GEOMETRIC APPROACH TO THE HÉNON-HEILES MODEL

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RESUMEN

As it is well known, the Hénon-Heiles model was introduced in 1964 with the purpose of studying the appearance of chaos in the motion of stars end galaxies. One of the features of this model is that it is not integrable. This is due to the fact there are not enough constants of the motion. In this contribution we analyze this situation within the framework of Eisenhart's geometric formulation of classical mechanics, based on the equivalence between the Lagrange equations of motion and the geodesic equations of a suitable riemannian manifold. This approach will enable us to make use of geometric tools to study the symmetries, and the associated constants of the motion, of mechanical systems. In particular wi will focus our attention on the Killing vectors fields, affine collineations, killing tensors and their associated conserved quantities.

INTRODUCCION

Once that General Relativity gained popularity, it was quite natural for theoretical physicists to become interested in the geometrization of mechanics. Therefore, the attempts to achieve a Riemannian formulation of classical dynamics date back to the first decades of the past century. The general idea of geometrization is based on the observation that the trajectories of classical systems in configuration space can be viewed geodesics of a suitable Riemannian manifold.

Hamilton's principle states that the natural motions of a hamiltonian system are the extremal curves of the functional

$$S = \int L \, dt \tag{1}$$

where L is the lagrangian function of the system. On the other hand, the geodesics of a riemannian manifold are the extremal curves of the length functional

$$l = \int ds \tag{2}$$

where s is the arc-length parameter.

If, by choosing a suitable metric, it is possible to establish a relationship between length and action, then it becomes possible to identify geodesics with physical trajectories.
Eisenhart metric has been used successfully to attain an understanding of the origin of chaos in Hamiltonian systems, relating the stability of the trajectories to the stability of the geodesics, which is completely determined by the curvature of the manifold.

In riemannian spaces various symmetries, described by the properties of infinitesimal transformation $x^a = x_0^a + \xi^a \ \delta s$, exist and some of them have a geometrical interpretation.

Motions, *i.e.* killing vectors ξ satisfying:

$$\pounds_\xi \ g_{ab} = \xi_{(a;b)} = 0 \tag{3}$$

preserve the metric tensor (isometry). As usual $\pounds_\xi$ denotes the Lie derivative, semicolon covariant derivative and $(,)$ symmetrization.

The conformal motions, ξ, satisfying

$$\$_{\xi}\; g_{ab} = 2\;\sigma(x)\; g_{ab} \tag{4}$$

preserve the angles between two directions at a point and map null geodesics into null geodesics. If $\sigma(x) = const.$ then we have a homothetic motion.

The affine collineation, ξ, satisfying:

$$\$_\xi \, \Gamma^a_{bc} = \xi_{(a;b;c)} = 0 \tag{5}$$

preserves the affine parameter on the geodesic, *i.e.* the geodesic equation structure remains unaltered by such a transformation.
Another type of symmetry of the metric is given by the existence of a symmetric tensor K_{ab}

The Killing Tensor, K_{ab}, satisfying:

$$K_{ab;c} = 0 \tag{6}$$

Associated to these symmetries there exist constants of the motion:

For Killing vectors

$$J = \xi^a P_a \tag{7}$$

For affine collineations

$$C_1 = m_{a;b}\; P^a P^b \tag{8}$$

$$C_2 = m_a\; P^a - s\; C_1 \tag{9}$$

are constant of the motion.

Associated to the Killing tensor there also exist conserved quantities

For the Killing Tensor

$$J_{(\mu)} = K^{\mu}_{ab} P^a P^b \tag{10}$$

Since, in this riemannian manifold geodesic lines coincide with the actual paths of a given dynamical system, we can establish a direct correspondence between the first integral, involving killing vectors or affine collineations on the geodesics, and the first integrals of the dynamical system.

Consider a system of n degrees of freedom described by $q^i(i = 1, ..., n)$ generalized coordinates, the kinetic energy has the general form

$$T = \frac{1}{2} g_{ij}\left(q^r, t\right)\ \dot{q}^i \dot{q}^j + D_i\left(q^r, t\right)\ \dot{q}^i + C\left(q^r, t\right) \tag{11}$$

As usual, repeated indices are summed. We assume that the $n \times n$ matrix g_{ij} is not degenerate, *i.e.* $\det g \neq 0$; then we may consider g_{ij} as a metric in the configuration space. The contravariant metric is defined by $g^{ij} g_{ij} = \delta^i_j$ to describe the kinetic energy in tensor language, we consider $n+1$ dimensional space V^{n+1} with coordinates $q^i,\ q^{n+1}$, where we set

$$q^{n+1} = t \implies \dot{q}^{n+1} = 1 \tag{12}$$

Then the kinetic energy is written as

$$T = \frac{1}{2} g_{ij} \left(q^r, t\right) \dot{q}^i \dot{q}^j + g_{i\ n+1} \left(q^r, t\right) \ \dot{q}^i + \frac{1}{2} g_{n+1\ n+1} \left(q^r, t\right) \tag{13}$$

where

$$g_{i\ n+1} = D_i, \qquad g_{n+1\ n+1} = 2C \tag{14}$$

The lagrangian of the system is

$$L(q^r, \dot{q}^r, t) = T - V = \frac{1}{2} g_{ij} \left(q^r, t\right) \dot{q}^i \dot{q}^j + g_{i\ n+1} \left(q^r, t\right) \ \dot{q}^i + \frac{1}{2} g_{n+1\ n+1} \left(q^r, t\right) - V \tag{15}$$

We can write Lagrange equations, $\frac{d}{dt}\left(\frac{\partial L}{\partial\ \dot{q}^i}\right) = \frac{\partial L}{\partial q^i}$, using the notation of riemannian geometry:

$$\ddot{q}^i + \Gamma^i_{jk}\ \dot{q}^j \dot{q}^k + g^{ir}\left[j\ \ n+1, r\right] \dot{q}^j + g^{ij} g_{j\ \ n+1,n+1} - \frac{1}{2} g^{ij}\left[g_{n+1\ \ n+1} - 2V\right]_{,j} = 0 \tag{16}$$

where "," indicates partial derivative,

$$\Gamma^i_{jk} = g^{il}\left[jk, l\right] \tag{17}$$

and

$$\left[jk, l\right] = \frac{1}{2}\left(g_{jl,k} + g_{kl,j} - g_{jk,l}\right) \tag{18}$$

The time t has no equation of motion because it is not a dynamical variable.

Let us consider now a $n+2$ dimensional space with coordinate functions (q^i, t, u) endowed with a metric defined by

$$ds^2 = g_{ij}\ dq^i dq^j + 2g_{i\ \ n+1} dq^i dt + A\ dt^2 + 2\ dt\ du \tag{19}$$

If we define

$$A = g_{n+1\ \ n+1} - 2V \tag{20}$$

then the geodesic equations

$$\frac{d^2 q^i}{ds^2} + \Gamma^i_{jk}\ \frac{dq^j}{ds}\ \frac{dq^k}{ds} = 0 \tag{21}$$

of the V^{n+2} space with metric (19) are:

Lagrange equations of motion of the dynamical system:

$$\ddot{q}^i + \Gamma^i_{jk}\ \dot{q}^j \dot{q}^k + g^{ir}\left[j\ \ t, r\right] \dot{q}^j + g^{ij} g_{j\ \ t,t} - \frac{1}{2} g^{ij}\left[g_{tt} - 2V\right]_{,j} = 0 \qquad (22)$$

where we have written t for the index $n+1$.

The equation which relates the coordinate t to the arc-length s along the geodesic

$$t = a\ s \qquad (23)$$

The equation which relates the coordinate u to the coordinates q^i and t

$$u(q^i, t) = \frac{1}{2}\frac{t}{a^2} - \int L dt + b \qquad (24)$$

For a two-dimensional Lagrangian with the following form

$$L = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - V(x, y)$$

the Eisenhart metric reads

$$ds^2 = dx^2 + dy^2 - 2V(x, t)dt^2 + 2dtdu \tag{25}$$

The Killing equations for this metric are

$$\xi^x_{,x} = 0$$

$$\xi^x_{,y} + \xi^y_{,x} = 0$$

$$\xi^x_{,t} - 2V(x, y)\xi^t_{,x} + \xi^u_{,x} = 0$$

$$\xi^x_{,u} + \xi^t_{,x} = 0$$

$$\xi^y_{,y} = 0$$

$$\xi^y_{,t} - 2V(x,y)\xi^t_{,y} + \xi^u_{,y} = 0$$

$$\xi^y_{,u} + \xi^t_{,y} = 0$$

$$-\xi^x V(x,y)_{,x} - \xi^y V(x,y)_{,y} - 2V(x,y)\xi^t_{,t} + \xi^u_{,t} = 0$$

$$-2V(x,y)\xi^t_{,u} + \xi^u_{,u} + \xi^t_{,t} = 0$$

$$\xi^t_{,u} = 0$$

For the special case of the Hénon-Heiles potential

The Hénon-Heiles potential

$$V(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + x^2 y - \frac{1}{3}y^2 \qquad (26)$$

Solving the Killing equations for the above potential, we obtain the following solution

Solution

$$\xi^x = 0 \tag{27}$$

$$\xi^y = 0 \tag{28}$$

$$\xi^t = \alpha \tag{29}$$

$$\xi^u = \beta \tag{30}$$

Substituting in (7) and reminding that $P^a = \frac{dx^a}{ds} = \frac{dx^a}{dt}$ we have that

$$P^x = \dot{x}$$

$$P^y = \dot{y}$$

$$P^t = 1$$

$$P^u = \dot{u} = \frac{1}{2} - L$$

therefore,

the associated coserved quantities are

$$J_{(1)} = \frac{1}{2} - \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + V = 1 - E \quad (31)$$

$$J_{(2)} = 1 \quad (32)$$

We can see that the only non-trivial conserved quantitie is the ENERGY of the system.

For the Hénon-Heiles potential the non-zero components of the conections are

$$\Gamma^x_{tt} = x + xy$$
$$\Gamma^y_{tt} = y + \frac{1}{2}x^2 - \frac{1}{2}y^2$$
$$\Gamma^u_{xt} = -x - 2xy$$
$$\Gamma^u_{yt} = y + x^2 - y^2$$

Substituting these expressions in (5), and solving the system of 40 equations we find

The Affine Collineations

$$\eta^x(x, y, t, u) = 0 \quad (33)$$
$$\eta^y(x, y, t, u) = 0 \quad (34)$$
$$\eta^t(x, y, t, u) = C \quad (35)$$
$$\eta^u(x, y, t, u) = k_1 t + k_2 \quad (36)$$

Substituting this in (8) and (9), we can see that

the associated coserved quantities are

$$C_1 = k_1 \tag{37}$$

$$C_2 = C[\frac{1}{2} - (T + V)] \tag{38}$$

Also in this case, we notice that the only non-trivial conserved quantity is the ENERGY.

Now, to compute the Killing tensor, we use (6) we obtain a system of twenty coupled PDEs whose solution will give ten components of the Killing tensor. After a very cumbersome computation we find that the Eisenhart metric associated to the Henon-Heiles potential admits three Killing tensors given by

1. $K_{xx} = c_1\,,\ K_{yy} = c_1,\ K_{tt} = c_1(x^2 + 2x^2y - \frac{2}{3}y^3 + y^2),\ K_{tu} = 0$
2. $K_{xx} = 0,\ K_{yy} = 0,\ K_{tt} = -2c_2(x^2 + 2x^2y - \frac{2}{3}y^3 + y^2),\ K_{tu} = c_2$
3. $K_{xx} = 0,\ K_{yy} = 0,\ K_{tt} = c_3,\ K_{tu} = 0$

Associated to the Killing tensors we have conserved quantities whose expression is given by

$$J_{(\mu)} = K_{ab}^{(\mu)} P^a P^b$$

Applying the above definition to the Henon-Heiles potential we find that the only conserved associated to the existence of Killing tensors is the ENERGY.

As it is well known, the only constant of the motion of the Henon-Heiles system is the energy. In this contribution, following a completely different approach, based on the equivalence between the geodesic equation and the Lagrange equations, find the same result. This fact provides, from a geometric point of view, an additional evidence of the non integrability of the system.