Linear response for σ_H in

$$j_1 = \sigma_H E_2 \quad :$$

- Consider

$$H(\varphi) = \frac{1}{2}(p - A(x) + \varphi_1 e_1 + \varphi_2 e_2)^2 + V(x)$$

$$= e^{-i\varphi \cdot x} H e^{i\varphi \cdot x} \quad \leftarrow \text{more general}$$

$\varphi = (\varphi_1, \varphi_2)$; e_i : unit vector in i-direction

- $$\left.\frac{\partial H}{\partial \varphi_1}\right|_{\varphi=0} = \int j_1 \, d^2x$$

- $\varphi_1 \equiv 0$, $\varphi_2 = \varphi_2(t)$ describes external field

$$(E_1, E_2) = (0, \dot{\varphi}_2)$$

- Result :

$$\sigma_H = -i \, tr'(P[\partial_1 P, \partial_2 P])$$

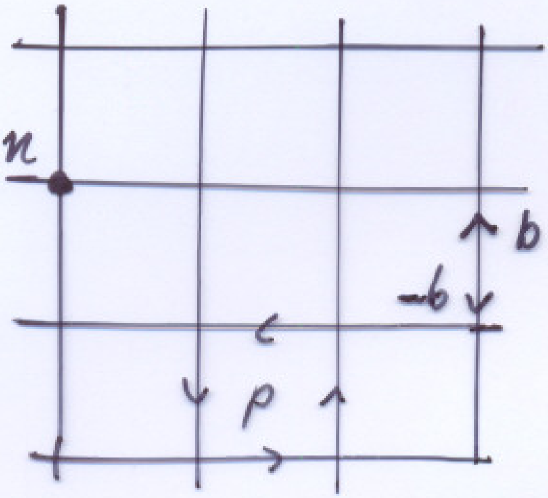
$$= i \, tr'(P[[P, x_1], [P, x_2]])$$

with $\partial_i = \partial / \partial \varphi_i$

tr' = trace per unit volume

<u>Example</u> The Harper model
(~ Landau Hamiltonian on lattice)

$$\mathcal{H} = \ell^2(\mathbb{Z}^2) \ni \psi = \{\psi(n)\}_{n \in \mathbb{Z}^2}$$



sites n
bonds b

$e(b)$ end, $i(b)$ initial } point of b

plaquette p.

$$(\psi_2, H\psi_1) := \sum_b e^{i\varphi(b)}\, \overline{\psi_2(e(b))}\, \psi_1(i(b))$$

with $\varphi(-b) = -\varphi(b)$; defines $H = H^* : \mathcal{H} \to \mathcal{H}$

Let

$$\sum_{b \in \partial p} \varphi(b) =: 2\pi\phi \qquad (\mathrm{mod}\ 2\pi)$$

be <u>independent</u> of p (constant magnetic field)

ϕ fixes $\varphi(b)$ and hence H up to unitary equivalence : $H(\phi)$

$\to$ $\sigma(H(\phi))$ depends on $0 \le \phi \le 1$

$H(\phi) \stackrel{\sim}{=} -H(\phi)$ (bipartite lattice)

$\overline{H(\phi)} = H(-\phi)$

$\rightarrow \quad \sigma(H(\phi)) = -\sigma(H(\phi)) = \sigma(H(1-\phi))$

If $\phi = \frac{p}{q}$, $(p, q \in \mathbb{Z})$, then $H(\phi)$ is periodic with unit cell $\Pi = \{q \text{ sites}\}$

$\rightarrow$ fiber $H(\phi, k)$ acts on $\ell^2(\Pi)$

dim $= q$

$\hookrightarrow$ q eigenvalues

$\rightarrow$ $H(\phi)$ has q bands

(some gaps may close; central gap does when q even

Linear response for σ_H in

$$I = \sigma_H V_H$$

- $\Lambda_0(\cdot)$ a switch function

$$\Lambda_i = \Lambda_0(x_i) \quad , \quad \Lambda = (\Lambda_1, \Lambda_2)$$

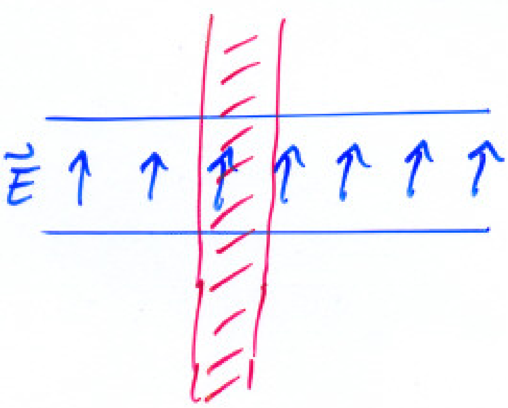
- Consider

$$H(\varphi) = e^{-i\varphi\cdot\Lambda} H e^{i\varphi\cdot\Lambda}$$

- $$\left.\frac{\partial H}{\partial \varphi_1}\right|_{\varphi=0} = i[H, \Lambda_1] = I$$

- $\varphi_1 \equiv 0$, $\varphi_2 = \varphi_2(t)$ describes external field

$$(E_1, E_2) = (0, -\Lambda_2') \quad \rightarrow V_H = 1$$



field confined to strip

I : current across this strip

$\rightarrow$ expect $\langle I \rangle$ finite

Result: $\quad \sigma_H = i\,\mathrm{tr}\left(P[[P,\Lambda_1],[P,\Lambda_2]]\right)$

The index of a pair of projections

P, Q orthogonal projections on a Hilbert space
$P - Q$ compact

<u>Definition</u>

$$\mathrm{Ind}(P,Q) = \dim\{\psi \mid P\psi = \psi,\ Q\psi = 0\}$$
$$\qquad - \dim\{\psi \mid Q\psi = \psi,\ P\psi = 0\}$$

- "compares dimensions"
- $\mathrm{Ind}(P,Q) = \mathrm{Ind}(P,R) + \mathrm{Ind}(R,Q)$
- $\|P-Q\| < 1 \Rightarrow \mathrm{Ind}(P,Q) = 0$
- If $(P-Q)^{2n+1}$ is trace class, then
  $$\mathrm{Ind}(P,Q) = \mathrm{tr}\,(P-Q)^{2n+1}$$
- In particular, if P, Q finite dimensional
  $$\mathrm{Ind}(P,Q) = \mathrm{tr}\,P - \mathrm{tr}\,Q$$
  $$= \dim \mathrm{Ran}\,P - \dim \mathrm{Ran}\,Q$$
- Hint:
  $$(P-Q) - (P-Q)^3 = [PQ, QP]$$

- One of Maxwell's equation (Faraday)

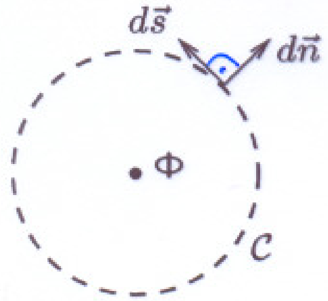
$$\underbrace{\frac{d}{dt}\int_S \vec{B}\cdot d\vec{\sigma}}_{\text{flux } \phi} = -\oint_{\partial S} \vec{E}\cdot d\vec{s}$$

S: surface

- $$\phi = \int_S \operatorname{rot}\vec{A}\cdot d\vec{\sigma} = \oint_{\partial S} \vec{A}\cdot d\vec{s}$$

IQHE from Flux insertion

$\vec{j}\ \ \vec{E}\quad (\sigma_H > 0)$



Flux increase from 0 to Φ
Charge Q traversing C

$$\frac{dQ}{dt} = \oint_C \vec{j} \cdot d\vec{n} = -\sigma_{\mathrm{H}} \oint_C \vec{E} \cdot d\vec{s} = \sigma_{\mathrm{H}} \frac{d\Phi}{dt}$$
$$Q = \sigma_{\mathrm{H}} \Phi$$

Charge Q, according to quantum mechanics:
Flux Φ generated by a gauge potential $\vec{A}$:

$$\oint_C \vec{A} \cdot d\vec{s} = \Phi, \text{ e.g. } \vec{A} = \vec{\nabla}\Big(\frac{\Phi}{2\pi} \arg \vec{x}\Big) \equiv \vec{\nabla}\chi$$

If $\chi(\vec{x})$ were single-valued:

gauge	$\vec{A} = 0$	equiv. to	$\vec{A} = \vec{\nabla}\chi$
	$\downarrow$		$\downarrow$
Hamiltonian	H_B		$U H_B U^*$

with $U = e^{i\chi}$, unitary. For $\Phi = 2\pi$, U is single-valued, though $\chi = \arg \vec{x}$ is not.

Fermi projection: $P_\mu = E_{(-\infty,\mu)}(H_B)$
So,

$$2\pi\sigma_{\mathrm{F}} = \text{"}\,\mathrm{tr}(P_\mu - U P_\mu U^*)\text{"}$$

(non existent) counts difference in number of electrons in the Fermi seas of $U H_B U^*$ and H_B.

Proposal: When the flux is increased by $\phi = 2\pi$ over a time t_0 (large), the transported charge is

$$Q = \mathrm{Ind}\,\big(U(t,0)\underbrace{P}_{\text{Fermi seas: for } \phi=0}U(t,0)^*,\ \underbrace{UPU^*}_{\text{for } \phi=2\pi}\big)$$

$$= \underbrace{\mathrm{Ind}\,(U(t,0)PU(t,0)^*, P)}_{=0} + \mathrm{Ind}\,(P, UPU^*)$$

Resulting definition of Hall conductance:

$$2\pi\,\sigma_F := \mathrm{Ind}\,(P, UPU^*)$$

- naturally an integer!
- $= \mathrm{tr}\,(P - UPU^*)^3$

  (uses $U = e^{i \arg \vec{x}}$)