) + \sqrt{A}\tilde{\gamma}_1\left(\partial_\chi + \tfrac{1}{2}\cot\chi\right)$$

Its square, after some changing of variable, is:

$$E^2 = \frac{A}{\sin^2\chi}\left(\partial_\psi + ieE\cos\chi\right)^2 + A\partial_\chi^2 + A\tilde{\gamma}_0\tilde{\gamma}_1\left(\frac{\cos\chi}{\sin^2\chi}\left(\partial_\psi + ieE\cos\chi\right) + ieE\right)$$

For the eigenvalues problem of $-E^2$ we obtain the following couple of hypergeometric differential equations:

$$z(1-z)\frac{d^2 g_+(z)}{dz^2} + \left(eE - \omega + \frac{1}{2} - (2eE+1)z\right)\frac{dg_+(z)}{dz} + \frac{w^2}{A}g_+(z) = 0$$

$$z(1-z)\frac{d^2 g_-(z)}{dz^2} + \left(-eE + \omega + \frac{1}{2} + (2eE-1)z\right)\frac{dg_-(z)}{dz} + \frac{w^2}{A}g_-(z) = 0$$

Obtained with the position: $\eta_\pm(t) = (1-t)^{\frac{\pm eE \pm \omega}{2}}(1+t)^{\frac{\pm eE \mp \omega}{2}} g_\pm(t)$

Solution

We are looking for solutions $\begin{pmatrix} \eta_+ \\ \eta_- \end{pmatrix} \in L^2[(0,1), \frac{dz}{z(1-z)}]^2$ and this condition depends on ω

Defining the three parameters of the hypergeometric functions as:

$$a_+ = eE + \sqrt{\frac{w^2}{A} + (eE)^2} \qquad a_- = -eE + \sqrt{\frac{w^2}{A} + (eE)^2}$$

$$b_+ = eE - \sqrt{\frac{w^2}{A} + (eE)^2} \qquad b_- = -eE - \sqrt{\frac{w^2}{A} + (eE)^2}$$

$$c_+ = eE - \omega + \frac{1}{2} \qquad c_- = -eE + \omega + \frac{1}{2}$$

Three regions can be identified!

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{(c)_k k!},$$

$$|z| < 1 \vee |z| = 1 \wedge \Re(c-a-b) > 0$$

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

$-eE < \omega < eE$	$\omega > eE$	$\omega < -eE$
$g_+(z) = {}_2F_1(a_+, b_+; c_+; z)$ $g_-(z) = z^{1-c_-}(1-z)^{c_- - (a_- + b_-)}$ ${}_2F_1(1-a_-, 1-b_-; 2-c_-; z)$	$g_+(z) = z^{1-c_+}$ ${}_2F_1(a_+ - c_+ + 1, b_+ - c_+ + 1; 2-c_+; z)$ $g_-(z) = (1-z)^{c_- - (a_- + b_-)}$ ${}_2F_1(c_- - a_-, c_- - b_-; c_-; z)$	$g_+(z) = (1-z)^{c_+ - (a_+ + b_+)} {}_2F_1(c_+ - b_+, c_+ - a_+;$ $c_+ - (a_+ + b_+) + 1; 1-z)$ $g_-(z) = z^{1-c_-} {}_2F_1(1 + b_- - c_-, 1 + a_- - c_-;$ $a_- + b_- + 1 - c_-; 1-z)$
$\lambda^2 = A(eE+n)^2 - A(eE)^2 + \mu^2 + Bk^2$	$\lambda^2 = A(\omega + n + \frac{1}{2})^2 - A(eE)^2 + \mu^2 + Bk^2$	$\lambda^2 = A(-\omega + n + \frac{1}{2})^2 - A(eE)^2 + \mu^2 + Bk^2$

Definition of the ς-function

For the heat kernel we obtain:

$$K(s) = \sum_k g(k) K_k(s)$$
$$= \sum_k g(k) 2\frac{\mathcal{T}}{2\pi}\left\{2\int_{eE}^{\infty} d\omega \sum_{n=0}^{\infty} \exp\left[-A\left((\omega+\frac{1}{2}+n)^2+\frac{\mu_k^2}{A}-(eE)^2\right)s\right]\right.$$
$$\left. + 2eE\sum_{n=0}^{\infty}\exp\left[-A\left((eE+n)^2+\frac{\mu_k^2}{A}-(eE)^2\right)s\right] - eE\exp\left(-\mu_k^2 s\right)\right\}$$

$$\mu_k^2 = \mu^2 + Bk^2$$

The zeta function is:

$$\frac{1}{2}\zeta_k(s) = \frac{\mathcal{T}}{2\pi}\left[2\int_{eE}^{\infty}\sum_n \frac{d\omega}{A^s\left[\left(n+\frac{1}{2}+\omega\right)^2+\frac{\mu_k^2}{A}-(eE)^2\right]^s} + \right.$$
$$\left. + (2eE)\left(\sum_n \frac{1}{A^s\left[\left(n+eE\right)^2+\frac{\mu_k^2}{A}-(eE)^2\right]^s} - \frac{1}{2}\frac{1}{\mu_k^{2s}}\right)\right]$$

The imaginary part of the ς-function

After some long computations, summations, integrations and turning back to the Lorentzian signature through $eE \to ieE$ the derivative of the ς-function evaluated in zero is:

$$\frac{1}{2}\zeta'(0) = \frac{\mathcal{T}}{2\pi}\left\{2(eE)^2\log A - eE\log A + 2eE\log\frac{\Gamma(eE+i\beta)\Gamma(eE-i\beta)}{2\pi}\right.$$
$$+ eE\log(\mu_k^2) + (2+\log A)\left[\zeta_H(\alpha+i\beta,-1)+\zeta_H(\alpha-i\beta,-1)\right]$$
$$\left. - 2\left[\zeta'_H(\alpha+i\beta,-1)+\zeta'_H(\alpha-i\beta,-1)\right]\right\}$$

Its imaginary part is:

$$\frac{1}{2}\mathrm{Im}\zeta'_k(0) = \frac{\mathcal{T}}{2\pi}\left\{-eE\log\left(2\cosh[2\pi\sqrt{\Delta}] - 2\cosh[2\pi eE]\right)\right.$$
$$-\frac{1}{4\pi}\left[-\mathrm{Li}_2(-\exp[-2\pi(\sqrt{\Delta}+eE)]) + \mathrm{Li}_2(-\exp[2\pi(\sqrt{\Delta}+eE)])\right.$$
$$\left.\left.-\mathrm{Li}_2(-\exp[2\pi(\sqrt{\Delta}-eE)]) + \mathrm{Li}_2(-\exp[-2\pi(\sqrt{\Delta}-eE)])\right]\right\}$$

The same result as the one obtained with the transmission coefficient approach!!!

Finite temperature effects I

Nariai geometry describes a BH manifold with non zero temperature

↓

Quantum instability not simply for a vacuum state
(Boulware-like state of standard Schwarzschild solution)

↓

Thermal vacuum state of thermofield approach with the temperature equal to the BH temperature
(corresponds to Hartle-Hawking state for the given solution)

↓

Thermofield dynamics gives a straightforward generalization of quantum instability to the case where "in" and "out" states are thermal states (at the same temperature) instead than vacuum one

Finite temperature effects II

Strategy to check if there is instability in the thermal state at the Hawking temperature:

we evaluate the thermal mean of the number of "out" particle (in the k-mode) minus the number of "in" particles (in the k-mode)

↓

see the net effect of quantum instability

↓

In our case we are considering BH background with a single temperature so β is the inverse of BH temperature

Finite temperature effects III

The "out" creator and annihilator operators are given by:

$$a_l^{out} = \mu_l \; a_l^{in} + \nu_l \; (b_l^{in})^\dagger$$ (Bogoliubov transformation)

$$b_l^{out} = \mu_l \; b_l^{in} - \nu_l \; (a_l^{in})^\dagger$$ CCR for fermions leads to: $|\mu_l|^2 + |\nu_l|^2 = 1$

Introduce thermal state operators $a_l(\beta), \tilde{a}_l(\beta), b_l(\beta), \tilde{b}_l(\beta)$ and thermal state $|O(\beta) >$ s.t.:

$$a_l(\beta)|O(\beta) >= \tilde{a}_l(\beta)|O(\beta) >= b_l(\beta)|O(\beta) >= \tilde{b}_l(\beta)|O(\beta) >= 0$$ "in" and "out"

Between standard state operators and thermal state operators the following relations hold:

$$a_l = s_l^+ a_l(\beta) + c_l^+ \tilde{a}_l^\dagger(\beta)$$

$$b_l = s_l^- b_l(\beta) + c_l^- \tilde{b}_l^\dagger(\beta)$$

$$c_l^+ := \frac{1}{\sqrt{1+\exp[\beta(\omega-\varphi^+)]}} \qquad c_l^- := \frac{1}{\sqrt{1+\exp[\beta(|\omega|+\varphi^-)]}}$$

$$s_l^+ := \frac{\exp[\frac{1}{2}\beta(\omega-\varphi^+)]}{\sqrt{1+\exp[\beta(\omega-\varphi^+)]}} \qquad s_l^- := \frac{\exp[\frac{1}{2}\beta(|\omega|+\varphi^-)]}{\sqrt{1+\exp[\beta(|\omega|+\varphi^-)]}}$$

With φ^+ and φ^- chemical potential for particle and antiparticle respectively

Nariai and finite temperature effects

Define:
$$\bar{N}_l^{out}(a) := (a_l^{out})^\dagger a_l^{out} - (a_l^{in})^\dagger a_l^{in}$$
$$= |\nu_l|^2(1 - (c_l^+)^2 - (c_l^-)^2)$$
$$= |\nu_l|^2 \frac{1}{2}\left(\tanh[\frac{1}{2}\beta(\omega - \varphi^+)] + \tanh[\frac{1}{2}\beta(|\omega| + \varphi^-)]\right)$$

↓ For the Nariai geometry

$$< \bar{N}_k^{out} >_\beta = |T_k(\omega)|^2 \frac{1}{2}\left(\tanh[\frac{1}{2}\beta(\omega - \varphi^+)] + \tanh[\frac{1}{2}\beta(|\omega| + \varphi^-)]\right)$$

where φ^+ is assumed for definiteness to be the chemical potential for particles in the case of a positively charged black hole, $\varphi^- = \varphi^+$, particles are electrons with charge -e and:

$$\varphi^+ = -e(A_0|_\pi - A_0|_0) = -2eQ\frac{B}{A}$$

In terms of physical (dimensionful) variables, by taking into account that $T_h = \dfrac{\hbar c\sqrt{A}}{2\pi k_b}$, and that $\omega_{phys} = \sqrt{A}\omega$, in such a way that $\beta_{phys}\omega_{phys} = 2\pi\omega$.

Conclusion

- We studied spontaneous emission of charged Dirac particles by the Nariai BH solution;
- the particular geometry allows an exact computation;
- the two different approaches give the same result;
- we performed the same exact computation also for other two geometries (ultracold I and II) obtaining perfect accord between the ζ-function approach and the transmission coefficient approach;
- we made analogous computation for the scalar case

References:

• Pair-production of charged Dirac particles on charged Nariai and ultracold black hole manifolds.
F. Belgiorno, (Milan U.) , S.L. Cacciatori, F. Dalla Piazza, (Insubria U., Como & INFN, Milan).
Jun 2009. 27pp. Published in JHEP 0908:028,2009. e-Print: arXiv:0906.1520 [gr-qc].

• Quantum instability for charged scalar particles on charged Nariai and ultracold black hole manifolds.
F. Belgiorno, (Milan U., Math. Dept.) , S.L. Cacciatori, F. Dalla Piazza, (Insubria U., Como & INFN, Milan).
Sep 2009. 20pp. Published in Class.Quant.Grav.27:055011,2010. e-Print: arXiv:0909.1454 [gr-qc]

Heuristically...

Let consider the eigenproblem: $(\not{D} - \mu)\Psi = \lambda\Psi$

Then we have $(\lambda_\pm + \mu) = \pm\sqrt{\not{D}^2}$ and thus we can formally write:

$$\begin{aligned}\log(\det(\not{D} - \mu)) &= \frac{1}{2}\log\left(\det\left(-\mu + \sqrt{\not{D}^2}\right)\right) + \frac{1}{2}\log\left(\det\left(-\mu - \sqrt{\not{D}^2}\right)\right) \\ &= \frac{1}{2}\log\left(\det\left(\mu^2 - \not{D}^2\right)\right)\end{aligned}$$

The factor 1/2 arises from the double degeneration of each eigenvalue, if $(\not{D} - \mu)\Psi_\pm = \lambda_\pm\Psi_\pm$ then, for example, $\left(-\mu + \sqrt{\not{D}^2}\right)\Psi_\pm = \lambda_+\Psi_\pm$.

It is convenient to define: $-\log\left(\det\left(\sqrt{\not{D}^2} - \mu\right)\right) = \frac{1}{2}\zeta'_{\mu^2 - \not{D}^2}(0)$

and then for the Euclidean effective action we get:

$$W = \frac{1}{2}\zeta'_{\mu^2 - \not{D}^2}(0)$$