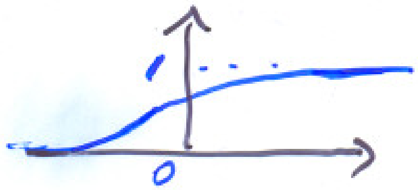
Linear response for σ_H in

$$I = \sigma_H V_H$$

- $\Lambda_0(\cdot)$ a switch function



$$\Lambda_i = \Lambda_0(x_i) \quad , \quad \Lambda = (\Lambda_1, \Lambda_2)$$

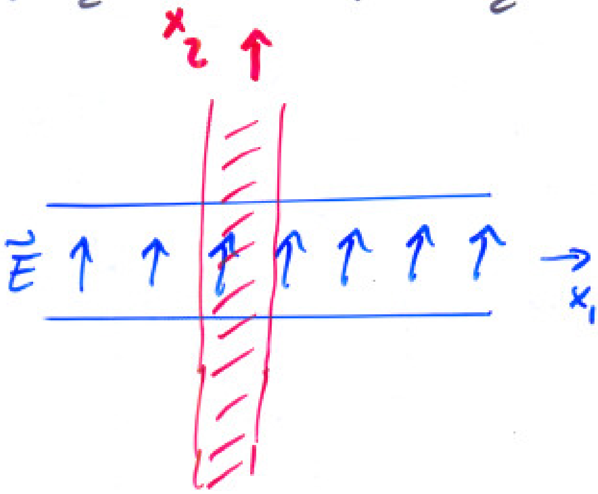
- Consider

$$H(\varphi) = e^{-i\varphi\cdot\Lambda} H e^{i\varphi\cdot\Lambda}$$

- $$\left.\frac{\partial H}{\partial \varphi_1}\right|_{\varphi=0} = i[H, \Lambda_1] = I$$

- $\varphi_1 \equiv 0$, $\varphi_2 = \varphi_2(t)$ describes external field

$$(E_1, E_2) = (0, -\Lambda_2') \quad \to V_H = 1$$



field confined to strip

I : current across this strip

$\to$ expect $\langle I \rangle$ finite

Result: $\sigma_H = i \operatorname{tr}(P[[P,\Lambda_1],[P,\Lambda_2]])$

The index of a pair of projections

P, Q orthogonal projections on a Hilbert space
$P - Q$ compact

<u>Definition</u>

$$\mathrm{Ind}(P,Q) = \dim \{\psi \mid P\psi = \psi,\ Q\psi = 0\} - \dim \{\psi \mid Q\psi = \psi,\ P\psi = 0\}$$

- "compares dimensions"
- $\mathrm{Ind}(P,Q) = \mathrm{Ind}(P,R) + \mathrm{Ind}(R,Q)$
- $\|P-Q\| < 1 \Rightarrow \mathrm{Ind}(P,Q) = 0$
- If $(P-Q)^{2n+1}$ is trace class, then
  $$\mathrm{Ind}(P,Q) = \mathrm{tr}\,(P-Q)^{2n+1}$$
- In particular, if P, Q finite dimensional
  $$\mathrm{Ind}(P,Q) = \mathrm{tr}\,P - \mathrm{tr}\,Q = \dim \mathrm{Ran}\,P - \dim \mathrm{Ran}\,Q$$
- Hint:
  $$(P-Q) - (P-Q)^3 = [PQ, QP]$$

- One of Maxwell's equation (Faraday)

$$\underbrace{\frac{d}{dt}\int_S \vec{B}\cdot d\vec{\sigma}}_{\text{flux } \phi} = -\oint_{\partial S} \vec{E}\cdot d\vec{s}$$

S: surface

- $$\phi = \int_S \operatorname{rot}\vec{A}\cdot d\vec{\sigma} = \oint_{\partial S} \vec{A}\cdot d\vec{s}$$

Proposal: When the flux is increased by $\phi = 2\pi$ over a time t_0 (large), the transported charge is

$$Q = \mathrm{Ind}\,(U(t_0,0)\underbrace{P}_{\text{for } \phi=0}U(t_0,0)^*,\ \underbrace{UPU^*}_{\text{for } \phi=2\pi})$$

Fermi seas: for $\phi = 0$ for $\phi = 2\pi$

$$= \underbrace{\mathrm{Ind}\,(U(t_0,0)PU(t_0,0)^*, P)}_{=0} + \mathrm{Ind}\,(P, UPU^*)$$

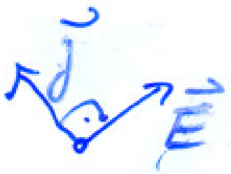
Resulting definition of Hall conductance:

$$2\pi\sigma_F := \mathrm{Ind}\,(P, UPU^*)$$

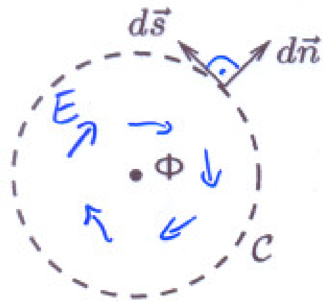
- naturally an integer!
- $= \mathrm{tr}\,(P - UPU^*)^3$

(uses $U = e^{i \arg \vec{x}}$)

IQHE from Flux insertion



$(\sigma_H > 0)$



Flux increase from 0 to Φ
Charge Q traversing C

$$\frac{dQ}{dt} = \oint_C \vec{\jmath} \cdot d\vec{n} = -\sigma_{\mathrm{H}} \oint_C \vec{E} \cdot d\vec{s} = \sigma_{\mathrm{H}} \frac{d\Phi}{dt}$$

$$Q = \sigma_{\mathrm{H}} \Phi$$

Charge Q, according to quantum mechanics:
Flux Φ generated by a gauge potential $\vec{A}$:

$$\oint_C \vec{A} \cdot d\vec{s} = \Phi, \text{ e.g. } \vec{A} = \vec{\nabla}\Big(\frac{\Phi}{2\pi} \arg \vec{x}\Big) \equiv \vec{\nabla}\chi$$

If $\chi(\vec{x})$ were single-valued:

gauge	$\vec{A} = 0$	equiv. to	$\vec{A} = \vec{\nabla}\chi$
	$\downarrow$		$\downarrow$
Hamiltonian	H_B		$U H_B U^*$

with $U = e^{i\chi}$, unitary. For $\Phi = 2\pi$, U is single-valued, though $\chi = \arg \vec{x}$ is not.

~~Fermi projection: $P_\mu = E_{(-\infty,\mu)}(H_B)$~~
~~So,~~

$$\cancel{2\pi\sigma_{\mathrm{F}} = \text{"}\operatorname{tr}(P_\mu - U P_\mu U^*)\text{"}}$$

~~(non existent) counts difference in number of electrons in the Fermi seas of $U H_B U^*$ and H_B.~~

- Earlier theorem (Bellissard, von Elst, Schulz-Baldes): for ergodic random Schrödinger operators; definition of σ_H included ensemble average; proof based on non-commutative geometry
- above deterministic version
  - spectral gap case (Avron, Seiler, Simon)
  - mobility gap case (Elgart, G., Schenker)

<u>Theorem</u> Let H be a Schrödinger operator and P the spectral projection associated to $(-\infty, \mu]$, where

- $\mu \notin \sigma(H)$ (i.e. μ lies in spectral gap)

or

- $\mu \in \sigma(H)$ and μ lies in mobility gap

Then

$$\sigma_H = \sigma_F \, ;$$

explicitely

$$i \, tr \, P[[\Lambda_1, P], [\Lambda_2, P]] = \frac{1}{2\pi} \, \mathrm{Ind}\,(P, UPU^*)$$
$$= \frac{1}{2\pi} \, tr \, (P - UPU^*)^3 \qquad (*)$$

where Λ_i are switch functions,

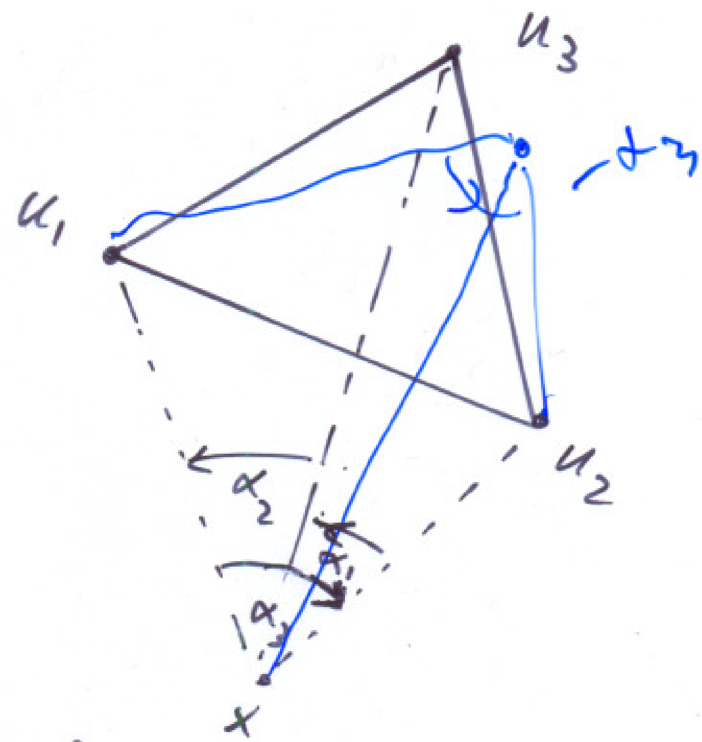
$$U(x) = e^{i \arg x}$$

In particular, $2\pi \sigma_H \in \mathbb{Z}$.

Note: both sides of $(*)$ exhibit 3 powers of P

Core of proof:

<u>Lemma</u> Let $u_i \in \mathbb{R}^2$ $(i=1,2,3)$



$$\int_{\mathbb{R}^2} d^2x \sum_{i=1}^{3} \sin \alpha_i(x) = 2\pi \underbrace{\mathrm{Area}(u_1, u_2, u_3)}_{\text{oriented area of triangle}}$$

(Lemma due to Connes, Non-commutative geometry)