Units: $m = \hbar = qB/2c = 1$

Landau Hamiltonian: On $L^2(\mathbb{R}^2)$

$$H = \frac{1}{2}(-\Delta + \vec{x}^2) - L$$

with $L = -i(x\partial_y - y\partial_x)$

Claims:

(1) $\sigma(H) = \{ 2k+1 \mid k = 0, 1, 2, \dots \}$

(2) each eigenvalue is ∞-degenerate with density:

π^{-1} eigenstates per unit area

<u>Sketch</u> of proof: Let $z = x + iy$ and

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$$

$$a = \partial_{\bar{z}} + \frac{z}{2} = e^{-|z|^2/2}\, \partial_{\bar{z}}\, e^{|z|^2/2}$$

Then

$$a^* = -\partial_z + \frac{\bar{z}}{2}, \quad [a, a^*] = 1$$

$$H = 2a^*a + 1$$

$\Rightarrow$ spectrum as claimed

Landau levels $\equiv$ corresponding eigenspaces

Landau level $k=0$: $\psi \in L^2(\mathbb{R}^2)$ such that

$$a\psi = 0,$$

where $a = e^{-|z|^2/2} \partial_{\bar{z}} e^{|z|^2/2}$. Hence

$$\psi(z) = f(z)\, e^{-|z|^2/2}$$

f analytic $(\partial_{\bar{z}} f = 0)$

Orthonormal states for LL $k=0$:

$$\psi_{0m}(z) = (\pi m!)^{-1/2} z^m e^{-|z|^2/2}$$

$(m = 0, 1, 2, \dots)$;

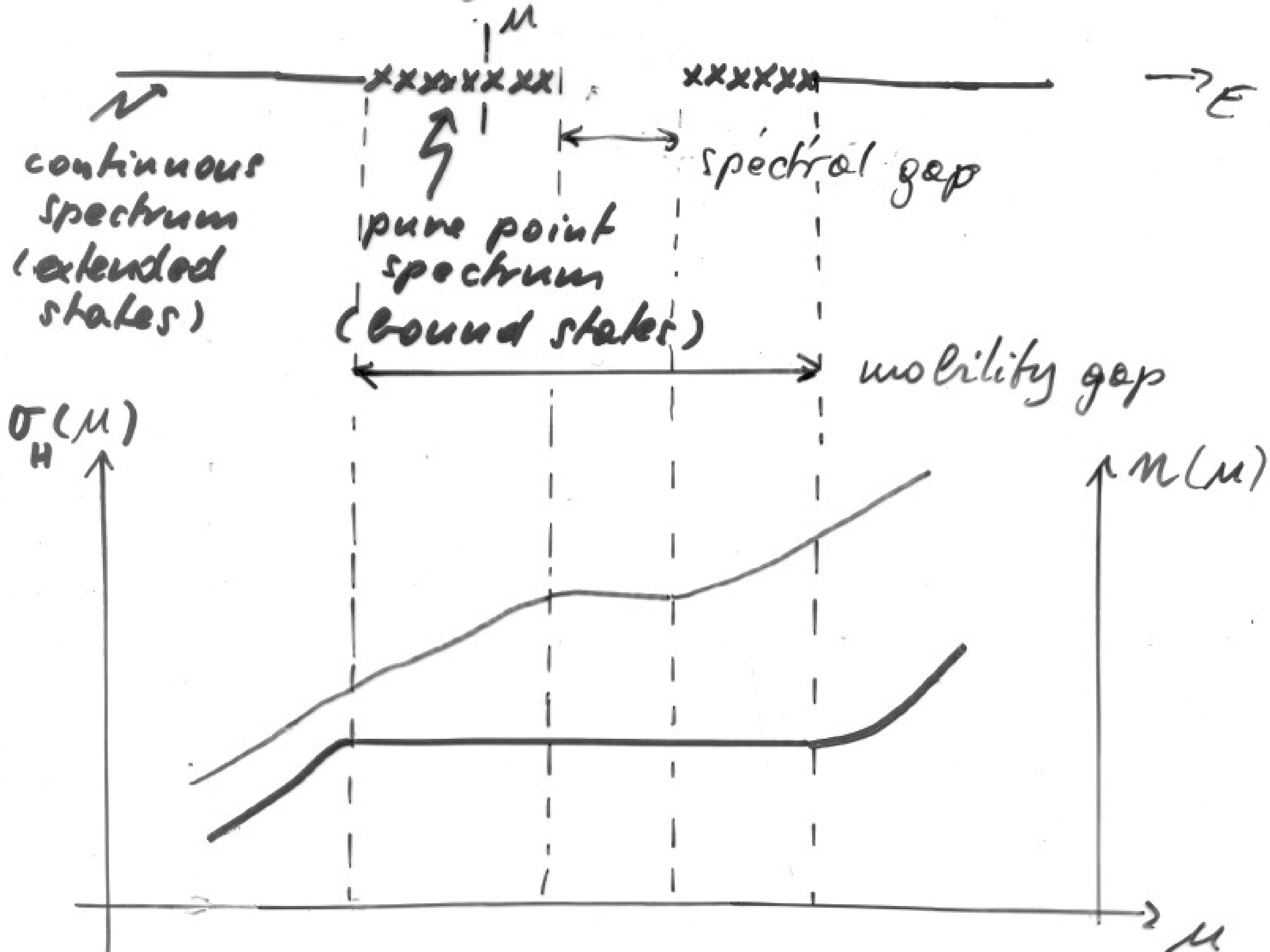
for LL k

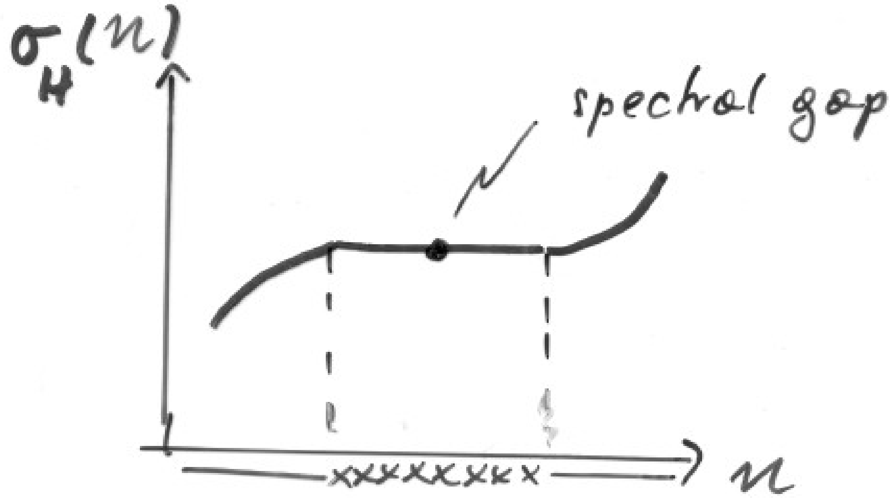
$$\psi_{km} = (k!)^{-1/2} (a^*)^k \psi_{0m} .$$

Density of eigenstates at $z = x + iy$

$$\sum_{m=0}^{\infty} |\psi_{km}(z)|^2 = \pi^{-1}$$

Spectrum of single-particle Hamiltonian





plateaux arise because $\sigma_H(\mu)$ is constant for μ in a mobility gap, not a spectral gap

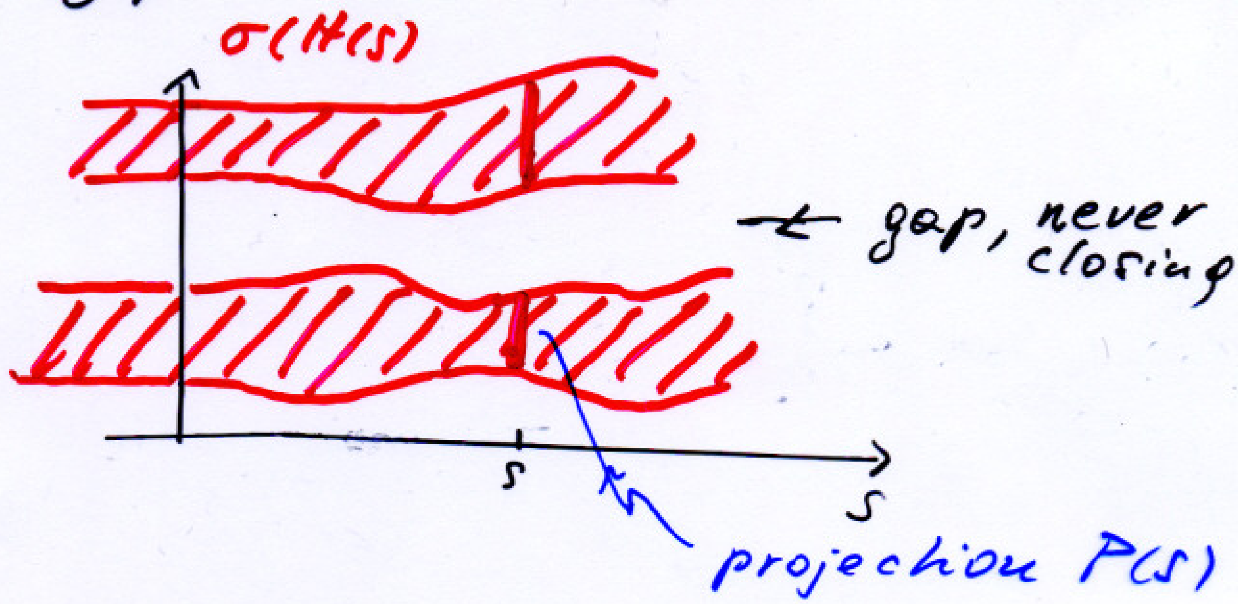
The adiabatic theorem of quantum mechanics

- Time-dependent Hamiltonian, slowly varying

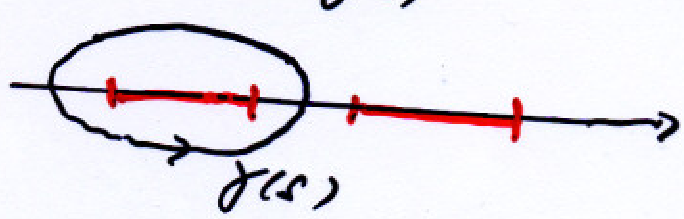
$$H(s) \quad , \quad s = \varepsilon t \quad (\varepsilon > 0, \text{small})$$

(independent of s for $s \le 0$)

- Spectrum $\sigma(H(s))$ has a part separated by a gap



$$P(s) = -\frac{1}{2\pi i} \int_{\gamma(s)} R(z,s)\, dz$$



$$R(z,s) = (H(s) - z)^{-1} \quad \text{resolvent}$$

Time-dependent Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = H(\varepsilon t)\,\psi(t) \quad , \quad \frac{\partial}{\partial t} = \varepsilon \frac{\partial}{\partial s}$$

with propagator $U_\varepsilon(s,s')$

$$i\varepsilon \frac{\partial}{\partial s} U_\varepsilon(s,s') = H(s)\, U_\varepsilon(s,s') \quad , \quad U_\varepsilon(s,s) = 1$$

- initial state $P(0)$ evolves into

$$U_\varepsilon(s,0)\, P(0)\, U_\varepsilon(s,0)^*$$

<u>Theorem</u>. Under the above assumptions

$$U_\varepsilon(s,0)\, P(0)\, U_\varepsilon(s,0)^* = P(s) + \varepsilon \tilde{P}(s) + O(\varepsilon^2)$$
$$(\varepsilon \to 0)$$

with

$$\tilde{P}(s) = -\frac{1}{2\pi} \oint_{\gamma(s)} R(z,s)\,[\dot{P}(s), P(s)]\, R(z,s)$$

(Kato ; Avron, Seiler, Yaffe)

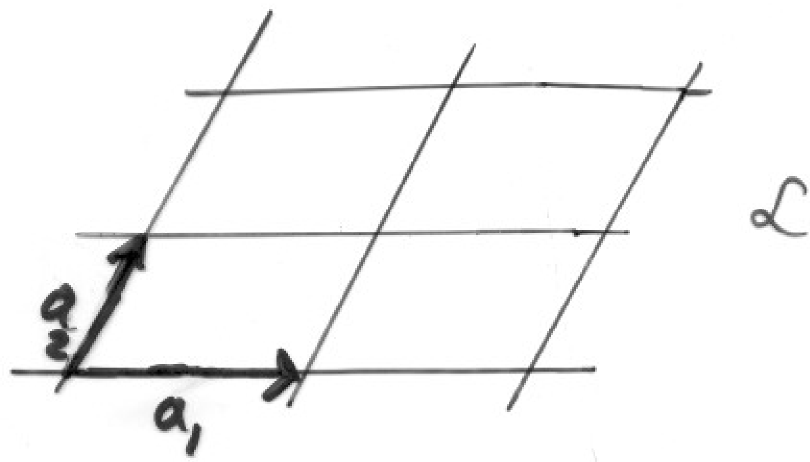
Block theory ($\vec{B}=0$, at first)

Hamiltonian with periodic potential

$$H = \frac{1}{2}p^2 + V(x) = -\frac{1}{2}\Delta + V(x) \quad \text{on } \mathcal{H} = L^2(\mathbb{R}^2)$$

with $V(x+a_i) = V(x)$

$a_1, a_2 \in \mathbb{R}^2$, linearly independent



lattice $\mathcal{L} = \{\sum_{i=1}^{2} n_i a_i \mid n_i \in \mathbb{Z}\}$

torus $\mathbb{T} = \mathbb{R}^2/\mathcal{L}$ (unit cell with opposite faces glued)

dual lattice $\mathcal{L}^* = \{\sum_{i=1}^{2} n_i b_i \mid n_i \in \mathbb{Z}\}$

$$a_i \cdot b_j = 2\pi \delta_{ij}$$

$$\mathbb{T}^* = \mathbb{R}^2/\mathcal{L}^* \quad \text{(Brillouin zone)}$$

Translation operators $T_i : \mathcal{H} \to \mathcal{H}$ (unitaries)

$$\psi \mapsto T_i \psi$$

$$(T_i \psi)(x) = \psi(x - a_i) \qquad (i = 1, 2)$$

$$[H, T_i] = 0$$

$$[T_1, T_2] = 0$$

"Eigenstates" of T_1, T_2 :

$$(T_i \psi)(x) = e^{-ik\cdot a_i} \psi(x) \ , \quad (k \in \mathbb{T}^*)$$

$$\begin{cases} \psi(x) = u_k(x) e^{ikx} \\ T_i u_k = u_k \qquad (u_k(x) \text{ periodic}) \end{cases}$$

Fourier decomposition of $\psi \in \mathcal{H}$

$$\psi(x) = \int_{\mathbb{T}^*} đk \ u_k(x) e^{ikx} \qquad \left(đk = \frac{d^2k}{(2\pi)^2}\right)$$

$$(\psi_1, \psi_2) = \int d^2x \ \overline{\psi_1(x)} \psi_2(x)$$

$$= \int_{\mathbb{T}^*} đk \ \langle u_{1k}, u_{2k} \rangle$$

$$\langle u_1, u_2 \rangle = \int_{\mathbb{T}} d^2x \ \overline{u_1(x)} u_2(x)$$

$$L^2(\mathbb{R}^2) \cong \mathbb{T}^* \times L^2(\mathbb{T})$$

(trivial) vector bundle with base $\mathbb{T}^*$ and fiber $L^2(\mathbb{T})$

Then

$$(H\psi)(x) = \int_{\mathbb{T}^*} dk \, (H(k)u_k)(x)\, e^{ikx}$$

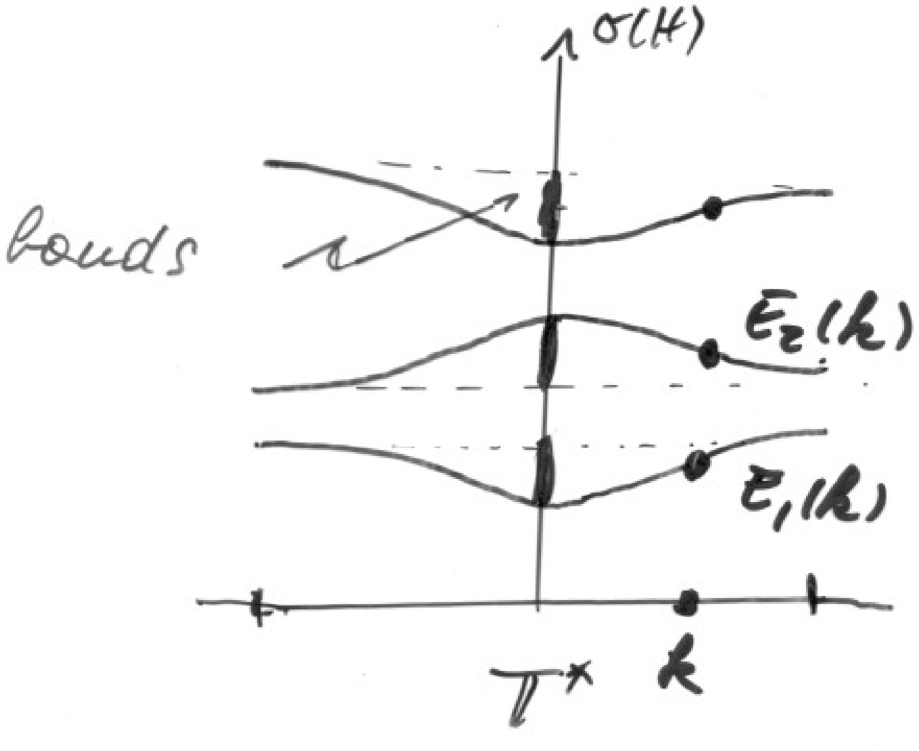
for some operator $H(k) : L^2(\mathbb{T}) \to L^2(\mathbb{T})$

$H(k)$ has discrete spectrum

$$E_1(k) \leq E_2(k) \leq \dots$$

H has continuous spectrum

$$\sigma(H) = \{ E_i(k) \mid i = 1, 2, \dots, k \in \mathbb{T}^* \}$$



P spectral projection associated to a separated part of $\sigma(H)$

Since $[P, T_i] = 0$,

$$(P\psi)(x) = \int_{\mathbb{T}^*} dk \, (P(k) u_k](x) e^{ikx}$$

Recall the context of Thouless' formula

$$H(\varphi) = \frac{1}{2}(p + \varphi_1 e_1 + \varphi_2 e_2)^2 + V$$

$$= e^{-i\varphi x} H e^{i\varphi x} \qquad (\varphi \in \mathbb{R}^2)$$

since $\quad p_i e^{i\varphi x} = e^{i\varphi x}(p_i + \varphi_i)$

Similarly, $\quad P(\varphi) = e^{-i\varphi x} P e^{i\varphi x}$

$$(P(\varphi)\psi)(x) = \int_{\mathbb{T}^*} dk \, (P(k+\varphi) u_k](x) e^{ikx}$$

i.e. $\quad P(\varphi, k) = P(k+\varphi)$

$$\frac{\partial}{\partial \varphi_i} P(\varphi, k) \Big|_{\varphi = 0} = \partial_i P(k)$$

But: For $B=0$ we have $\sigma_H = 0$

Proof. $(H - E(k))(u_k(x) e^{ikx}) = 0$

Since H has **real coefficents**, $\overline{H} = H$

$$(H - E(k))(\overline{u_k(x)}\, e^{-ikx}) = 0$$

$$\rightarrow \quad u_{-k}(x) = \overline{u_k(x)}$$

$$u(-k) = \overline{u(k)} \qquad (\text{as } \in L^2(\mathbb{T}^*))$$

$$(\partial_\alpha u)(-k) = -\overline{(\partial_\alpha u)(k)}$$

$$\begin{aligned} v_\alpha(-k) &= \langle u(-k), (\partial_\alpha u)(-k) \rangle \\ &= -\langle \overline{u(k)}, \overline{(\partial_\alpha u)(k)} \rangle \\ &= -\langle (\partial_\alpha u)(k), u(k) \rangle \\ &= + \langle u(k), (\partial_\alpha u)(k) \rangle = v_\alpha(k) \end{aligned}$$

$\partial_1 v_2 - \partial_2 v_1$ odd in k

$$Ch(P) = -\frac{i}{2\pi} \int_{\mathbb{T}^*} (\partial_1 v_2 - \partial_2 v_1)\, dk_1\, dk_2 = 0$$

Bloch theory ($B \neq 0$, const)

$$H(A) = \frac{1}{2}(p - A(x))^2 + V(x)$$

$$A(x) = \frac{1}{2} B \wedge x = \frac{B}{2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

Let $V(x)$ be $\mathcal{L}$-periodic. Nevertheless

$$[H, T_i] \neq 0$$

$$T_i^{-1} H T_i = \frac{1}{2}(p - \underbrace{A(x + a_i)}_{A(x) + \frac{1}{2} B \wedge a_i})^2 + \underbrace{V(x + a_i)}_{V(x)}$$

$$\neq H$$

Gauge transformations:

$A + \nabla\chi$ ($\chi = \chi(x)$ arb.) and A

- generate same $B = \operatorname{rot}(A + \nabla\chi) = \operatorname{rot} A$ (since $\operatorname{rot} \nabla\chi = 0$)

- associated Hamiltonians are unitarily equivalent:

$$e^{-i\chi} H(A + \nabla\chi) e^{i\chi} = H(A)$$

since $e^{-i\chi}(p_i - A_i - \partial_i \chi) e^{i\chi} = p_i - A_i$

$$\tfrac{1}{2} B \wedge a_i = D\left(\tfrac{1}{2}(B \wedge a_i)\cdot x\right)$$

$$\tilde{T}_i := T_i \, e^{\frac{i}{2}(B \wedge a_i)\cdot x} \qquad (i = 1,2)$$

$$\rightarrow \quad \tilde{T}_i^{-1} H(A) \tilde{T}_i = H(A).$$

Moreover,

$$\tilde{T}_2 \tilde{T}_1 = e^{-iB\underbrace{(a_1 \wedge a_2)}_{\substack{\text{area} \\ \text{of unit cell}}}} \tilde{T}_1 \tilde{T}_2$$

Hence

$$[\tilde{T}_1, \tilde{T}_2] = 0$$

if <u>flux</u> $B(a_1 \wedge a_2)$ through unit cell $\in 2\pi\mathbb{Z}$.

(If $\in 2\pi\mathbb{Q}$, i.e. $= 2\pi \frac{p}{q}$, just redefine unit cell)

Conclusion: As before, $2\pi\sigma_H$ is a Chern number. In general $\neq 0$,

since $\overline{H(A)} = H(-A)$